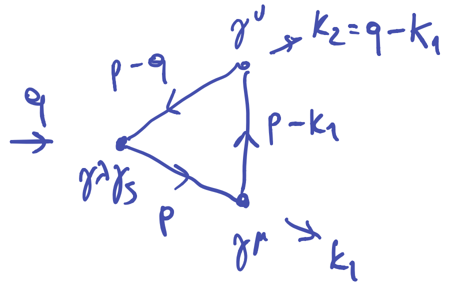
# CHIRAL ANOMALY IN 4d

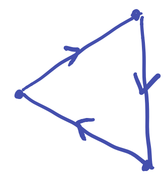
$P = \bar{\Psi}\gamma_5\psi$

Ci calcoliamo i correlatori $\underbrace{\langle j^\mu j^\nu j_A^\lambda \rangle}_{T^{\mu\nu\lambda}}$ e $\underbrace{\langle P j^\mu j^\nu \rangle}_{T^{\mu\nu}}$

AWI : $q_\lambda T^{\mu\nu\lambda} = 2m\, T^{\mu\nu}$

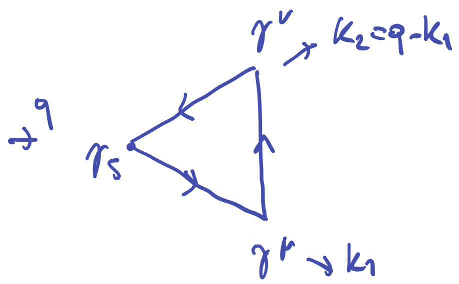
VWI : $k_{1\mu} T^{\mu\nu\lambda} = 0 = k_{2\nu} T^{\mu\nu\lambda}$

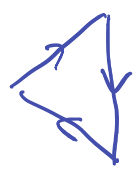
$\langle j^\mu j^\nu j_A^\lambda \rangle$ 

$+$ 

$\downarrow$

$$T^{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\left[\frac{i}{\not p - m}\gamma^\lambda\gamma_5 \frac{i}{\not p - \not q - m}\gamma^\nu \frac{i}{\not p - \not k_1 - m}\gamma^\mu\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$\langle j^\mu j^\nu P \rangle$ 

$+$ 

$\downarrow$

$$T^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}(-)\,\mathrm{tr}\left[\frac{i}{\not p - m}\gamma_5 \frac{i}{\not p - \not q - m}\gamma^\nu \frac{i}{\not p - \not k_1 - m}\gamma^\mu\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Avremo bisogno della seguente identità:

$$\not q \gamma_5 = \gamma_5(\not p - \not q - m) + (\not p - m)\gamma_5 + 2m\gamma_5$$

$$q_\lambda T^{\mu\nu\lambda} = 2m\, T^{\mu\nu} + R_1^{\mu\nu} + R_2^{\mu\nu}$$

$$R_1^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left( \overbrace{\frac{1}{\not p - \not k_2 - m}\, \gamma_5 \gamma^\nu\, \frac{1}{\not p - \underbrace{\not q}_{=k_1+k_2} - m}\, \gamma^\mu}^{1°} - \overbrace{\frac{1}{\not p - m}\, \gamma_5 \gamma^\nu\, \frac{1}{\not p - \not k_1 - m}\, \gamma^\mu}^{2°} \right)$$

$$R_2^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left( \frac{1}{\not p - \not k_1 - m}\, \gamma_5 \gamma^\mu\, \frac{1}{\not p - \not q - m}\, \gamma^\nu - \frac{1}{\not p - m}\, \gamma_5 \gamma^\mu\, \frac{1}{\not p - \not k_2 - m}\, \gamma^\nu \right)$$

Se $R_1 = R_2 = 0 \Rightarrow$ AWI è soddisfatta

→ Formalmente $R_1 \to 0$ se shiftiamo $p \to p + k_2$ nel 1° integrale e $p \to p - k_2$ nel 2° integrale

$$R_1^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left( \frac{1}{\not p - m}\, \gamma_5 \gamma^\nu\, \frac{1}{\not p - \not k_1 - m}\, \gamma^\mu - \frac{1}{\not p - \not k_2 - m}\, \gamma_5 \gamma^\nu\, \frac{1}{\not p - \not q - m}\, \gamma^\mu \right)$$

$$= - R_1^{\mu\nu} \Rightarrow R_1^{\mu\nu} = 0$$

Stesso conto per $R_2^{\mu\nu}$ $(p \to p \pm k_1)$

→ siccome l'integrale diverge (div. lineare), le manipolazioni interne all'integrale non sono giuste $(\infty - \infty)$

→ dobbiamo regolarizzare la teoria.

Per questo conto useremo la regolarizzaz. di PAULI-VILLARS: Introduciamo una specie ausiliaria di fermioni $\Psi$ con STATISTICA OPPOSTA e massa $M$; alla fine del conto bisogna prendere il lim. $M \to \infty$ (statistica errata)

$\hookrightarrow$ • $T^{\mu\nu\lambda}_{reg} = T^{\mu\nu\lambda}(m) - T^{\mu\nu\lambda}(M)$

$$T^{\mu\nu\lambda}_{phys} \equiv \lim_{M\to\infty} T^{\mu\nu\lambda}_{reg}$$

• $T^{\mu\nu}_{phy} = \lim_{M\to\infty} T^{\mu\nu}_{reg} = \lim_{M\to\infty} \left(T^{\mu\nu}(m) - T^{\mu\nu}(M)\right) = T^{\mu\nu}(m)$

$T^{\mu\nu}$ è convergente e $T^{\mu\nu}(M) \sim \frac{1}{M} \xrightarrow[M\to\infty]{} 0$

Questa regolarizzazione fa una scelta specifica in di preservare tra AWI e VWI :

$$\left.\begin{array}{l} K_{1\mu} T^{\mu\nu\lambda}_{phy} = 0 \\ K_{2\nu} T^{\mu\nu\lambda}_{phys} = 0 \end{array}\right\} \longrightarrow \text{VWI è PRESERVATA}$$

(Euristicam. $\mathcal{L}_{\Psi}$ non rompe $U(1)_V$ )

AWI ?

Ora $T^{\mu\nu\lambda}_{reg}$ è finito ; in particolare $R_1{}^{\mu\nu}_{reg}$ e $R_2{}^{\mu\nu}_{reg}$ sono finiti $\longrightarrow$ quindi ora posso fare le manipolazioni all'interno dell'integrale $\Longrightarrow R_1{}^{\mu\nu}_{reg} = 0 = R_2{}^{\mu\nu}_{reg}$

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda}_{reg} = 2m\, T^{\mu\nu}(m) - 2M\, T^{\mu\nu}(M)$$

$$\Rightarrow \quad q_\lambda T^{\mu\nu\lambda}_{phys} = 2m\, T^{\mu\nu}(m) - \underbrace{\lim_{M\to\infty} 2M\, T^{\mu\nu}(M)}_{\substack{\text{ora calcoleremo qta} \\ \text{(violazione di AWI} \\ \text{se} \neq 0)}}$$

$$T^{\mu\nu}(M) = \int \frac{d^4p}{(2\pi)^4} (-) \, \mathrm{tr} \frac{1}{\not p - M} \gamma_5 \frac{1}{\not p - \not q - M} \gamma^\nu \frac{1}{\not p - \not k_1 - M} \gamma^\mu + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$\frac{1}{a_1 a_2 a_3} = 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{\left[a_1 x_2 + a_2(1-x_1-x_2) + a_3 x_1\right]^3}$$

$$= - \int \frac{d^4p}{(2\pi)^4} 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{\mathrm{tr}\left[(\not p + M)\gamma_5(\not p - \not q + M)\gamma^\nu(\not p - \not k_1 + M)\gamma^\mu\right]}{\left[(p^2 - M^2)x_2 + \left((p-q)^2 - M^2\right)(1-x_1-x_2) + \left((p-k_1)^2 - M^2\right)x_1\right]^3} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$\mathrm{Tr}\, \gamma_5 \gamma^{\alpha_1} \cdots \gamma^{\alpha_n} = 0 \quad \text{für } n \neq 4$$
$$= 4i\, \epsilon^{\alpha_1 \cdots \alpha_4} \quad \text{für } n = 4$$

$$\mathrm{tr}(\not p \gamma_5 (\not p - \not q)\gamma^\nu \gamma^\mu) + \mathrm{tr}(\not p \gamma_5 \gamma^\nu (\not p - \not k_1)\gamma^\mu) + \mathrm{tr}(\gamma_5 (\not p - \not q)\gamma^\nu (\not p - \not k_1)\gamma^\mu)$$

$$= \mathrm{tr}(\gamma_5 \cancel{\not p \not q} \gamma^\nu \gamma^\mu) + \mathrm{tr}(\gamma_5 \cancel{\not p} \gamma^\nu \not k_1 \gamma^\mu) - \mathrm{tr}(\cancel{\gamma_5 \not p \gamma^\nu \not k_1 \gamma^\mu})$$

$$- \mathrm{tr}(\cancel{\gamma_5 \not q \gamma^\nu \not p \gamma^\mu}) + \mathrm{tr}(\gamma_5 \underbrace{\not q}_{\cancel{\not k_1} + \not k_2} \gamma^\nu \not k_1 \gamma^\mu) =$$

$$= 4i\, \epsilon^{\beta\nu\alpha\mu} k_{1\alpha} k_{2\beta} \qquad \epsilon^{\alpha\nu\beta\mu} = \epsilon^{\beta\alpha\nu\mu} = \epsilon^{\beta\mu\alpha\nu}$$

$$= 4i\, \epsilon^{\beta\mu\alpha\nu} k_{2\alpha} k_{1\beta}$$

$$= - \int \frac{d^4p}{(2\pi)^4} 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{M(-4i\,\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta})}{\left[x_2 p^2 - x_2 M^2 + (1-x_1-x_2)(p^2 - 2pq + q^2 - M^2) + x_1(p^2 - 2pk_1 + k_1^2 - M^2)\right]^3} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$p^2 - M^2 - 2p\left(q(1-x_1-x_2) + k_1 x_1\right) + q^2(1-x_1-x_2) + k_1^2 x_1$$
$$= p^2 - 2pK - \bar M^2 \qquad K \equiv q(1-x_1-x_2) + k_1 x \qquad \bar M \equiv M^2 - q^2(1-x_1-x_2) - k_1^2 x_1$$

$$= 8iM\,\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 dx_1\int_0^{1-x_1}dx_2\underbrace{\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-2pk-\bar{M}^2)^3}} + \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

$p^0 = il^0$

$\Downarrow$

$p^2 = -l^2$

$$\to -i\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+2pk+\bar{M}^2)^3} = \frac{-i\,\Gamma(3-2)}{(\bar{M}-k^2)^{3-2}(4\pi)^2\,\Gamma(3)}$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+2kp+b^2)^A} = \frac{\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}(4\pi)^{d/2}\Gamma(A)}$$

$$= \frac{-i}{(4\pi)^2}\frac{1}{2(\bar{M}^2-k^1)}$$

$$= \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{M}{M^2+\underbrace{f(x_1,x_2)}_{\text{non dip da } M}} + \begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}$$

stessa espressione del primo termine ma con $f(x_1,x_2)$ sostituita da una diversa $g(x_1,x_2)$

$$\lim_{M\to\infty}2M\,T^{\mu\nu}(M) = \frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 dx_1\int_0^{1-x_1}dx_2\underbrace{\lim_{M\to\infty}\left(\frac{2M^2}{M^2+f(x_1,x_2)}+\frac{2M^2}{M^2+g(x_1,x_2)}\right)}_{=4}$$

$$= \frac{1}{\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\underbrace{\int_0^1 dx_1\int_0^{1-x_1}dx_2}_{=\int_0^1 dx_1(1-x_1) = 1-\frac{1}{2}=\frac{1}{2}}$$

$$q_\lambda T^{\mu\nu\lambda}_{phys} = 2m\,T^{\mu\nu}(m)\underbrace{-\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}}_{\text{ANOMALIA}}$$

$$\partial^z_\lambda \langle j^\mu(x)\, j^\nu(y)\, j^\lambda_A(z)\rangle = -i\int \frac{dq}{(2\pi)^4}\int\frac{dk_1}{(2\pi)^4}\int\frac{dk_2}{(2\pi)^4} e^{-ik_1x-ik_2y+iqz} \cdot iq_\lambda T^{\mu\nu\lambda}(q,k_1,k_2)$$

$$= \int\frac{dq}{(2\pi)^4}\int\frac{dk_2}{(2\pi)^4} e^{-ik_2(y-x)+iq(z-x)}\left(-\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta} iq_\alpha(-i)k_{2\beta}\right)$$

$$= -\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial^z_\alpha\,\delta(z-x)\,\partial^y_\beta\,\delta(y-x)$$

$$\partial^z_\lambda\langle j^\lambda_A(z)\rangle_{\underset{\uparrow}{A}} = \partial^z_\lambda\langle j^\lambda_A(z)\rangle_{\text{free}} + \dots -\frac{1}{2}\int dx_1\, dx_2\, A_{\rho_1}(x_1)A_{\rho_2}(x_2)\cdot$$

$$\cdot\,\partial_\lambda\langle j^{\rho_1}(x_1)\, j^{\rho_2}(x_2)\, j^\lambda_A(z)\rangle + \dots$$

campo di gauge esterno accoppiato alla corrente vettoriale $j^\mu$

$$= -\int dx_1\, dx_2\, A_{\rho_1}(x_1)A_{\rho_2}(x_2)\left(-\frac{1}{4\pi^2}\right)\epsilon^{\rho_1\rho_2\alpha\beta}\partial^z_\alpha\,\delta(z-x_1)\,\partial^{x_2}_\beta\,\delta(x_2-\overset{=z}{\overset{\downarrow}{x_1}})$$

$$= \frac{1}{16\pi^2}\epsilon^{\rho_1\rho_2\alpha\beta}F_{\alpha\rho_1}F_{\beta\rho_2}$$

$$= -\frac{1}{16\pi^2}\epsilon^{\alpha\rho_1\beta\rho_2}F_{\alpha\rho_1}F_{\beta\rho_2}$$

$$= -\frac{1}{16\pi^2}\epsilon^{\mu_1\nu_1\mu_2\nu_2}F_{\mu_1\nu_1}F_{\mu_2\nu_2}$$