1 Schrödinger equations

sec:Segal

For $u_0 \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the linear homogeneous Schrödinger equation is

$$\mathrm{i}u_t + \triangle u = 0\,, u(0,x) = u_0(x). \tag{1.1}$$ linear_SE

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + \mathrm{i}|\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi)$. We have $e^{-\mathrm{i}t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t\mathrm{i})^{-\frac{d}{2}} e^{\frac{\mathrm{i}|x|^2}{4t}}$. This follows from the following generalization of (**??**) for $\mathrm{Re}\, z > 0$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2z}}\, dx.$$

This formula follows from the fact that both sides are holomorphic in $\mathrm{Re}\, z > 0$ and coincide for $z \in \mathbb{R}_+$. Then taking the limit $z \to 2\mathrm{i}$ for $\mathrm{Re}\, z > 0$ and using the continuity of $\mathcal{F}$ in $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ we get

$$e^{-\mathrm{i}|\xi|^2} = (4\pi \mathrm{i})^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{\frac{\mathrm{i}|x|^2}{4}}\, dx.$$

Then $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * u_0(x)$. In particular, for $u_0 \in L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and by Reisz's interpolation defines for any $t > 0$ an operator which we denote by

$$e^{\mathrm{i}\triangle t} u_0(x) = (4\pi \mathrm{i}t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} u_0(y)dy \tag{1.2}$$ eq:schsgroup

which is s.t. $e^{\mathrm{i}\triangle t} : L^p(\mathbb{R}^d, \mathbb{C}) \to L^{p'}(\mathbb{R}^d, \mathbb{C})$ for $p \in [1,2]$ and $p' = \frac{p}{p-1}$ with $\|e^{\mathrm{i}\triangle t} u_0\|_{L^{p'}} \le (4\pi t)^{-d(\frac{1}{2} - \frac{1}{p'})} \|u_0\|_{L^p}$ by Riesz interpolation.

rem:scr *Remark* 1.1. Notice that for no $p \neq 2$ and $t > 0$ we have that $e^{\mathrm{i}\triangle t}$ defines a bounded operator $L^p(\mathbb{R}^d, \mathbb{C}) \to L^p(\mathbb{R}^d, \mathbb{C})$, see hormander [8].

rem:heat *Remark* 1.2. Notice that $e^{\triangle t} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator for all $1 \le p \le q \le \infty$.

Notice that (1.1) is time reversible. and if $u(t,x) = e^{\mathrm{i}\triangle t} u_0(x)$, then $v(t,x) = \overline{u(-t,x)} = e^{\mathrm{i}\triangle t}\overline{u_0}(x)$ is a solution.

Let now $u(t,x) = e^{\mathrm{i}\triangle t} u_0(x)$, and for $\mathbf{v}, D \in \mathbb{R}^d$ consider $v_0(x) = e^{\mathrm{i}\frac{\mathbf{v}}{2}\cdot x} u_0(x - D)$. Then

$$v(t,x) := e^{\mathrm{i}\triangle t} v_0(x) = e^{\frac{\mathrm{i}}{2}\mathbf{v}\cdot x - \mathrm{i}\frac{\mathbf{v}^2}{4}t} u(t, x - t\mathbf{v} - D).$$

In the sequel, given $v, w \in L^2(\mathbb{R}^d, \mathbb{C})$ we will use the notation

$$\langle v, w \rangle = \mathrm{Re} \int_{\mathbb{R}^d} v(x)\overline{w}(x) dx. \tag{1.3}$$ eq:scalar_Sch

In the sequel we will reinterpret the equation

$$\mathrm{i}u_t + \triangle u = f\,, \quad u(0) = u_0 \in H^1(\mathbb{R}^d) \tag{1.4}$$
eq:LinSh

in the integral form

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\int_0^t e^{\mathrm{i}(t-t')\triangle}f(t')dt'. \tag{1.5}$$
eq:sduhxsidw

To understand this formula we will need Strichartz's inequalities.
We say that a pair (q,r) is *admissible* when

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \tag{1.6}$$
admissiblepair1

$$2 \leq r \leq \frac{2d}{d-2} \ (2 \leq r \leq \infty \text{ if } d = 1,\ 2 \leq r < \infty \text{ if } d = 2). \tag{1.7}$$
admissiblepair2

rem:admiss *Remark* 1.3. The pair $(\infty, 2)$ is always admissible. The *endpoint* $(2, \frac{2d}{d-2})$ is admissible for $d \geq 3$ but the point $(2,\infty)$ is not for $d = 2$. The equality (1.6) needs to be true by the parabolic scaling $u(t,x) \rightsquigarrow u(\lambda^2 t, \lambda x)$, which preserves the set of solutions to (1.1).

We have the following important result.

thm:strich **Theorem 1.4** (Strichartz's estimates)**.** *The following facts hold.*

(1) For every $u_0 \in L^2(\mathbb{R}^d)$ we have $e^{\mathrm{i}\triangle t}u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$ for every admissible (q,r). Furthermore, there exists a C s.t.

$$\|e^{\mathrm{i}\triangle t}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \leq C\|u_0\|_{L^2}. \tag{1.8}$$
strich1

(2) Let I be an interval and let $t_0 \in \overline{I}$. If (γ,ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then for any admissible pair (q,r) the function

$$\mathcal{T}f(t) = \int_{t_0}^t e^{\mathrm{i}\triangle(t-s)}f(s)ds \tag{1.9}$$
strich2

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\overline{I}, L^2(\mathbb{R}^d))$ and there exists a constant C independent of I and f s.t.

$$\|\mathcal{T}f\|_{L^q(I,L^r(\mathbb{R}^d))} \leq C\|f\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{R}^d))}. \tag{1.10}$$
strich3

2 Keel and Tao's proof of Strichartz estimates

Strichartz

We will follow the argument by Keel and Tao [7] (Kl-Tao). We will assume that (X,dx) is a measurable space and that H is a Hilbert space. We consider a family of operators $U(t) : H \to L^2(X)$. We assume the following two hypotheses.

(1) There exists a $C > 0$ s.t.

$$\|U(t)f\|_{L^2} \leq C\|f\|_H \text{ for all } f \in H;$$

(2) there exist a $\sigma > 0$ and a $C > 0$ s.t. for all $t \neq s$ and all $g \in L^1(X)$ we have

$$\|U(t)(U(s))^* g\|_{L^\infty} \leq C|t-s|^{-\sigma}\|g\|_{L^1}.$$

We say that a pair (q, r) is σ*–admissible* when

$$\begin{aligned} &\frac{2}{q} + \frac{2\sigma}{r} = \sigma \\ &r, q \geq 2 \text{ and } (q, r, \sigma) \neq (2, \infty, 1). \end{aligned} \tag{2.1}$$ sigadmissiblepair

Particularly important, for $\sigma > 1$, is the point $P = \left(2, \frac{2\sigma}{\sigma - 1}\right)$.

Notice that (1) implies $\|U^*(t)F\|_{L^2} \leq C\|F\|_{L^2}$ by duality and that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, (U(t))^* f\rangle_H$ [1]

:taostrich **Theorem 2.1** (Keel and Tao's Strichartz estimates)**.** *If $U(t)$ satisfies (1) and (2), and if furthermore there exists an appropriate scaling operator in X and H, then we have*

(3)

$$\|U(t)u_0\|_{L^q(\mathbb{R}, L^r(X))} \leq C_{q,r}\|u_0\|_H.$$

(4)

$$\|\int_{\mathbb{R}} (U(s))^* F(s) ds\|_H \leq C\|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

(5)

$$\|\int_{t>s} U(t)(U(s))^* F(s) ds\|_{L^q(\mathbb{R}, L^r(X))} \leq C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{\widetilde{q}'}(\mathbb{R}, L^{\widetilde{r}'}(X))}.$$

for all admissible pairs (q, r) and $(\widetilde{q}, \widetilde{r})$.

(3) is called the homogeneous estimate and (5) the non–homogeneous estimate or also the retarded estimate. (3) and (4) are equivalent by duality. The scaling operators are used only in Sect. 2.2.

[1] Notice that since $h \to \langle U(t)h, f\rangle_{L^2(X)}$ is continuous, an element $f^* \in H$ remains defined such that $\langle U(t)h, f\rangle_{L^2(X)} = \langle h, f^*\rangle_H$. The map $f \to f^*$ is linear, bounded and $(U(t))^* f := f^*$.

2.1 Proof of the nonendpoint homogeneous estimate

onendpoint

We consider the case $(q,r) \neq P$. The proof of this case predates the paper by Keel and Tao.
It is elementary that (4) is equivalent to

$$\left| \int_{\mathbb{R}^2} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds \right| \leq C \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

(indeed, if $T : X \to H$ is an operator from a Banach space X to a Hilbert space H, we have $\|Tx\|_H \leq C\|x\|_X$ for all $x \in X$ if and only if $| \langle Tx, Ty \rangle_H | \leq C^2 \|x\|_X \|y\|_X$ for all $x, y \in X$). So we have to prove the above estimate. Furthermore, it is enough to prove the above bound for

$$T(F,G) := \int_{t>s} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds. \tag{2.2}$$

eq:TFG

By (1) we know that (3) holds for $q = \infty$ and $r = 2$. So pointwise

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = \left| \langle U(t)(U(s))^* F(s), G(t) \rangle_{L^2(X)} \right| \\ &\leq \|U(t)(U(s))^* F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)} \leq C^2 \|F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)}. \end{aligned}$$

Furthermore

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = \left| \langle U(t)(U(s))^* F(s), G(t) \rangle_{L^2(X)} \right| \leq \|U(t)(U(s))^* F(s)\|_{L^\infty(X)} \|G(t)\|_{L^1(X)} \\ &\leq C|t-s|^{-\sigma} \|F(s)\|_{L^1(X)} \|G(t)\|_{L^1(X)}. \end{aligned}$$

From the Riesz–Thorin Interpolation Theorem, see Theorem **??**, we have (omitting the constant) for any $r \in 2, \infty]$

$$\|U(t)(U(s))^* F(s)\|_{L^r(X)} \lesssim |t-s|^{-\sigma\left(1-\frac{2}{r}\right)} \|F(s)\|_{L^{r'}(X)} = |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)}$$

$$\text{where } \beta(r, \widetilde{r}) := \sigma - 1 - \frac{\sigma}{r} - \frac{\sigma}{\widetilde{r}}.$$

Then we conclude

$$|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| \lesssim |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} \|G(t)\|_{L^{r'}(X)}.$$

Then for $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$, using the Hardy,Littlewood Sobolev inequality, see Theorem **??**
, which requires $q > q'$,

$$|T(F,G)| \lesssim \| \int_{\mathbb{R}} |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} ds \|_{L^q(\mathbb{R})} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \lesssim \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

Notice that $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$ means

$$1 - \frac{2}{q} = -\sigma + 1 + 2\frac{\sigma}{r} \Leftrightarrow \frac{2}{q} + \frac{2\sigma}{r} = \sigma$$

and $-\beta(r,r) > 0$ means

$$r < \frac{2\sigma}{\sigma - 1}.$$

2.2 Proof of the endpoint homogeneous estimate

c:endpoint

Here we consider the endpoint case $(q,r) = P = (2, \frac{2\sigma}{\sigma-1})$, when $\sigma > 1$.

The introduction of a scaling operator will simplify considerably the discussion. We will denote it by D_λ for $\lambda > 0$. We assume the following:

1. there exist operators $D_\lambda : H \to H$ s.t. $\langle D_\lambda f, D_\lambda g\rangle_H = \lambda^{-\sigma} \langle f, g\rangle_H$
2. there exist operators $D_\lambda : L^r(X) \to L^r(X)$ s.t $\|D_\lambda f\|_{L^r(X)} = \lambda^{-\frac{\sigma}{r}} \|f\|_{L^r(X)}$
3. in all cases $D_\lambda^{-1} = D_{\lambda^{-1}}$ and $D_\lambda^* = \lambda^{-\sigma} D_{\lambda^{-1}}$.

Notice that for $\sigma = \frac{d}{2}$, $H = L^2(\mathbb{R}^d)$ and $X = \mathbb{R}^d$ with $L^r(X)$ the standard Lebesgue spaces, then $D_\lambda f(x) := f(\lambda^{\frac{1}{2}} x)$ satisfies the desired requirements. Notice that we used the same notation for dilation operators in H and $L^r(X)$, but they are distinct operators.

lem:resc **Lemma 2.2.** *Let the function $t \to U(t)$ satisfy (1) and (2) in Sect. 2. Then $t \to D_\lambda U(\lambda t) D_{\lambda^{-1}}$ satisfies (1) and (2) in Sect. 2 with exactly the same constants C.*

Proof. Indeed

$$\|D_\lambda U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} = \lambda^{-\frac{\sigma}{2}} \|U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} \le C\lambda^{-\frac{\sigma}{2}} \|D_{\lambda^{-1}} f\|_H = C\|f\|_H$$

and from $(D_\lambda U(\lambda s) D_{\lambda^{-1}})^* = D_\lambda (U(\lambda s))^* D_{\lambda^{-1}}$,

$$\begin{aligned}&\|D_\lambda U(\lambda t) D_{\lambda^{-1}} (D_\lambda U(\lambda s) D_{\lambda^{-1}})^* f\|_{L^\infty} \|D_\lambda U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \\ &= \|U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \le C\lambda^{-\sigma} |t-s|^{-\sigma} \|D_{\lambda^{-1}} f\|_{L^1} = C|t-s|^{-\sigma} \|f\|_{L^1}.\end{aligned}$$

□

After the above preliminary on scaling operators, expand

$$T(F,G) = \sum_{j\in\mathbb{Z}} T_j(F,G) \text{ where } T_j(F,G) := \int_{t-2^j > s > t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds. \tag{2.3}$$

eq:keel 21

We will prove

$$\sum_{j\in\mathbb{Z}} |T_j(F,G)| \lesssim \|F\|_{L^2 L^{r'}} \|G\|_{L^2 L^{r'}}. \tag{2.4}$$

eq:keel 22

We will prove the following.

m:keel 4.1 **Lemma 2.3.** *For a fixed constant C dependent only on the constants in (1) –(2) Sect. 2. we have*

$$|T_j(F,G)| \le C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}. \tag{2.5}$$

eq:keel 23

with $(1/a, 1/b)$ in a sufficiently small, but fixed neighborhood of $(1/r, 1/r)$, dependent only on σ.

Proof. Notice that

$$T_j(F,G) = \int_{t-2^j>s>t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds$$
$$= 2^{2j} 2^{j\sigma} \int_{t-1>s>t-2} \left\langle D_{2^j}(U(2^j s))^* D_{2^{-j}} D_{2^j} F(2^j s), D_{2^j}(U(2^j t))^* D_{2^{-j}} D_{2^j} G(2^j t)\right\rangle_H dtds.$$

Suppose now that we have (2.4) in the particular case $j=0$. Then we have

$$|T_j(F,G)| \le C 2^{2j} 2^{j\sigma} \|D_{2^j} F(2^j s)\|_{L^2 L^{a'}} \|D_{2^j} G(2^j t)\|_{L^2 L^{b'}} = C 2^{2j} 2^{j\sigma} 2^{-j\left(1+\frac{\sigma}{a'}+\frac{\sigma}{b'}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}$$
$$= C 2^{j\left(2+\sigma-1-2\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{j\left(1-\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}$$

where we recall $\beta(a,b) = \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}$.

So we have reduced to the case $j=0$. Next we do another reduction. We claim that to prove the case $j=0$ it is enough to assume that F and G are supported in time intervals of length 1. Indeed, assuming this case, then we have

$$|T_0(F,G)| \le \sum_{n\in\mathbb{Z}} \left| \int_{n+1>t>n} dt \int_{t-1>s>t-2} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, ds \right|$$
$$\le C \sum_{n\in\mathbb{Z}} \|F\|_{L^2((n,n+1),L^{a'})} \|G\|_{L^2((n-2,n),L^{b'})} \le C \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-2,n),L^{a'})}\right)^{\frac{1}{2}}$$
$$= C\sqrt{2} \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-1,n),L^{b'})}\right)^{\frac{1}{2}} = C\sqrt{2} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}.$$

Hence, in the rest of the proof we will assume that F and G are supported in time intervals of length 1. To prove (2.5) for $j=0$ we consider three cases:

(i) $a = b = \infty$;

(ii) $2 \le a < r$ and $b = 2$;

(iii) $a = 2$ and $2 \le b < r$.

Then the desired result follows by interpolation.

Let us start with (i). The proof is elementary and straightforward, because we have

$$|T_0(F,G)| \le \int dt \int_{t-1>s>t-2} |\langle U(t)(U(s))^* F(s), G(t)\rangle_{L^2(X)}| ds$$
$$\le C \int dt \int_{t-1>s>t-2} |t-s|^{-\sigma} \|F(s)\|_{L^1} \|G(t)\|_{L^1} \le C \int dt \int_{t-1>s>t-2} \|F(s)\|_{L^1} \|G(t)\|_{L^1}$$
$$\le C \|F\|_{L^1 L^1} \|G\|_{L^1 L^1} \le C \|F\|_{L^2 L^1} \|G\|_{L^2 L^1}.$$

Let us now consider (ii). Here we will use the Strichartz estimates in Sect. 2.1. We have

$$\begin{aligned}
|T_0(F,G)| &\le \int \left| \left\langle \int_{t-1>s>t-2} (U(s))^* F(s) ds, (U(t))^* G(t) \right\rangle_H \right| dt \\
&\le \int \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \|(U(t))^* G(t)\|_H dt \\
&\le \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \int \|(U(t))^* G(t)\|_H dt \\
&\le C \|G\|_{L^1 L^2} \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H ,
\end{aligned}$$

where we used (1) in Sect. 2. Now, using the non endpoint Strichartz estimates in Sect. 2.1 (notice here $2 \le a < r$) we have, for $(q(a), a)$ admissible,

$$\sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s) ds \right\|_H \le C \|F\|_{L^{q(a)'} L^{a'}} \le C \|F\|_{L^2 L^{a'}}.$$

This proves (ii) and by symmetry yields also (iii). □

Now we need to show that (2.5) implies (2.4). Obviously, we cannot just take $a = b = r$ and sum up, since $\beta(r,r) = 0$. To give an intuition on how to overcome this problem, Keel and Tao consider functions of the form

$$F(t) = 2^{-\frac{k}{r'}} f(t) \chi_{E(t)}(x) \text{ and } G(s) = 2^{-\frac{\tilde{k}}{r'}} g(s) \chi_{\tilde{E}(s)}(x), \tag{2.6}$$ eq:atoms0

with scalar functions $f(t), g(s)$ and $E(t)$ resp. $\tilde{E}(s)$ sets of size 2^k resp. $2^{\tilde{k}}$. Applying (2.5) we obtain

$$\begin{aligned}
|T_j(F,G)| &\le C 2^{-j\left(\sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}\right)} 2^{-\frac{k}{r'}} 2^{\frac{k}{a'}} 2^{-\frac{\tilde{k}}{r'}} 2^{\frac{\tilde{k}}{b'}} \|f\|_{L^2} \|g\|_{L^2} \\
&= C 2^{-j\left(\frac{2\sigma}{r} - \frac{\sigma}{a} - \frac{\sigma}{b}\right)} 2^{-(k+\tilde{k})\left(\cancel{1} - \frac{1}{r}\right) + \cancel{(k+\tilde{k})} - \frac{k}{a} - \frac{\tilde{k}}{b}} \|f\|_{L^2} \|g\|_{L^2} \\
&= C 2^{-j\left(\frac{2\sigma}{r} - \frac{\sigma}{a} - \frac{\sigma}{b}\right) + k\left(\frac{1}{r} - \frac{1}{a}\right) + \tilde{k}\left(\frac{1}{r} - \frac{1}{b}\right)} \|f\|_{L^2} \|g\|_{L^2} \\
&= C 2^{(k - j\sigma)\left(\frac{1}{r} - \frac{1}{a}\right) + (\tilde{k} - j\sigma)\left(\frac{1}{r} - \frac{1}{b}\right)} \|f\|_{L^2} \|g\|_{L^2}.
\end{aligned} \tag{2.7}$$ eq:atoms1

Notice now that we can adjust (a,b) s.t. for a fixed small $\varepsilon > 0$ the last term equals

$$C 2^{-\varepsilon|k - j\sigma| - \varepsilon\left|\tilde{k} - j\sigma\right|} \|f\|_{L^2} \|g\|_{L^2} \tag{2.8}$$ eq:atoms2

whose sum for $j \in \mathbb{Z}$ is finite.

To convert the above intuition in a proof we consider the following preliminary lemma.

m:keel 5.1 **Lemma 2.4.** *Let $p \in (0, \infty)$. Then any $f \in L^p_x$ can be written as*

$$f = \sum_{k \in \mathbb{Z}} c_k \chi_k$$

where $meas(supp \chi_k) \le 2\ 2^k$, $|\chi_k| \le 2^{-\frac{k}{p}}$ and $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}} \|f\|_{L^p}$.

Proof. Consider the distribution function $\lambda(\alpha) = \text{meas}(\{|f(x)| > \alpha\})$. Then for each k consider

$$\alpha_k := \inf_{\lambda(\alpha)<2^k} \alpha\,, \quad c_k := 2^{\frac{k}{p}} \alpha_k\,, \quad \chi_k := \frac{1}{c_k}\chi_{(\alpha_{k+1},\alpha_k]}(|f|)f.$$

Notice that $\{\alpha_k\}_{k\in\mathbb{Z}}$ is decreasing in k (since, the larger k, the larger is the set $\{\alpha : \lambda(\alpha) < 2^k\}$).
We show the desired properties. We have

$$\text{supp}\chi_k \subseteq \{x : \alpha_{k+1} < |f(x)| \le \alpha_k\} \subseteq \{x : |f(x)| > \alpha_{k+1}\}.$$

Then we get the 1st inequality:

$$\text{meas}(\text{supp}\chi_k) \le \text{meas}(\{x : |f(x)| > \alpha_{k+1}\}) = \lim_{\alpha\to\alpha_{k+1}^+} \lambda(\alpha) = \sup\{\lambda(\alpha) : \alpha > \alpha_{k+1}\}) \le 2^{k+1}.$$

Next, by $|f(x)| \le \alpha_k$ in $\text{supp}\chi_k$, we have

$$|\chi_k(x)| \le 2^{-\frac{k}{p}} \frac{|f(x)|}{\alpha_k} \le 2^{-\frac{k}{p}}.$$

Let now $\lim_{k\to+\infty} \alpha_k = \inf_{k\in\mathbb{Z}} \alpha_k = \underline{\alpha}$ and $\lim_{k\to-\infty} \alpha_k = \sup_{k\in\mathbb{Z}} \alpha_k = \overline{\alpha}$. Then we claim that $\underline{\alpha} = 0$ and that $|f(x)| \le \overline{\alpha}$ a.e. Indeed, suppose that $|f(x)| > \overline{\alpha}$ on a set of positive measure. There there is $\alpha > \overline{\alpha}$ with $\lambda(\alpha) > 2^k$ for some $k \in \mathbb{Z}$. Then $\alpha_k \ge \alpha > \overline{\alpha}$, which is a contradiction. On the other hand, suppose we have $0 < \alpha < \underline{\alpha}$. Then $\lambda(\alpha) = \infty$, since otherwise $\lambda(\alpha) < 2^k$ for a k, and then $\alpha \ge \alpha_k \ge \underline{\alpha} > \alpha$, getting a contradiction. But by Chebyshev's inequality,

$$\infty > \|f\|_{L^p}^p \ge \alpha^p \lambda(\alpha),$$

hence getting a contradiction. The above claim and the obvious fact that for any x we have $|f(x)| \in (\alpha_{k+1}, \alpha_k]$ for at most one k, prove $f = \sum_{k\in\mathbb{Z}} c_k\chi_k$ (the claim guarantees the existence of one such k).
We have $\|f\|_{L^p} \le 2^{\frac{1}{p}} \|c_k\|_{\ell^p}$ by

$$\|f\|_{L^p}^p = \int |f|^p dx = \int \sum_{k\in\mathbb{Z}} |c_k|^p |\chi_k|^p dx = \sum_{k\in\mathbb{Z}} |c_k|^p \int |\chi_k|^p dx \le \sum_{k\in\mathbb{Z}} |c_k|^p 2^{-k} \text{meas}(\text{supp}\chi_k)$$
$$\le 2 \sum_{k\in\mathbb{Z}} |c_k|^p$$

Next we have

$$\sum_{k\in\mathbb{Z}} |c_k|^p = \sum_{k\in\mathbb{Z}} 2^k \alpha_k^p = \int_{\mathbb{R}_+} \alpha^p \left(\sum 2^k \delta(\alpha - \alpha_k)\right) d\alpha = \int_{\mathbb{R}_+} \alpha^p (-F'(\alpha)) d\alpha$$

where

$$F(\alpha) := \sum_{k\in\mathbb{Z}} 2^k H(\alpha_k - \alpha) = \sum_{\alpha_k > \alpha} 2^k \le \sum_{2^k \le \lambda(\alpha)} 2^k \le 2\lambda(\alpha).$$

Then, integrating by parts and using (**??**),

$$\sum_{k\in\mathbb{Z}} |c_k|^p = p\int_{\mathbb{R}_+} \alpha^{p-1} F(\alpha) d\alpha \le 2p \int_{\mathbb{R}_+} \alpha^{p-1}\lambda(\alpha) d\alpha = 2\|f\|_{L^p}^p,$$

so that $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}} \|f\|_{L^p}$.

□

Furthermore we have the following.

:keel 5.1b **Lemma 2.5.** *Let $1 \le q, r < \infty$ and let $f \in L^q(I, L^r_x)$ with I an interval. Then we can write the expansion of Lemma 2.4*

$$f = \sum_{k\in\mathbb{Z}} c_k(t)\chi_k(t) \tag{2.9}$$ lem:keel 5.1b1

with $t \to \{c_k(t)\}$ a map in $L^q(I, \ell^r)$.

Proof. Formally this follows immediately from

$$\|\{c_k(t)\}\|_{L^q(I,\ell^r)} = \|\|\{c_k(t)\}\|_{\ell^r}\|_{L^q(I)} \le 2^{\frac{1}{p}} \|\|f\|_{L^r_x}\|_{L^q(I)}.$$

However one needs to argue that the function $t \to \{c_k(t)\}$ is measurable. By a density argument it is enough to consider the case of simple functions $f = \sum_{j=1,\dots,n} \chi_{E_j}(t) g_j(x)$ with E_j mutually disjoints sets. Then $\lambda(t,\alpha) = \text{meas}(\{|f(t,x)| > \alpha\}) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\lambda_j(\alpha)$ with λ_j the distribution function of g_j. Then $\alpha_k(t) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\alpha_k^{(j)}$ with $\alpha_k^{(j)}$ defined like in Lemma 2.4 for each g_j. Then

$$\{c_k(t)\} = \sum_{j=1,\dots,n} \chi_{E_j}(t)\{c_k^{(j)}\} \text{ for } c_k^{(j)} = 2^{\frac{k}{p}}\alpha_k^{(j)}.$$

This is measurable in t.

□

Consider now the

$$F(t) = \sum_{k\in\mathbb{Z}} f_k(t)\chi_k(t)\,, \quad G(s) = \sum_{k\in\mathbb{Z}} g_k(s)\widetilde{\chi}_k(s). \tag{2.10}$$ eq;kt29

By (2.6)–(2.8) e have

$$\sum_j |T_j(F,G)| \le \sum_{j,k,\widetilde{k}} |T_j(f_k\chi_k, g_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})| \le C \sum_{j,k,\widetilde{k}} 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \|f_k\|_{L^2_t} \|g_{\widetilde{k}}\|_{L^2_t}$$

$$= C \sum_{k,\widetilde{k}} \left(\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \right) \|f_k\|_{L^2_t} \|g_{\widetilde{k}}\|_{L^2_t}.$$

We claim that for a fixed $C = C(\sigma, \varepsilon)$

$$\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \le C 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|). \tag{2.11}$$ eq:youngKT

To prove this inequality, it is not restrictive to assume $k \le \widetilde{k}$. Then the summation on the left can be rewritten as

$$\sum_{j\sigma \le k} 2^{2\varepsilon j\sigma - \varepsilon(k+\widetilde{k})} + \sum_{k<j\sigma\le \widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} + \sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma}.$$

Then (here $[t] \in \mathbb{Z}$ is the integer part of $t \in \mathbb{R}$, defined by $[t] \le t < [t]+1$)

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j\le\left[\frac{k}{\sigma}\right]} 2^{2\varepsilon j\sigma} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{2\varepsilon\sigma\left(\left[\frac{k}{\sigma}\right]-j\right)} = C_{\varepsilon\sigma}\ 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\left[\frac{k}{\sigma}\right]}$$

$$\le C_{\varepsilon\sigma} 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\frac{k}{\sigma}} = C_{\varepsilon\sigma} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma} 2^{-\varepsilon|k-\widetilde{k}|} \text{ where } C_{\varepsilon\sigma} = \frac{1}{1-2^{-2\varepsilon\sigma}}.$$

We have

$$\sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma} \le 2^{\varepsilon(k+\widetilde{k})} \sum_{j\ge\left[\frac{\widetilde{k}}{\sigma}\right]+1} 2^{-2\varepsilon j\sigma} = 2^{\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1+j\right)} = C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1\right)}$$

$$\le C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\frac{\widetilde{k}}{\sigma}} = C_{\varepsilon\sigma} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma} 2^{-\varepsilon|k-\widetilde{k}|}.$$

Finally

$$\sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} = 2^{-\varepsilon(\widetilde{k}-k)} \sum_{\left[\frac{k}{\sigma}\right]+1\le j\sigma\le\left[\frac{\widetilde{k}}{\sigma}\right]} 1 = 2^{-\varepsilon(\widetilde{k}-k)} \left(\left[\frac{\widetilde{k}}{\sigma}\right] - \left[\frac{k}{\sigma}\right] - 1 \right) \le \sigma^{-1} 2^{-\varepsilon(\widetilde{k}-k)} (\widetilde{k}-k)$$

Hence (2.11) is proved. From this we conclude that for a fixed C

$$\sum_j |T_j(F,G)| \le C \sum_{k,\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}$$

$$\le C\|\{\|f_k\|_{L^2_t}\}\|_{\ell^2(\mathbb{Z})} \left\| \{\sum_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|g_{\widetilde{k}}\|_{L^2_t}\} \right\|_{\ell^2(\mathbb{Z})}$$

$$\le C\left(\sum_k 2^{-\varepsilon|k|}(1+|k|)\right)\|\{\|f_k\|_{L^2_t}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L^2_t}\}\|_{\ell^2(\mathbb{Z})}$$

where we used Lemma **??**. So, using $r' \le 2$,

$$\sum_j |T_j(F,G)| \le C'\|\{\|f_k\|_{L^2_t}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L^2_t}\}\|_{\ell^2(\mathbb{Z})} = C'\|\|\{f_k\}\|_{\ell^2(\mathbb{Z})}\|_{L^2_t}\|\|\{g_k\}\|_{\ell^2(\mathbb{Z})}\|_{L^2_t}$$

$$\le C''\|\|\{f_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L^2_t}\|\|\{g_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L^2_t} \le C'''\|\|F\|_{L^{r'}_x}\|_{L^2_t}\|\|\|G\|_{L^{r'}_x}\|_{L^2_t}$$

which completes the proof of (2.4).

ec:non hom

2.3 Proof of the non homogeneous estimate

We need to prove that for all admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$ we have

$$|T(F,G)| \le C_{q,r,\widetilde{q},\widetilde{r}}\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}\|G\|_{L^{\widetilde{q}'}(\mathbb{R},L^{\widetilde{r}'}(X))}. \qquad (2.12)$$

eq:29KT

We have already proved that this is true for $(q,r)=(\widetilde{q},\widetilde{r})$. Furthermore, proceeding like in Lemma 2.3

$$|T(F,G)| \le \int \left| \left\langle \int_{t>s}(U(s))^*F(s)ds, (U(t))^*G(t)\right\rangle_H \right| dt$$

$$\le \int \|\int_{t>s}(U(s))^*F(s)ds\|_H\|(U(t))^*G(t)\|_H dt \le \sup_t \|\int_{t>s}(U(s))^*F(s)ds\|_H \int \|(U(t))^*G(t)\|_H dt$$

$$\le C\|G\|_{L^1L^2}\sup_t\|\int_{t>s}(U(s))^*F(s)ds\|_H,$$

Then, by (4) in Theorem 2.1 (that is the dual homogenous estimates, which are already proved) for any admissible pair (q,r)

$$\sup_t\|\int_{t>s}(U(s))^*F(s)ds\|_H = \sup_t\|\int_{\mathbb{R}}(U(s))^*F(s)\chi_{(-\infty,t)}(s)ds\|_H \le C\|F\chi_{(-\infty,t)}\|_{L^{q'}(\mathbb{R},L^{r'})} \le C\|F\|_{L^{q'}(\mathbb{R},L^{r'})}.$$

So (2.12) holds for $(\widetilde{q},\widetilde{r})=(\infty,2)$ and any admissible pair (q,r). Obviously, symmetrically (2.12) holds for $(q,r)=(\infty,2)$ and any admissible pair $(\widetilde{q},\widetilde{r})$. Finally, let us consider (q,r)

and $(\widetilde{q},\widetilde{r})$ not in one of the cases already covered. Then it is not restrictive to assume that $(\widetilde{q},\widetilde{r}) = (a_{t_0}, b_{t_0})$ for $t_0 \in (0,1)$ where

$$\left(\frac{1}{a_t}, \frac{1}{b_t}\right) = t\left(\frac{1}{q}, \frac{1}{r}\right) + (1-t)\left(\frac{1}{\infty}, \frac{1}{2}\right).$$

In the cases $t = 0, 1$ the inequality holds, because these are cases considered above. By a generalization of Riesz–Thorin, Theorem **??**, the inequality holds also for the intermediate t's. $\square$

3 The semilinear Schrödinger equation

sec:NLS

There is a vast literature on semilinear Schrödinger equations. For a survey, with a concise discussion of some physical motivations, we refer to [13] (sulem). Here though, we consider only the mathematical formalism and only the pure power semilinear Schrödinger equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + \lambda |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{3.1}$$

eq:NLS

for $\lambda \in \mathbb{R}\backslash\{0\}$ and $p > 1$. Here $p < d^*$ with $d^* = \infty$ for $d = 1, 2$ and $d^* = \frac{d+2}{d-2}$ for $d \geq 3$. We collect here a number of facts needed later.

lem:misc **Lemma 3.1.** *We have the following facts.*

(1) For $1 < p < d^*$ we have the Gagliardo–Nirenberg inequality:

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \leq C_p \|\nabla u\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}. \tag{3.2}$$

eq:gn

(2) The map $u \to |u|^{p-1}u$ is a locally Lipschitz from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.

(3) For $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have $\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla \overline{u}$ and belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$.

Proof. For (1) see Theorem **??**.

We turn (2). By (3.2) we know that $u \to |u|^{p-1}u$ maps $H^1(\mathbb{R}^d) \to L^{p+1}(\mathbb{R}^d) \to L^{\frac{p+1}{p}}(\mathbb{R}^d)$. Furthermore this map is locally Lipschitz:

$$\begin{aligned} &\| |u|^{p-1}u - |v|^{p-1}v \|_{L^{\frac{p+1}{p}}} \leq C \| (|u|^{p-1} + |v|^{p-1})(u-v) \|_{L^{\frac{p+1}{p}}} \\ &\leq C' (\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}}) \|u - v\|_{L^{p+1}} \end{aligned}$$

where we have used, for $w = v - u$,

$$|u|^{p-1}u - |v|^{p-1}v = \int_0^1 \frac{d}{dt}\left(|u+tw|^{p-1}(u+tw)\right)dt =$$

$$\int_0^1 |u+tw|^{p-1}dtw + \int_0^1 (u+tw)\frac{d}{dt}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-1}{2}}dt = \int_0^1 |u+tw|^{p-1}dtw +$$

$$\sum_{j=1}^{2}\int_0^1 (u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)dtw_j$$

which from $|u+tw| \le |u|+|v|$ for $t \in [0,1]$ and

$$\left|(u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2+(u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)w_j\right| \le (p-1)|u+tw|^{p-1}|w|$$

yields

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \le p(|u|+|v|)^{p-1}|u-v| \le p2^{p-1}(|u|^{p-1}+|v|^{p-1})|u-v|,$$

where in the last step we used, for $|u| \ge |v|$,

$$(|u|+|v|)^{p-1} \le 2^{p-1}|u|^{p-1} \le 2^{p-1}(|u|^{p-1}+|v|^{p-1}).$$

Next, we show that we have an embedding $L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$. Indeed, this is equivalent to $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ with in turn is a consequence of (3.2).

We turn (3). First of all we claim that if $G \in C^1(\mathbb{C},\mathbb{C})$ with $G(0)=0$ and $|\nabla G| \le M < \infty$, then $\nabla(G(u)) = \partial_u G(u)\nabla u + \partial_{\overline{u}} G(u)\nabla \overline{u}$ in the sense of distributions. This claim can be proved like Proposition 9.5 in brezis [2] and we skip the proof here.
Let us now consider an increasing function $g \in C^\infty(\mathbb{R}_+,\mathbb{R})$ s.t.

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} & \text{for } 0 \le s \le 1 \\ 2^{\frac{p-1}{2}} & \text{for } s \ge 2 \end{cases}$$

and let us define $G_m(u) = m^{p-1} g\left(\frac{|u|^2}{m^2}\right)u$ for $m \in \mathbb{N}$. Then, by the claim, for all $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int G_m(u)\ \partial_j\varphi = \int \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j \overline{u}\right)\ \varphi. \tag{3.3}$$

eq:chain d--1

Let us take now the limit for $m \to \infty$. We have

$$\int G_m(u)\ \partial_j\varphi = \int |u|^{p-1}u\ \partial_j\varphi - \int_{|u|\ge m} |u|^{p-1}u\ \partial_j\varphi + \int_{|u|\ge m} G_m(u)\ \partial_j\varphi.$$

Now we have

$$\int_{|u|\geq m} |u|^{p-1}u\ \partial_j\varphi \xrightarrow{m\to\infty} 0 \text{ by Dominated Convergence}$$

since $\chi_{\{|u|\geq m\}}(x) \xrightarrow{m\to\infty} 0$ a.e. by Chebyshev's inequality. Similarly

$$\left|\int_{|u|\geq m} G_m(u)\ \partial_j\varphi\right| \leq \int_{|u|\geq m} |G_m(u)\ \partial_j\varphi| \leq 2^{p-1}\int_{|u|\geq m} m^{p-1}|u|\ \partial_j\varphi|$$
$$\leq 2^{p-1}\int_{|u|\geq m} |u|^p|\partial_j\varphi| \xrightarrow{m\to\infty} 0$$

Next, we consider the limit of the r.h.s. of (3.3). For $G(u)=|u|^{p-1}u$ we have

$$\int (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u})\ \varphi = \int (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u})$$
$$-\int_{|u|\geq m} (\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u})\ \varphi + \int_{|u|\geq m} (\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u})\ \varphi.$$

Then, like before, the terms of the 2nd line converge to 0 as $m\to\infty$ and so we conclude that all $\varphi\in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u\in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int |u|^{p-1}u\ \partial_j\varphi = \int \left(p|u|^{p-1}\partial_j u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2\partial_j\overline{u}\right)\ \varphi.$$

The fact of belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d,\mathbb{C})$ follows immediately from Hölder inequality.

□

Important are the following quantities:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}dx \\ P_j(u) &= \frac{1}{2}\operatorname{Im}\int_{\mathbb{R}^d} \partial_j u\,\overline{u}dx \\ Q(u) &= \int_{\mathbb{R}^d} |u|^2 dx. \end{aligned} \tag{3.4}$$

eq:energyfunctional

Here $E(u)$ is the energy, $P_j(u)$ for $j=1,...,d$ are the linear momenta and $Q(u)$ is the mass or charge.

functional *Remark* 3.2. Notice that $Q, P_j\in C^\infty(H^1(\mathbb{R}^d),\mathbb{R})$ while $E\in C^1(H^1(\mathbb{R}^d),\mathbb{R})$. We will show that the above quantities are conserved for solutions in $H^1(\mathbb{R}^d,\mathbb{C})$. Here E is the hamiltonian. The system is invariant under the transformation $u\to e^{\mathrm{i}\vartheta}u$ for $\vartheta\in\mathbb{R}$ and the transformations $u(x_1,...x_{j-1},x_j,x_{j+1},...,x_d)\to u(x_1,...x_{j-1},x_j-\tau,x_{j+1},...,x_d)$ for $\tau\in\mathbb{R}$. The related Noether invariants are Q and P_j.

3.1 The local existence

c:NLSlocal

We will consider the following integral formulation of (3.1):

$$u(t) = e^{\mathrm{i}t\triangle}u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{3.5}$$

eq:INLS

thm:lwpL2 **Proposition 3.3** (Local well posedness in $L^2(\mathbb{R}^d)$)**.** *For any $p \in (1, 1+4/d)$ and any $u_0 \in L^2(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of* (3.5) *with*

$$u \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.6}$$

eq:lwpL2

Furthermore, there exists a (decreasing) function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{L^2}) > 0$.
Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'], L^2(\mathbb{R}^d)) \cap L^q([-T',T'], L^{p+1}(\mathbb{R}^d)) \tag{3.7}$$

eq:lwpL2--

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], L^b(\mathbb{R}^d))$ for all admissible pairs (a, b).

Remark 3.4. We will prove later that for $p \in (1, 1+2/d)$ that we can take $T = \infty$ always.

Proof. The proof is a fixed point argument. We set

$$E(T,a) = \Big\{ v \in C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T], L^{p+1}(\mathbb{R}^d)) : \\ \|v\|_T := \|v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))} \leq a \Big\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Our first aim is to show that for $T = T(\|u_0\|_{L^2})$ sufficiently small, then $\Phi : E(T,a) \to E(T,a)$ and it is a contraction.
By Strichartz's estimates

$$\begin{aligned} \|\Phi(u)\|_T &\leq c_0\|u_0\|_{L^2} + c_0|\lambda| \||u|^{p-1}u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ &= c_0\|u_0\|_{L^2} + c_0|\lambda| \|u\|^p_{L^{pq'}([-T,T],L^{p+1})} \end{aligned}$$

We will see in a moment that

$$p \in (1, 1+4/d) \Longleftrightarrow pq' < q. \tag{3.8}$$

eq:lwpL21

Assuming this for a moment, by Hölder we conclude that for a $\theta > 0$

$$\|\Phi(u)\|_T \leq c_0\|u_0\|_{L^2} + c_0(2T)^\theta|\lambda|\|u\|^p_{L^q([-T,T],L^{p+1})} \leq c_0\|u_0\|_{L^2} + c_0(2T)^\theta|\lambda|a^p.$$

So for $c_0(2T)^\theta|\lambda|a^{p-1} < 1/2$, which can be obtained by picking T small enough, we have

$$\|\Phi(u)\|_T \le c_0\|u_0\|_{L^2} + \frac{a}{2} \le a$$

if $a \ge 2c_0\|u_0\|_{L^2}$. Hence $\Phi\left(E(T,a)\right) \subseteq E(T,a)$. Let us fix here $a = 2c_0\|u_0\|_{L^2}$.
Now let us show that Φ is a contraction for T small enough. We have

$$\begin{aligned}
&\|\Phi(u)-\Phi(v)\|_T \le c_0|\lambda|\||u|^{p-1}u - |v|^{p-1}v\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\
&\le c_0C|\lambda|\|(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}}\|_{L^{q'}(-T,T)} \\
&\le c_0C|\lambda|(\|u\|^{p-1}_{L^q([-T,T],L^{p+1}} + \|v\|^{p-1}_{L^q([-T,T],L^{p+1})})\|u-v\|_{L^\rho([-T,T],L^{p+1})}
\end{aligned}$$

where $\frac{p-1}{q} + \frac{1}{\rho} = \frac{1}{q'}$. Since we are still assuming (3.8), we must have $\rho < q$, for $\rho \ge q$ would imply $pq' \ge q$, contrary to (3.8). Then by Hölder and for an appropriate $\theta > 0$

$$\|\Phi(u)-\Phi(v)\|_T \le c_0C|\lambda|2a^{p-1}T^\theta\|u-v\|_{L^q([-T,T],L^{p+1})} \le c_0C|\lambda|2a^{p-1}T^\theta\|u-v\|_T.$$

So, for $c_0C|\lambda|2a^{p-1}T^\theta < 1$, where $a = 2c_0\|u_0\|_{L^2}$, we obtain that Φ is a contraction and we obtain the existence and uniqueness of the solution.

Next, let us prove (3.8). Obviously $pq' < q$ is equivalent to $p/q < 1 - 1/q$, in turn to $(p+1)/q < 1$, that is to $1/q < 1/(p+1)$. But $1/q = d/4 - d/(2p+2)$, so the last inequality is equivalent to

$$d/4 < \left(\frac{d}{2}+1\right)/(p+1) \Leftrightarrow p+1 < \frac{2d+4}{d} = 2 + \frac{4}{d}$$

and this yields the desired result.

We have proved the existence of a $T = T(\|u_0\|_{L^2})$ with the desired properties. Fix $T' \in (0,T)$. Then there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ such that for any $v_0 \in V$ the corresponding solution $v(t)$ is in $C([-T',T'],L^2(\mathbb{R}^d)) \cap L^q([-T',T'],L^{p+1}(\mathbb{R}^d))$ with $\|v\|_{T'} \le 2c_0\|v_0\|_{L^2}$. This is clear because with v_0 sufficiently close to u_0, by $T' < T$ we can assume

$$\begin{aligned}
&c_0(2T')^\theta|\lambda|(2c_0\|v_0\|_{L^2})^{p-1} < 1/2 c_0(2T)^\theta|\lambda|(2c_0\|u_0\|_{L^2})^{p-1} < 1/2 \text{ and} \\
&c_0C|\lambda|2(2c_0\|v_0\|_{L^2})^{p-1}(T')^\theta < c_0C|\lambda|2(2c_0\|u_0\|_{L^2})^{p-1}T^\theta.
\end{aligned}$$

Using the equation and proceeding like above,

$$\begin{aligned}
&\|u-v\|_{T'} \le c_0\|u_0-v_0\|_{L^2} + c_0C|\lambda|(2T')^\theta\left(\|u\|^{p-1}_{T'} + \|v\|^{p-1}_{T'}\right)\|u-v\|_{T'} \\
&\le c_0\|u_0-v_0\|_{L^2} + c_0C|\lambda|(2T')^\theta 2\left((2c_0\|v_0\|_{L^2})^{p-1} + (2c_0\|u_0\|_{L^2})^{p-1}\right)\|u-v\|_{T'}.
\end{aligned}$$

Adjusting T, we can assume that, in addition to the previous inequalities, T satisfies also

$$4c_0C|\lambda|(2T)^\theta(2c_0\|u_0\|_{L^2})^{p-1} < 1/2.$$

Adjusting V, we can assume that,

$$(2T')^{\theta}(2c_0\|v_0\|_{L^2})^{p-1} < (2T)^{\theta}(2c_0\|u_0\|_{L^2})^{p-1}.$$

Then from the above we get

$$\|u-v\|_{T'} \leq 2c_0\|u_0-v_0\|_{L^2}$$

and this give the desired Lipschitz continuity.

Finally, the last statement follows from (3.5) and the Strichartz Estimates.

$\square$

thm:lwpH1 **Proposition 3.5** (Local well posedness in $H^1(\mathbb{R}^d)$)**.** *For any $p \in (1, d^*)$ and any $u_0 \in H^1(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of (3.5) with*

$$u \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{3.9}$$
eq:lwpH11

Furthermore, there exists a (decreasing) function $T(\cdot) : [0,+\infty) \to (0,+\infty]$ such that the above T satisfies $T \geq T(\|u_0\|_{H^1}) > 0$.
Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^1(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'], L^2(\mathbb{R}^d)) \cap L^q([-T',T'], W^{1,p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a,b).

Proof. The proof is similar to that of Proposition 3.3. The proof is a fixed point argument.This time we set

$$E^1(T,a) = \Big\{ v \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) : \\ \|v\|_T^{(1)} := \|v\|_{L^\infty([-T,T],H^1(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],W^{1,p+1}(\mathbb{R}^d))} \leq a \Big\}$$

and, as before, use $\Phi(u)$ for the r.h.s. of (3.5). We need to show that by taking T sufficiently small then $\Phi : E^1(T,a) \to E^1(T,a)$ and is a contraction. The argument is similar to the one in Proposition 3.3 and is based on the Strichartz estimates. We will only consider some of the estimates. By Lemma 3.1 and Strichartz's estimates, we have

$$\begin{aligned} \|\nabla\Phi(u)\|_T &\leq c_0\|u_0\|_{H^1} + c_0|\lambda| \||u|^{p-1}\nabla u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ &= c_0\|u_0\|_{L^2} + c_0|\lambda|\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})}\|\nabla u\|_{L^q([-T,T],L^{p+1})}. \end{aligned}$$

where $\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$. Notice that if $\beta < q$, we can proceed exactly like in Proposition 3.3. However this works only for $p \in (1, 1+4/d)$, which is not necessarily true here. Instead, using the Sobolev Embedding we bound

$$\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})} \lesssim \|u\|^{p-1}_{L^\beta([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}\|u\|^{p-1}_{L^\infty([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^{p-1}.$$

So, inserting this in the previous inequality we get

$$\|\nabla\Phi(u)\|_T \le c_0\|u_0\|_{H^1} + c_0|\lambda|(2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^p. \tag{3.10}$$

eq:lwpH12

Here it is important to remark that the admissible pair $(q, p+1)$ is s.t. $q > 2$. Indeed, for $d = 1, 2$ it is always true that, if $p + 1 < \infty$, then the q in (3.27) is $q > 2$. On the other hand, for $d \ge 3$ recall that

$$p + 1 < d^* + 1 = \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}.$$

And so again, since $(q, p+1)$ differs from the endpoint admissible pair $(2, \frac{2d}{d-2})$, we necessarily have $q > 2$ also if $d \ge 3$.
In turn, the fact that $q > 2$ implies that the β in the above formulas is $\beta < \infty$. This implies that we can pick T small enough s.t. $(2T)^{p-1}a^{p-1} < 1/2$, which from (3.10) yields $\|\Phi(u)\|_T^{(1)} \le c_1\|u_0\|_{H^1} + a/2 \le a$ for $a \ge 2c_1\|u_0\|_{H^1}$. From these arguments, it is easy to conclude that there exists a $T(\|u_0\|_{H^1})$ s.t. for $T \in (0, T(\|u_0\|_{H^1}))$ we have $\Phi\left(E^1(T,a)\right) \subseteq E^1(T,a)$. Proceeding similarly and like in Proposition 3.3, it can be shown that there exists a $T_1(\|u_0\|_{H^1})$ s.t. for $T \in (0, T_1(\|u_0\|_{H^1}))$ and $a \ge 2c_1\|u_0\|_{H^1}$ the map Φ is a contraction inside $E^1(T,a)$. The Lipschitz continuity in terms of the initial data can be shown like in Proposition 3.3 and the last statement follows from the Strichartz estimates.

□

thm:consl

Proposition 3.6 (Conservation laws). *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.5. Then all the three quantities in* (3.4) *are constant in t.*

Proof. For $u \in C((-T_2, T_1), H^1(\mathbb{R}^d))$ a maximal solution of (3.5) we will show that there exists $[-T, T] \subset (-T_2, T_1)$ where $E(u(t)) = E(u(0))$, $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$. In fact this shows that $E(u(t))$, $Q(u(t))$ and $P_j(u(t))$ are locally constant in t. Since these functions are continuous in t, the set of $t \in (-T_2, T_1)$ where $E(u(t)) = E(u(0))$ is closed in $(-T_2, T_1)$; on the other hand, it is also open in $(-T_2, T_1)$ since $E(u(t))$ is locally constant, and hence we have $E(u(t)) = E(u(0))$ for all $t \in (-T_2, T_1)$. Similarly $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$ for all $t \in (-T_2, T_1)$.

Step 1: truncations of the NLS. For $\varphi \in C_c^\infty(\mathbb{R}, [0,1])$ a function with $\varphi = 1$ near 0 and with support contained in the ball $B_{\mathbb{R}^d}(0, r_0)$, consider [2] the operators $\mathbf{Q}_n = \varphi(\sqrt{-\triangle}/n)$. The truncations $\mathbf{Q}_n(|u|^{p-1}u)$ are locally Lipschitz functions from $H^1(\mathbb{R}^d)$ into itself as they are compositions $H^1(\mathbb{R}^d) \overset{|u|^{p-1}u}{\to} H^{-1}(\mathbb{R}^d) \overset{\mathbf{Q}_n}{\to} H^1(\mathbb{R}^d))$ of a locally Lipschitz function, Lemma 3.1, and of bounded linear maps.

[2]Notice that using everywhere the projections $\mathbf{P}_n = \chi_{[0,n]}(\sqrt{-\triangle})$ would be a bad choice for this proof. Difficulties would arise from the fact proved by C.Feffermann [6] that $\mathbf{P}_n$ for $d \ge 2$ is bounded from $L^p(\mathbb{R}^d)$ into itself only if $p = 2$. On the other hand it is elementary that the $\mathbf{Q}_n$ are of the form $\rho_{\frac{1}{n}} *$ for a $\rho \in \mathcal{S}(\mathbb{R}^d)$ and so are uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for all p and form a sequence converging strongly to the identity operator.

We consider the following truncations of the NLS

$$\begin{cases} \mathrm{i}u_{nt} = -\mathbf{P}_{nr_0}\triangle u_n + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) \text{ for } (t,x)\in\mathbb{R}\times\mathbb{R}^d \\ u_n(0) = \mathbf{Q}_n u_0. \end{cases} \tag{3.11}$$

eq:NLStr

By the theory of ODE's, there exists a maximal solution $u_n(t)\in C^1(-T_1(n),T_2(n)),H^1(\mathbb{R}^d))$ of (3.11) . Furthermore, if $T_2(n)<\infty$ then we must have blow up

$$\lim_{t\nearrow T_2(n)} \|u_n(t)\|_{H^1} = +\infty \text{ if } T_2(n)<\infty \tag{3.12}$$

eq:bup1

with a similar blow up phenomenon if $T_1(n)<\infty$.
To get bounds on this sequence of functions we consider invariants of motion. The following will be proved later.

m:inv_NLS1 **Claim 3.7.** The following functions are invariants of motion of (3.11):

$$\begin{aligned} &E_n(v) := \frac{1}{2}\|P_{nr_0}\nabla v\|^2_{L^2} + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n v|^{p+1}dx \\ &P_j(v) \text{ with } j=1,...,d, \\ &Q(v). \end{aligned} \tag{3.13}$$

eq:inv_NLS

We assume Claim 3.7 and proceed. It is easy to check that $u_n = \mathbf{P}_{nr_0}u_n$. We claim that $T_1(n)=T_2(n)=\infty$. Indeed by $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)\le Q(u_0)$ we have

$$\|u_n(t)\|_{H^1} = \|\mathbf{P}_{nr_0}u_n(t)\|_{H^1} \le nr_0\|u_n(t)\|_{L^2} = nr_0\|\mathbf{Q}_n u_0\|_{L^2} \le nr_0\|u_0\|_{L^2}. \tag{3.14}$$

eq:bounder

Let us now fix M such that $\|u_0\|_{H^1}<M$ and let us set

$$\theta_n := \sup\{\tau>0 : \|u_n(t)\|_{H^1} < 2M \text{ for } |t|<\tau.\} \tag{3.15}$$

deftheta

Our main focus is now to prove that there exists a fixed $T(M)>0$ s.t. $\theta_n\ge T(M)$ for all n.
First of all we prove that $u_n\in C^{0,\frac{1}{2}}((-\theta_n,\theta_n),L^2)$ with a fixed Hölder constant $C(M)$. By an interpolation similar to Lemma **??**

$$\begin{aligned} &\|u_n(t)-u_n(s)\|_{L^2} \lesssim \|u_n(t)-u_n(s)\|^{\frac{1}{2}}_{H^1}\|u_n(t)-u_n(s)\|^{\frac{1}{2}}_{H^{-1}} \\ &\le \sqrt{2}\|u_n\|^{\frac{1}{2}}_{L^\infty((-\theta_n,\theta_n),H^1)}\|u_{nt}\|^{\frac{1}{2}}_{L^\infty((-\theta_n,\theta_n),H^{-1})}\sqrt{|t-s|} \\ &\le C(M)\sqrt{|t-s|} \text{ for } t,s\in(-\theta_n,\theta_n) \end{aligned} \tag{3.16}$$

eq:planch1

Now we want to prove

$$\|u_n(t)\|^2_{H^1} \le \|u_0\|^2_{H^1} + C(M)t^b \text{ for some fixed } b>0 \text{ and for } t\in(-\theta_n,\theta_n). \tag{3.17}$$

eq:engr

From $E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0)$ and $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ we get

$$\|u_n(t)\|^2_{H^1} + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n u_n|^{p+1}dx = \|\mathbf{Q}_n u_0\|^2_{H^1} + \frac{2\lambda}{p+1}\int_{\mathbb{R}^n}|\mathbf{Q}_n^2 u_0|^{p+1}dx.$$

Hence using Hölder and Gagliardo–Nirenberg

$$\|u_n(t)\|_{H^1}^2 \le \|u_0\|_{H^1}^2 + \frac{2|\lambda|}{p+1}\int_{\mathbb{R}^d} \Big| \ |\mathbf{Q}_n u_n(t)|^{p+1} - |\mathbf{Q}_n^2 u_0|^{p+1}\Big| \, dx$$
$$\le \|u_0\|_{H^1}^2 + C\int_{\mathbb{R}^d} (|\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p)|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0| dx$$
$$\le \|u_0\|_{H^1}^2 + C\||\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p\|_{L^{\frac{p+1}{p}}} \|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0\|_{L^{p+1}}$$
$$\le \|u_0\|_{H^1}^2 + C_1 \left(\|\mathbf{Q}_n u_n(t)\|_{L^{p+1}}^p + \|\mathbf{Q}_n^2 u_0\|_{L^{p+1}}^p\right) \|u_n(t) - \mathbf{Q}_n u_0\|_{\dot H^1}^\alpha \|u_n(t) - \mathbf{Q}_n u_0\|_{L^2}^{1-\alpha}$$

Then by (3.16) with $s=0$, the Sobolev Embedding Theorem and (3.15) we get (3.17). Now for $T(M)$ defined s.t. $C(M)T(M)^b = 2M^2$ (for the $C(M)$ in (3.17)) from (3.17) we get

$$\|u_n(t)\|_{L^\infty([-T(M),T(M)],H^1)} \le \sqrt{3}M. \tag{3.18}$$

eq:conv--0

Since $\sqrt{3}M < 2M$ this obviously means that $T(M) < \theta_n$ since, if we had $\theta_n \le T(M)$ then, by the fact that $u_n \in C^1(\mathbb{R}, H^1)$, the definition of θ_n in (3.15) would be contradicted. Hence we have

$$\|u_n\|_{L^\infty([-T(M),T(M)],H^1)} < 2M \tag{3.19}$$

eq:conv0

This completes step 1, up to Claim 3.7.

The proof of Claim 3.7 is rather elementary and involves applying to (3.11) $\langle \ , u_{nt}\rangle$, $\langle \ , \mathrm{i}u_n\rangle$ and $\left\langle \ , \partial_{x_j} u_n\right\rangle$ and integration by parts. We will do this now, but then we will discuss also the fact that Claim 3.7 is just a consequence of the fact that (3.11) is a hamiltonian system with hamiltonian E_n and that the invariance of Q resp. P_j just due to Nöther principle and the invariance with respect to multiplication by $e^{\mathrm{i}\vartheta}$ resp. translation.

Indeed, applying $\langle \cdot, u_{nt}\rangle$ to (3.11)

$$0 = -\langle \mathbf{P}_{nr_0} \triangle u_n, u_{nt}\rangle + \lambda \langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), u_{nt}\rangle$$
$$= -\langle \triangle u_n, u_{nt}\rangle + \lambda\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathbf{Q}_n u_{nt}\rangle = \frac{d}{dt}E_n(u_n).$$

Notice furthermore that, by $u_n = \mathbf{P}_{nr_0} u_n$, we have

$$E_n(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx.$$

Similarly when we apply $\langle \cdot, \mathrm{i}u_n\rangle$ to (3.11) we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2} = -\langle \mathbf{P}_{nr_0}\triangle u_n, \mathrm{i}u_n\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle. \tag{3.20}$$

eq:truncmass

We have to show that r.h.s. are equal to 0. We observe that the the 1st term is 0 because the bounded operator $\mathrm{i}\mathbf{P}_{nr_0}\triangle$ of $L^2(\mathbb{R}^d)$ into itself is antisymmetric: $(\mathrm{i}\mathbf{P}_{nr_0}\triangle)^* = -\mathrm{i}\mathbf{P}_{nr_0}\triangle$. For the 2nd term we use

$$\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle = \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle = \lambda \,\mathrm{Re}\,\mathrm{i}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = 0.$$

This yields $\frac{d}{dt}Q(u_n(t)) = 0$. In a similar fashion we can prove $\frac{d}{dt}P_j(u_n(t)) = 0$.

These computations obscure somewhat the following simple facts. First of all, (3.11) and, in a somewhat formal sense also (3.1), is a hamiltonian system. First of all, the symplectic form is

$$\Omega(X,Y) := \langle \mathrm{i}X, Y\rangle \tag{3.21}$$ eq:Omega

where

$$\langle f, g\rangle = \mathrm{Re}\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx. \tag{3.22}$$ eq:bilf

Notice that Ω satisfies the following definition for $X = L^2(\mathbb{R}^d, \mathbb{C})$ or $X = H^1(\mathbb{R}^d, \mathbb{C})$.

Definition 3.8. Let X be a Banach space on $\mathbb{R}$ and let X' be its dual. A strong symplectic form is a 2-form ω on X s.t. $d\omega = 0$ (i.e. ω is closed) and s.t. the map $X \ni x \to \omega(x, \cdot) \in X'$ is an isomorphism.

Definition 3.9 (Gradient)**.** Let $F \in C^1(L^2(\mathbb{R}^d, \mathbb{C}), \mathbb{R})$. Then the gradient $\nabla F \in C^0(L^2(\mathbb{R}^d, \mathbb{C}), L^2(\mathbb{R}^d, \mathbb{C}))$ is defined by

$$\langle \nabla F(u), Y\rangle = dF(u)Y \text{ for all } u, Y \in L^2(\mathbb{R}^d, \mathbb{C}).$$

Notice that

$$\begin{aligned}\langle \nabla E_n(u), Y\rangle &= \frac{d}{dt}\left(\frac{1}{2}\|P_{nr_0}\nabla(u+tY)\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n(u+tY)|^{p+1}dx\right)\bigg|_{t=0} \\ &= \left\langle -\mathbf{P}_{nr_0}\triangle u + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u), Y\right\rangle .\end{aligned} \tag{3.23}$$ eq:grad E_n

We are interested in hamiltonian vector fields.

Definition 3.10 (Hamiltonian vector field)**.** Let ω be a strong symplectic form on the Banach space X and $F \in C^1(X, \mathbb{R})$. We define the Hamiltonian vector field X_F with respect to ω by

$$\omega(X_F(u), Y) := dF(u)Y \text{ for all } u, Y \in X.$$

From $\Omega(X_F, Y) = \langle \mathrm{i}X_F, Y\rangle = \langle \nabla F, Y\rangle$ we conclude $X_F = -\mathrm{i}\nabla F$. Then from (3.23) it is straightforward to conclude that (3.11) is a hamiltonian system with hamiltonian E_n.

Definition 3.11 (Poisson bracket)**.** Let ω be a strong symplectic form in a Banach space X and let $F, G \in C^1(X, \mathbb{R})$. Then the Poisson bracket $\{F, G\}$ is given by

$$\{F, G\}(u) := \omega(u)(X_F(u), X_G(u)) = dF(u)X_G(u).$$

So, for Ω we have $\{F, G\} = \langle \nabla F, -\mathrm{i}\nabla G\rangle = \langle \mathrm{i}\nabla F, \nabla G\rangle$. Now notice that if $F \in C^1(X, \mathbb{R})$ then

$$\frac{d}{dt}\left(F(u_n(t))\right) = \langle \nabla F(u_n(t)), \dot{u}_n(t)\rangle = \langle \nabla F(u_n(t)), -\mathrm{i}\nabla E_n(u_n(t))\rangle = \{F, E_n\}\big|_{u_n(t)} \tag{3.24}$$ eq:var ham

Notice now that the map $u \in e^{\mathrm{i}\vartheta}u$ leaves E_n invariant. In particular the last assertion implies that

$$\begin{aligned}0 &= \frac{d}{d\vartheta}E_n(u)\Big|_{\vartheta=0} = \frac{d}{d\vartheta}E_n(e^{\mathrm{i}\vartheta}u)\Big|_{\vartheta=0}\\ &= \langle \nabla E_n(u), \mathrm{i}u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla Q(u)\rangle = \langle \mathrm{i}\nabla Q(u), \nabla E_n(u)\rangle = \{Q, E_n\}|_u\end{aligned}$$

But then, since $\{Q, E_n\} = 0$, by (3.24) we obviously have $\frac{d}{dt}(Q(u_n(t))) = 0$.

Let us consider now, for $\{\overrightarrow{e}_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, the transformation $(\tau_{\lambda\overrightarrow{e}_j}F)(x) := F(x - \lambda\overrightarrow{e}_j)$. Obviously E_n is invariant by this transformation and

$$\begin{aligned}0 &= \frac{d}{d\lambda}E_n(u)\Big|_{\lambda=0} = \frac{d}{d\lambda}E_n(\tau_{\lambda\overrightarrow{e}_j}u)\Big|_{\lambda=0}\\ &= -\langle \nabla E_n(u), \partial_j u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla P_j(u)\rangle = \langle \mathrm{i}\nabla P_j(u), \nabla E_n(u)\rangle = \{P_j, E_n\}|_u\end{aligned}$$

But then, since $\{P_j, E_n\} = 0$, by (3.24) we obviously have $\frac{d}{dt}(P_j(u_n(t))) = 0$.
The above argument gives a link between group actions and invariants.

Step 2: Convergence $u_n \to u$. Let us consider $I := [-T, T] \subseteq [-T(M), T(M)] \cap (-T_2, T_1)$. Obviously we have

$$u_n(t) = e^{\mathrm{i}t\triangle}\mathbf{Q}_n u_0 - \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n(|\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s))ds.$$

Taking the difference with (3.5) we obtain

$$\begin{aligned}u(t) - u_n(t) &= e^{\mathrm{i}t\triangle}(1-\mathbf{Q}_n)u_0 - \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}(1-\mathbf{Q}_n)|u(s)|^{p-1}u(s)ds\\ &- \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n\left(|u(s)|^{p-1}u(s) - |\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s)\right)ds\\ &- \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n\left(|\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s) - |\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s)\right)ds.\end{aligned}$$

Then we have

$$\begin{aligned}&\|u - u_n\|_{L^q(I,W^{1,p+1})} + \|u - u_n\|_{L^\infty(I,H^1)} \le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}\\ &+ c_0|\lambda|\||u|^{p-1}u - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}\\ &+ c_0|\lambda|\||\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}.\end{aligned}$$

and so, for a fixed $\vartheta > 0$

$$\|u-u_n\|_{L^q(I,W^{1,p+1})} + \|u-u_n\|_{L^\infty(I,H^1)} \le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+ c_0C|\lambda||I|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + \|\mathbf{Q}_nu\|^{p-1}_{L^\infty(I,H^1)}\right)\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+ c_0C|\lambda||I|^\vartheta\left(\|\mathbf{Q}_nu\|^{p-1}_{L^\infty(I,H^1)} + \|\mathbf{Q}_nu_n\|^{p-1}_{L^\infty(I,H^1)}\right)\|\mathbf{Q}_n(u-u_n)\|_{L^q(I,W^{1,p+1})}$$
$$\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}$$
$$+ c_0C|\lambda||I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}$$
$$+ c_0C|\lambda||2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + (C(M))^{p-1}\right)\|u-u_n\|_{L^q(I,W^{1,p+1})}.$$

Then, taking T small so that $c_0C|\lambda||2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + (C(M))^{p-1}\right) < 1/2$ we conclude

$$\|u-u_n\|_{L^q(I,W^{1,p+1})} + \|u-u_n\|_{L^\infty(I,H^1)} \le 2c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} +$$
$$2c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} + 2c_0C|\lambda||I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}.$$

But now we have r.h.s.$\overset{n\to\infty}{\to} 0$. Hence we have proved that there exist $T>0$ s.t.

$$\lim_{n\to+\infty}\|u-u_n\|_{L^\infty([-T,T],H^1)} = 0. \tag{3.25}$$ eq:convT

Now, taking the limit for $n\to+\infty$ in $Q(u_n(t)) = Q(\mathbf{Q}_nu_0)$ and $P_j(u_n(t)) = P_j(\mathbf{Q}_nu_0)$ we obtain $Q(u(t)) = Q(u_0)$ and $P_j(u(t)) = P_j(u_0)$ for all $t\in[-T,T]$. Similarly, taking the limit for $n\to+\infty$ in $E_n(u_n) = E_n(\mathbf{Q}_nu_0)$ and with a little bit of work, we obtain $E(u(t)) = E(u_0)$ for all $t\in[-T,T]$.

□

thm:consL2 **Corollary 3.12.** *Let $u(t)$ be a solution* (3.5) *as in Proposition 3.3. Then $Q(u(t)) = Q(u_0)$. In particular, the solutions in in Proposition 3.3 are globally defined.*

Proof. As above it is enough to show that $Q(u(t)) = Q(u_0)$ for $t\in[-T,T]$ for some $T>0$. So let us take the T in the statement of Proposition 3.3 and let us take $T'\in(0,T)$. There exists a sequence $u_0^{(n)}\in H^1(\mathbb{R}^d,\mathbb{C})$ with $u_0^{(n)}\overset{n\to\infty}{\to} u_0$ in $L^2(\mathbb{R}^d,\mathbb{C})$. So for $n\gg 1$ we have $u_0^{(n)}\in V$, the V in (3.7). In particular, for the corresponding solutions u_n we have $u^{(n)}\overset{n\to\infty}{\to} u$ in $C([-T',T'],L^2(\mathbb{R}^d))$. Then, since $Q(u^{(n)}(t)) = Q(u_0^{(n)})$ for $t\in([-T',T']$, taking the limit we obtain $Q(u(t)) = Q(u_0)$ for $t\in([-T',T']$. Since $T'\in(0,T)$ is arbitrary and $t\to Q(u(t))$ is continuous, we have $Q(u(t)) = Q(u_0)$ for $t\in([-T,T]$. This implies that $t\to Q(u(t))$ is locally constant, and hence it is constant.

□

:NLSglobal

3.2 The global existence

We start with the following observation.

em:blowup1 **Lemma 3.13.** *Let $u \in C^0((-S,T), H^1(\mathbb{R}^d))$ be a maximal solution as of Proposition 3.5. Then if $T < \infty$ we have*

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty. \tag{3.26}$$ eq:blowup11

Analogously, $\lim_{t \searrow -S} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$ *if* $S < \infty$.

em:subcrit *Remark* 3.14. Notice that it is very important for this lemma that $p < d^*$. Indeed, in the *energy* critical case $p = d^*$, the above statement is false.

Proof. Suppose by contradiction that there exists a solution with $T < \infty$ for which there is a sequence $t_j \nearrow T$ s.t. $\|u(t_j)\|_{H^1(\mathbb{R}^d)} \leq M < \infty$. Then by Proposition 3.5 one can extend $u(t)$ beyond $t_j + T(M) > T$ and get a contradiction.

□

hm:cor1nls **Corollary 3.15.** *If $\lambda > 0$ the solutions of Proposition 3.5 are globally defined.*

Proof. Indeed if a solution has maximal interval of existence $(-S,T)$ with $T < \infty$, we must have (3.26). But for $\lambda > 0$ we have $\|\nabla u(t)\|_{L^2} \leq 2E(u(t)) = 2E(u_0)$.

□

hm:cor2nls **Corollary 3.16.** *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.5 are globally defined.*

Proof. We have

$$2E(u(t)) \geq \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}.$$

Notice that

$$\alpha(p+1) = \frac{d}{2}(p+1) - d < 2 \Longleftrightarrow (p+1) - 2 < \frac{4}{d} \Longleftrightarrow p < 1 + \frac{4}{d}.$$

But then, if (3.26) happens, we have

$$2E(u_0) = \lim_{t \nearrow T} 2E(u(t)) \geq \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \left(1 - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)-2}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)}\right)$$
$$= \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = +\infty,$$

which is absurd. □

hm:cor3nls **Corollary 3.17.** *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 3.5 are globally defined.*

LScritical

3.3 The L^2 critical cases

We consider now equation (3.5) for $p = 1 + \frac{4}{d}$. Notice that in this case $(p+1, p+1)$ is an admissible pair.

thm:critL2

Theorem 3.18. *For any $u_0 \in L^2(\mathbb{R}^d)$ there exists a unique maximal solution of (3.5) with $p = 1 + \frac{4}{d}$ with*

$$u \in C([0,T^*), L^2(\mathbb{R}^d)) \cap L^{p+1}_{loc}([0,T^*), L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \qquad (3.27)$$

eq:lwpL2

Furthermore, the mass is preserved, we have $u \in L^a([0,T], L^b(\mathbb{R}^d))$ for any admissible pair, if $T \in (0, T^*)$.
There is continuity with respect to the initial data. And finally, if $T^* < \infty$, then

$$\lim_{T \to T^*} \|u\|_{L^a([0,T], L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } b \geq p+1. \qquad (3.28)$$

eq:crit_blow_up_L2

op:crtiL21

Proposition 3.19. *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{it}\triangle} u_0\|_{L^{p+1}([0,T), L^{p+1}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0,T], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T), L^{p+1}(\mathbb{R}^d)).$$

The mass is constant. Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0,T'], L^2(\mathbb{R}^d)) \cap L^{p+1}([0,T'], L^{p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([0,T], L^b(\mathbb{R}^d))$ for all admissible pairs (a,b).

Proof. The proof is a fixed point argument. We set like before

$$E(T,\delta) = \left\{ v \in L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d)) : \|v\|_{L^{p+1}([0,T], L^{p+1}(\mathbb{R}^d))} \leq 2\delta \right\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5).
By Strichartz's estimates

$$\begin{aligned} \|\Phi(u)\|_{L^{p+1}([0,T] \times \mathbb{R}^d))} &< \delta + c_0 |\lambda| \| |u|^{p-1} u \|_{L^{\frac{p+1}{p}}([0,T] \times \mathbb{R}^d))} \\ &= \delta + c_0 |\lambda| \|u\|^p_{L^{p+1}([0,T] \times \mathbb{R}^d))} \leq \delta + c_0 |\lambda| 2^p \delta^p < 2\delta, \end{aligned}$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T,\delta)$. Now we show that Φ is a contraction in $E(T,\delta)$. We have

$$\begin{aligned}
&\|\Phi(u)-\Phi(v)\|_{L^{p+1}([0,T]\times\mathbb{R}^d))} \le c_0|\lambda|\||u|^{p-1}u-|v|^{p-1}v\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))}\\
&\le c_0C|\lambda|\|(|u|^{p-1}+|v|^{p-1})|u-v|\|_{L^{\frac{p+1}{p}}([0,T]\times\mathbb{R}^d))}\\
&\le c_0C|\lambda|\left(\|u\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))}+\|v\|^{p-1}_{L^{p+1}([0,T]\times\mathbb{R}^d))}\right)\|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))}\\
&\le c_0C|\lambda|2^{p-1}\delta^{p-1}\|u-v\|_{L^{p+1}([0,T]\times\mathbb{R}^d))},
\end{aligned}$$

which is a contraction for $\delta > 0$ small enough. The remaining part is also similar to that in Proposition 3.3. In particular, let us now discuss the fact that the conservation of mass. The first observation is that if $u_0 \in H^1(\mathbb{R}^d)$ then we have $u \in C([0,T],H^1(\mathbb{R}^d))$. In fact we have $u \in C([0,\tau],H^1(\mathbb{R}^d))$ by Proposition 3.5 and if it is not possible to take $\tau \ge 0$, then we will have a maximal interval of existence $u \in C([0,\tau),H^1(\mathbb{R}^d))$ with $\tau \in (0,T)$ and blow up $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. But

$$\|\nabla u\|_{L^{p+1}([0,s]\times\mathbb{R}^d))} < \|\nabla e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}([0,s]\times\mathbb{R}^d))} + c_0|\lambda|\|u\|^{p-1}_{L^{p+1}([0,s]\times\mathbb{R}^d))}\|\nabla u\|_{L^{p+1}([0,s]\times\mathbb{R}^d))}$$

by Gronwall's inequality implies

$$\|\nabla u\|_{L^{p+1}([0,s]\times\mathbb{R}^d))} < \|\nabla e^{it\triangle}u_0\|_{L^{p+1}([0,s]\times\mathbb{R}^d))}\exp\left(c_0|\lambda|\int_0^s \|u\|^{p-1}_{L^{p+1}([0,s']\times\mathbb{R}^d))}ds'\right).$$

This inequality implies

$$\lim_{s\to\tau^-}\|\nabla u\|_{L^{p+1}([0,s]\times\mathbb{R}^d))} = \|\nabla u\|_{L^{p+1}([0,\tau]\times\mathbb{R}^d))}.$$

feeding this back in Strichartz inequality, we have

$$\|\nabla u\|_{L^\infty([0,s],L^2(\mathbb{R}^d)))} < \|\nabla u_0\|_{L^2(\mathbb{R}^d)} + c_0|\lambda|\|u\|^{p-1}_{L^{p+1}([0,s]\times\mathbb{R}^d))}\|\nabla u\|_{L^{p+1}([0,s]\times\mathbb{R}^d))},$$

which implies that it is not true that $\|\nabla u(s)\|_{H^1} \xrightarrow{s\to\tau} +\infty$. So we conclude that $u \in C([0,T],H^1(\mathbb{R}^d))$ and that, energy, momenta and mass of $u(t)$ are constant in $[0,T]$. If now $u_0 \notin H^1(\mathbb{R}^d)$, we consider a sequence $u_{0n} \in H^1(\mathbb{R}^d)$ with $u_{0n} \xrightarrow{n\to\infty} u_0$ in $L^2(\mathbb{R}^d)$. For any $T' \in (0,T)$, we have by well posedness that for the corresponding solutions we have $u_n \xrightarrow{n\to\infty} u$ in $C([0,T'],L^2(\mathbb{R}^d))$. Then $Q(u_n) \xrightarrow{n\to\infty} Q(u)$ in $C([0,T'],\mathbb{R})$. Since $Q(u_n)$ are constant functions, also $Q(u)$ is constant in $[0,T']$ for all $T' < T$.

□

Proof of Theorem 3.18. Clearly we have $\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}([0,T),L^{p+1}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.19 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.28). Suppose that it is false, and that there is a maximal solution in $[0,T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),L^b(\mathbb{R}^d))} < +\infty \text{ for an admissible pair with } b \ge p+1. \quad (3.29)$$ eq:crit_blow_up_L2pr

Then if $b > p+1$, we have

$$\|u\|_{L^{p+1}([0,T^*),L^{p+1}(\mathbb{R}^d))} \leq \|u\|^{\mu}_{L^\infty([0,T^*),L^2(\mathbb{R}^d))} \|u\|^{1-\mu}_{L^a([0,T^*),L^b(\mathbb{R}^d))} \text{ for } \mu = \frac{\frac{1}{p+1} - \frac{1}{b}}{\frac{1}{2} - \frac{1}{b}}.$$

So (3.29) holds also for $b = p+1$. Now, for s close to T^* we have from (3.5)

$$e^{\mathrm{i}(t-s)\triangle} u(s) = u(t) + \mathrm{i}\lambda \int_s^t e^{\mathrm{i}(t-t')\triangle} |u(t')|^{p-1} u(t') dt'.$$

This yields

$$\|e^{\mathrm{i}(t-s)\triangle} u(s)\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \leq \|u\|_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} + C|\lambda| \|u\|^p_{L^{p+1}([s,T],L^{p+1}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0.$$

So we conclude that $\|e^{\mathrm{i}(t-s)\triangle} u(s)\|_{L^{p+1}([s,T_*+\varepsilon],L^{p+1}(\mathbb{R}^d))} < \delta$, for s close enough to T^* and for $\varepsilon > 0$ small enough. But then the solution u can be extended beyond T^*.

□

Example 3.20. In the case $\lambda = -1$ of the L^2- critical focusing NLS

$$\mathrm{i}u_t = -\triangle u - |u|^{\frac{4}{d}} u \text{ in } \mathbb{R} \times \mathbb{R}^d, \tag{3.30}$$ critical focusing NL

there are related solutions in $H^1\left(\mathbb{R}^d, [0,+\infty)\right)$ to

$$-\Delta\phi + \phi - |\phi|^{p-1}\phi = 0. \tag{3.31}$$ solitond

In 1–d they are explicit,

$$\phi(x) = \frac{(\frac{p-1}{2} + 1)^{\frac{4}{p-1}}}{\cosh^{\frac{2}{p-1}}(\frac{p-1}{2} x)}. \tag{3.32}$$ eq:sol

For $d \geq 2$ there are many types of solitons. For example, the ones in (3.32) are *ground states*, and they are the only ones in $d = 1$. But in $d \geq 2$ there are also excited states. Notice that if $u(t,x)$ is a solution of (3.30), then also the following is a solution,

$$v(t,x) = t^{-\frac{d}{2}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{\mathrm{i}\frac{x^2}{4t}}.$$

Since now, given a solution $\phi(x)$ of (3.31), then $u(t,x) = e^{\mathrm{i}t + \frac{\mathrm{i}}{2}\mathbf{v}\cdot x - \mathrm{i}\frac{\mathbf{v}^2}{4}t} \phi(x - t\mathbf{v} - D)$ is a solution of (3.30), it follows, choosing $\mathbf{v} = D = 0$, that

$$S(t,x) := t^{-\frac{d}{2}} \phi\left(\frac{x}{t}\right) e^{\mathrm{i}\frac{x^2}{4t}} e^{-\frac{\mathrm{i}}{t}} \text{ so also } S(T-t,x) := (T-t)^{-\frac{d}{2}} \phi\left(\frac{x}{T-t}\right) e^{\mathrm{i}\frac{x^2}{4(T-t)}} e^{-\frac{\mathrm{i}}{T-t}}.$$

Obviously this for $T > 0$ has maximal positive lifespan T. Then, for any admissible pair (q,r) with $r > 2$, we have

$$\|S(T-t,x)\|_{L^r(\mathbb{R}^d)} = (T-t)^{-\frac{d}{2}+\frac{d}{r}} \|\phi\|_{L^r(\mathbb{R}^d)} = (T-t)^{-\frac{2}{q}} \|\phi\|_{L^r(\mathbb{R}^d)} \notin L^q(0,T).$$

3.4 The H^1 critical cases

riticalH^1

We consider now equation (3.5) for $p = 1 + \frac{4}{d-2}$. We will consider the admissible pair

$$\rho = \frac{2d^2}{d^2 - 2d + 4}, \quad \gamma = \frac{2d}{d-2}.$$

thm:critH1 **Theorem 3.21.** *For any $u_0 \in H^1(\mathbb{R}^d)$ there exists a unique maximal solution of* (3.5) *with $p = 1 + \frac{4}{d-2}$ with*

$$u \in C([0, T^*), H^1(\mathbb{R}^d)) \cap C^1([0, T^*), H^{-1}(\mathbb{R}^d)). \tag{3.33}$$

eq:lwpH1

Furthermore, the mass and energy are preserved, we have $u \in L^a([0, T], W^{1,b}(\mathbb{R}^d))$ for any admissible pair, if $T \in (0, T^*)$.
There is continuity with respect to the initial data in the following sense. If $0 < T' < T^*$ and if $u_{0n} \xrightarrow{n\to\infty} u_0$ in $H^1(\mathbb{R}^d)$ then for the corresponding solutions we have we have $u_n \xrightarrow{n\to\infty} u$ in $L^p([0, T'], H^1(\mathbb{R}^d))$ for any $p < \infty$.

And finally, if $T^* < \infty$, then

$$\lim_{T \to T^*} \|u\|_{L^a([0,T], L^b(\mathbb{R}^d))} = +\infty \text{ for any admissible pair with } d > b > 2. \tag{3.34}$$

eq:crit_blow_up_H1

op:crtiH11 **Proposition 3.22.** *There exists a $\delta > 0$ such that if for some $T > 0$ we have*

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} < \delta,$$

then there exists a unique solution

$$u \in C([0, T], H^1(\mathbb{R}^d)) \cap L^\gamma([0, T), W^{1,\rho}(\mathbb{R}^d)).$$

Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([0, T'], L^2(\mathbb{R}^d)) \cap L^\gamma([0, T'), W^{1,\rho}(\mathbb{R}^d)$$

and is Lipschitz.

Finally, we have $u \in L^a([0, T], W^{1,b}(\mathbb{R}^d))$ for all admissible pairs (a, b) and mass and energy are preserved.

Proof (sketch). The proof is by a contraction argument. We set like before

$$E(T, \delta) = \left\{ v \in L^\gamma([0, T), W^{1,\rho}(\mathbb{R}^d)) : \|v\|_{L^\gamma([0,T), W^{1,\rho}(\mathbb{R}^d))} \le 2\delta \right\}$$

and we denote by $\Phi(u)$ the r.h.s. of (3.5). Let us open a small parenthesis now, and let us pick an admissible pair (a, b) with $b \in (2, d)$. Notice that (γ, ρ) has this property. Now let us set $b^* = \frac{bd}{d-b} = \frac{1}{b} - \frac{1}{d}$ and let (α, β) be an admissible pair such that $\frac{1}{\beta'} = \frac{1}{\beta} + \frac{\frac{4}{d-2}}{b^*}$

$$\frac{1}{\beta'} = \frac{1}{\beta} + \frac{\frac{4}{d-2}}{b^*} \text{ or}$$

$$1 = \frac{2}{\beta} + \frac{4}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right). \tag{3.35}$$

claim:crtiH110

Here notice that for $b^* = \infty$, that is when $b = d$, then $\beta = 2$, and if $b^* = \frac{2d}{d-2}$, for $b = 2$, we have $\beta = \frac{2d}{d-2}$, which is the endpoint. So for $b \in (2, d)$ we have the intermediate cases $2 < \beta < \frac{2d}{d-2}$. We claim that

$$\begin{aligned} \frac{1}{\alpha'} &= \frac{1}{\alpha} + \frac{\frac{4}{d-2}}{a}. \qquad (3.36) \\ \text{or } 1 &= \frac{2}{\alpha} + \frac{4}{d-2}\,\frac{d}{2}\,\left(\frac{1}{2} - \frac{1}{b}\right). \end{aligned}$$

claim:crtiH11

This can be checked by considering the endpoints, since from the 2nd line in (3.35)–(3.36) we see that the curve with parameter $1/b$

$$\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) = \left(\frac{1}{2} - \frac{d}{d-2}\left(\frac{1}{2} - \frac{1}{b}\right), 1 - \frac{2}{d-2}\left(\frac{1}{b} - \frac{1}{d}\right)\right)$$

is straight.

Looking at $b = 2$, then as we mentioned, we have the endpoint $(\alpha, \beta) = \left(2, \dfrac{2d}{d-2}\right)$, which makes (3.36) true because $\alpha' = 2$ and $a = 0$.

For $b = d$ and the corresponding value $a = \frac{4}{d-2}$, then as we mentioned $b^* = \infty$, so $\beta = \beta' = 2$, which implies $\alpha = \infty$, and so (3.36) becomes

$$1 = \frac{\frac{4}{d-2}}{a} = \frac{\frac{4}{d-2}}{\frac{4}{d-2}},$$

which is obviously correct.

The implication of this numbers is that by Strichartz estimates and by the Chain Rule in Lemma 3.1, we have

$$\begin{aligned} &\|\Phi(u)\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} \leq \|e^{\mathrm{i}t\triangle}u_0\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} + c_0|\lambda|\|u^{p-1}\langle\nabla\rangle u\|_{L^{\alpha'}([0,T),W^{1,\beta'}(\mathbb{R}^d))} \\ &\leq \|e^{\mathrm{i}t\triangle}u_0\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} + c_0|\lambda|\|u\|^{p-1}_{L^a([0,T],L^{b^*})}\|u\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} \\ &\leq \|e^{\mathrm{i}t\triangle}u_0\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} + c_0'|\lambda|\|u\|^{p-1}_{L^a([0,T],W^{1,b})}\|u\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))} \end{aligned}$$

Now, returning to case (ρ, γ), it turns out that for $(a, b) = (\rho, \gamma)$ we have $(\alpha, \beta) = (\rho, \gamma)$, which is left to be checked as an exercise. So, in this case

$$\|\Phi(u)\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} \leq \|e^{\mathrm{i}t\triangle}u_0\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} + c_0'|\lambda|\|u\|^p_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))}$$

Hence in $E(T, \delta)$ we have

$$\|\Phi(u)\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} < \delta + c_0'|\lambda|2^p\delta^p < 2\delta,$$

for $\delta > 0$ small enough, so that the map Φ preserves $E(T, \delta)$. In a similar fashion we prove that Φ is a contraction in $E(T, \delta)$. We skip the proof on the conservation of mass, energy and momenta.

Proof of Theorem 3.21. Clearly we have $\|e^{\mathrm{i}t\triangle}u_0\|_{L^\gamma([0,T),W^{1,\rho}(\mathbb{R}^d))} \xrightarrow{T\to 0^+} 0$, so we can apply Proposition 3.22 for $T > 0$ sufficiently small. There will be a maximal interval of existence. We now prove the blow up result (3.34), but only in the case $(a, b) = (\gamma, \rho)$. Suppose that it is false, and that there is a maximal solution in $[0, T^*)$ with $T^* < \infty$ and

$$\|u\|_{L^a([0,T^*),W^{1,b}(\mathbb{R}^d))} < +\infty. \tag{3.37}$$ eq:crit_blow_up_H1pr

But then

$$\|u\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le \|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} + c_0'|\lambda|\|u\|^{p-1}_{L^a([0,T],W^{1,b})}\|u\|_{L^\alpha([0,T),W^{1,\beta}(\mathbb{R}^d))}$$

and the fact that $\|u\|^p_{L^a([s,T],W^{1,b}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0$, implies

$$\|u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))} \le 2\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\alpha([s,T],W^{1,\beta}(\mathbb{R}^d))}$$

for $s < T < T^*$ close to s. This implies in fact that also

$$\|u\|_{L^\alpha([0,T^*),W^{1,\beta}(\mathbb{R}^d))} < +\infty. \tag{3.38}$$ eq:crit_blow_up_H1pr

Then, by

$$e^{\mathrm{i}(t-s)\triangle}u(s) = u(t) + \mathrm{i}\lambda \int_s^t e^{\mathrm{i}(t-t')\triangle}|u(t')|^{p-1}u(t')dt',$$

$$\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T],W^{1,\rho}(\mathbb{R}^d))} \le \|u\|_{L^\gamma([s,T],W^{1,\rho}(\mathbb{R}^d))} + c_0'|\lambda|\|u\|^{p-1}_{L^a([s,T],W^{1,b})}\|u\|_{L^\alpha([s,T),W^{1,\beta}(\mathbb{R}^d))} \xrightarrow{s<T\to T^{*-}} 0.$$

So we can arrange $\|e^{\mathrm{i}(t-s)\triangle}u(s)\|_{L^\gamma([s,T_*+\varepsilon],W^{1,\rho}(\mathbb{R}^d))} < \delta$, for s close enough to T^* and for $\varepsilon > 0$ arbitrarily small. But then the solution u can be extended beyond T^*.

We skip here the discussion of the well posedness.

□

4 The dispersive equation

dispersive

Here we will consider dispersive equations

$$\begin{cases} \mathrm{i}u_t = -\triangle u + |u|^{p-1}u \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{4.1}$$ eq:NLSdispersive

with $1 + 4/d < p < d^*$.

scatt_disp **Theorem 4.1.** *Consider the unique solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$. Then*

$$u \in L^a(\mathbb{R}, W^{1,b}(\mathbb{R}^d)) \text{ for any admissible pair} \tag{4.2}$$ eq:scatt1

and there exist $u_\pm \in H^1(\mathbb{R}^d)$ s.t.

$$\lim_{t\to\pm\infty} \|u(t) - e^{\mathrm{i}t\triangle} u_\pm\|_{H^1(\mathbb{R}^d)} = 0. \tag{4.3}$$ eq:scatt2

Here the key deep statement is (4.2). In fact, (4.2) implies easily (4.3), as we show now in the case $+$. So, assume (4.2), and in particular let

$$u \in L^q(\mathbb{R}_+, W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{4.4}$$ eq:scatt11

From (3.5) with $\lambda = 1$, we have

$$e^{-\mathrm{i}t\triangle} u(t) = u_0 - \mathrm{i} \int_0^t e^{-\mathrm{i}s\triangle} |u(s)|^{p-1} u(s) ds,$$

so that, for $t_1 < t_2$, we have

$$e^{-\mathrm{i}t_2\triangle} u(t_2) - e^{-\mathrm{i}t_1\triangle} u(t_1) = -\mathrm{i} \int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle} |u(s)|^{p-1} u(s) ds.$$

Then

$$\begin{aligned} &\|e^{-\mathrm{i}t_2\triangle} u(t_2) - e^{-\mathrm{i}t_1\triangle} u(t_1)\|_{H^1} \le \left\| \int_{t_1}^{t_2} e^{-\mathrm{i}s\triangle} |u(s)|^{p-1} u(s) ds \right\|_{H^1} \\ &\le \|u\|^{p-1}_{L^\alpha([t_1,t_2],L^{p+1})} \|u\|_{L^q([t_1,t_2],W^{1,p+1})} \end{aligned} \tag{4.5}$$ scatt16

where $\frac{p-1}{\alpha} + \frac{1}{q} = \frac{1}{q'}$. It can be checked that $\alpha > q$. Otherwise $\alpha \le q$ and so

$$\frac{p}{q} \le \frac{1}{q'} \Leftrightarrow p + 1 \le q.$$

So, from $p > 1 + \frac{4}{d}$, $(q, p+1)$ is an admissible pair with both entries $> 2 + \frac{4}{d}$. But $\left(2 + \frac{4}{d}, 2 + \frac{4}{d}\right)$ is an admissible pair, so we get an absurd and we conclude $\alpha > q$.
So, let us (α, β) be admissible. We claim that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} \text{ with } \tau \in [0,1]. \tag{4.6}$$ admissiblepair31

Assuming this, (4.5) can be majorized yielding

$$\|e^{-\mathrm{i}t_2\triangle} u(t_2) - e^{-\mathrm{i}t_1\triangle} u(t_1)\|_{H^1} \le c_0 \|u\|^{p-1}_{L^\alpha([t_1,t_2],W^{1,\beta})} \|u\|_{L^q([t_1,t_2],W^{1,p+1})} \xrightarrow{t_1 < t_2 \to +\infty} 0.$$

This implies that there exists

$$u_+ = \lim_{t \to +\infty} e^{-\mathrm{i}t\triangle} u(t) \text{ in } H^1(\mathbb{R}^d).$$

Then we have

$$e^{\mathrm{i}t\triangle} u_+ - u(t) = -\mathrm{i} \int_t^{\infty} e^{\mathrm{i}(t-s)\triangle} |u(s)|^{p-1} u(s) ds.$$

As above,

$$\|e^{\mathrm{i}t\triangle} u_+ - u(t)\|_{H^1} \le \|u\|^{p-1}_{L^{\alpha}([t,\infty),W^{1,\beta})} \|u\|_{L^q([t,\infty),W^{1,p+1})} \xrightarrow{t\to+\infty} 0,$$

which proves the limit (4.3).
Turning to the proof of (4.6), obviously $\alpha > q$ implies $\beta < p+1$ so that

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d}$$

with $\tau > 0$. Since $2 \le \beta < p+1 < +\infty$, for $d = 1, 2$ we have $\tau < 1$. For $d \ge 3$ we have $2 \le \beta < p+1 < \frac{2d}{d-2}$. Since $\frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$,

$$\frac{1}{p+1} = \frac{1}{\beta} - \frac{\tau}{d} > \frac{d-2}{2d} = \frac{1}{2} - \frac{1}{d}$$

which implies $\tau < 1$ by

$$\frac{1-\tau}{d} > \frac{1}{2} - \frac{1}{\beta}.$$

As we indicated above, in Theorem 4.1, the deep statement in (4.2). The proof is rather complicated. For this we will need the following which we will discuss only for dimension $d \ge 3$.

thm:decay **Theorem 4.2.** *Let $d \ge 3$. Then given a solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}^d))$ we have*

$$\lim_{t \to \pm\infty} \|u(t)\|_{L^r(\mathbb{R}^d)} = 0 \text{ for all } 2 < r < \frac{2d}{d-2}. \tag{4.7}$$ eq:decay1

This deep result implies (4.2) rather easily as we see now. We will use the following elementary lemma.

_lem_zeros **Lemma 4.3.** *consider a function $f(x) = a - x + bx^{\alpha}$ for $x \ge 0$, $a, b > 0$, $\alpha > 1$. We assume that there are $0 < x_0 < x_1$ s.t. $f(x_0) = f(x_1) = 0$, which is the case if b is small. Let now $\phi \in C(I, [0, +\infty))$ be such that $\phi(t) \le a + b\phi^{\alpha}(t)$ for all $t \in I$ and that there exists a point $t_0 \in I$ s.t. $\phi(t_0) \le x_0$. Then $\phi(t) \le x_0$ for all $t \in I$*

Proof. Since $f(\phi(t)) \geq 0$ for all t, and ϕ is continuous, the image of ϕ is either in $[0, x_0]$ or in $[x_1, +\infty)$. Obviously, the first case needs to occur. □

Proof that Theorem 4.2 implies (4.2) *(sketch).* Consider

$$u(t) = e^{\mathrm{i}(t-S)\triangle}u(S) - \mathrm{i}\int_S^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds,$$

Then by the Strichartz estimates

$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq C\|u(S)\|_{H^1} + C\left\| \|u\|^{p-1}_{L^{p+1}_x}\|u\|_{W^{1,p+1}_x}\right\|_{L^{q'}(S,t)}$$
$$= C\|u(S)\|_{H^1} + C\left(\int_S^t \|u\|^{(p-1)q'-(q-q')}_{L^{p+1}_x}\|u\|^{q-q'}_{L^{p+1}_x}\|u\|^{q'}_{W^{1,p+1}_x}ds\right)^{\frac{1}{q'}} \leq$$
$$C\|u(S)\|_{H^1} + C\|u\|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)}\|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.$$

Here

$$p - \frac{q}{q'} = p + 1 - q > 0 \Leftrightarrow p > 1 + 4/d.$$

From Theorem 4.2, applied to $r = p+1$, we know $\|u\|^{p-\frac{q}{q'}}_{L^\infty((S,t),L^{p+1}_x)} \xrightarrow{S\to+\infty} 0$. Furthermore, using conservation of mass and energy, there is a uniform upper bound for $\|u(S)\|_{H^1}$. There exists a constant $C_0 > 0$ s.t. for any $\epsilon > 0$ there is $S_0 > 0$ such that for any $S_0 < S < t$,

$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq C_0 + \epsilon\|u\|^{\frac{q}{q'}}_{L^q([S,t],W^{1,p+1})}.$$

Picking $\epsilon > 0$ sufficiently small, by Lemma 4.3 we conclude that there exists a fixed constant X_0 s.t.

$$\|u\|_{L^q((S,t),W^{1,p+1})} \leq X_0 \text{ for any } S_0 < S < t.$$

In particular we can take $t = \infty$. Since we know that $u \in L^q_{loc}(\mathbb{R}, W^{1,p+1})$, we conclude that $\|u\|_{L^q(\mathbb{R}_+,W^{1,p+1})} < +\infty$. Time reversibility of the NLS, yields the same result for negative times. The Strichartz estimates, yield $u \in L^\alpha(\mathbb{R}, W^{1,\beta})$ for any admissible pair.

5 Proof of Theorem 4.2

roof_moraw

lem:mor1 **Lemma 5.1.** *Let $p \in [1,\infty)$ and $q < d$ with $0 \leq q \leq p$. Then we have*

$$\int_{\mathbb{R}^d}\frac{|u(x)|^p}{|x|^q}dx \leq \left(\frac{p}{d-q}\right)\|u\|^{p-q}_{L^p(\mathbb{R}^d)}\|\nabla u\|^q_{L^p(\mathbb{R}^d)}. \quad (5.1)$$ eq:mor11

Proof. The general case $u \in W^{1,p}(\mathbb{R}^d)$ reduces to the special case $u \in C_c^\infty(\mathbb{R}^d)$. In fact, if (5.1) is valid for all $u \in C_c^\infty(\mathbb{R}^d)$, then for a $u \in W^{1,p}(\mathbb{R}^d)$ with $u \notin C_c^\infty(\mathbb{R}^d)$, we can consider a sequence $C_c^\infty(\mathbb{R}^d) \ni u_n \xrightarrow{n\to+\infty} u$ in $W^{1,p}(\mathbb{R}^d)$. Then, up to subsequence, we have $u_n(x)(\xrightarrow{n\to+\infty} u(x)$ for a.a. $x \in \mathbb{R}^d$, see p. 94 brezis [2]. Then, by Fathou's Lemma

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^q} dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{|u_n(x)|^p}{|x|^q} dx \\
&\le \lim_{n\to\infty} \left(\frac{p}{d-q}\right) \|u_n\|_{L^p(\mathbb{R}^d)}^{p-q} \|\nabla u_n\|_{L^p(\mathbb{R}^d)}^q = \left(\frac{p}{d-q}\right) \|u\|_{L^p(\mathbb{R}^d)}^{p-q} \|\nabla u\|_{L^p(\mathbb{R}^d)}^q .
\end{aligned}
$$

So we will prove (5.1) for $u \in C_c^\infty(\mathbb{R}^d)$. Let $z(x) := |x|^{-q} x$. Then

$$
\nabla \cdot z = \nabla(|x|^{-q}) \cdot x + |x|^{-q} \nabla \cdot x = -q|x|^{-q-1} \frac{x}{|x|} \cdot |x| + d|x|^{-q} = (d-q)|x|^{-q}.
$$

Integrating the identity

$$
|u|^p \nabla \cdot z = \nabla \cdot (|u|^p z) - p|u|^{p-1} \nabla |u| \cdot z,
$$

we obtain for arbitrary $r > 0$

$$
\begin{aligned}
&(d-q) \int_{|x|>r} \frac{|u(x)|^p}{|x|^q} dx = \int_{|x|>r} \nabla \cdot (|u|^p z)\, dx - p \int_{|x|>r} |u|^{p-1} \nabla |u| \cdot z dx \\
&\le -p \int_{|x|>r} |u|^{p-1} \nabla |u| \cdot z dx \le p \int_{|x|>r} \frac{|u|^{p-1} |\nabla u|}{|x|^{q-1}} dx,
\end{aligned}
$$

where we used

$$
\int_{|x|>r} \nabla \cdot (|u|^p z)\, dx = -\int_{|x|=r} |u|^p z \cdot \frac{x}{|x|} dS = -\int_{|x|=r} |u|^p |x|^{-q+1} dS \le 0.
$$

Using $1 - \frac{1}{q} + \frac{p-q}{pq} + \frac{1}{p} = 1$ and Hölder inequality, we have

$$
\begin{aligned}
&p \int_{|x|>r} \frac{|u|^{p-1} |\nabla u|}{|x|^{q-1}} dx = p \int_{|x|>r} \frac{|u|^{\frac{p(q-1)}{q}}}{|x|^{q-1}} |u|^{\frac{p-q}{q}} |\nabla u| dx \\
&\le p \left(\int_{|x|>r} \frac{|u|^p}{|x|^q} dx \right)^{\frac{q-1}{q}} \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p-q}{q}} \|\nabla u\|_{L^p(\mathbb{R}^d)} .
\end{aligned}
$$

This yields

$$
\int_{|x|>r} \frac{|u(x)|^p}{|x|^q} dx \le \left(\frac{p}{d-q}\right) \|u\|_{L^p(\mathbb{R}^d)}^{p-q} \|\nabla u\|_{L^p(\mathbb{R}^d)}^q
$$

and, taking $r \to 0^+$, we obtain (5.1).

□

lem:mor2 **Lemma 5.2.** *For $d \geq 4$ there exists a C_d s.t. we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^3} dx \leq C_d \|u\|^2_{H^2(\mathbb{R}^d)}. \tag{5.2}$$ eq:mor21

Proof. We proceed as above for $q = 3$ and $p = 2$, to obtain

$$(d-3) \int_{|x|>r} \frac{|u(x)|^2}{|x|^3} dx \leq -p \int_{|x|>r} |u|^{p-1} \nabla |u| \cdot z dx \leq 2 \int_{|x|>r} \frac{|u||\nabla u|}{|x|^2} dx$$
$$\leq 2 \left(\int_{|x|>r} \frac{|u|^2}{|x|^2} dx \right)^{\frac{1}{2}} \left(\int_{|x|>r} \frac{|\nabla u|^2}{|x|^2} dx \right)^{\frac{1}{2}}.$$

In the 2nd line we apply (5.1) for $p = q = 2$ to both u and ∇u, to obtain

$$(d-3) \int_{|x|>r} \frac{|u(x)|^2}{|x|^3} dx \leq 2 \left(\int_{|x|>r} \frac{|u|^2}{|x|^2} dx \right)^{\frac{1}{2}} \left(\int_{|x|>r} \frac{|\nabla u|^2}{|x|^2} dx \right)^{\frac{1}{2}} \leq 2 \left(\frac{2}{d-2} \right) \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla^2 u\|_{L^2(\mathbb{R}^d)}$$

Then (5.2) follows sending $r \to 0$. □

Let $u_0 \in H^2$. Then $u \in C^0([0,T), H^2)$ by the theory by Kato. Then equation (4.1) holds also in a differential sense as

$$\mathrm{i} u_t = -\triangle u + |u|^{p-1} u \text{ in } L^2(\mathbb{R}^d, \mathbb{C}).$$

Notice that $u \in C^1([0,T), L^2)$. Let us now consider the quadratic form

$$\frac{1}{2} \left\langle \mathrm{i} \left(\partial_r + \frac{d-1}{2r} \right) u, u \right\rangle. \tag{5.3}$$ lem:quadrmor_1

Notice that it is well defined and self–adjoint. Then, taking the derivative for $u \in C^0([0,T), H^2) \cap C^1([0,T), L^2)$ we have

$$\frac{d}{dt} 2^{-1} \left\langle \mathrm{i} \left(\partial_r + \frac{d-1}{2r} \right) u, u \right\rangle = - \left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, \mathrm{i} \dot{u} \right\rangle.$$

which can be proved assuming first $u \in C^\infty([0,T), H^2)$ and then proceeding by a density argument. In our case we get

$$\frac{d}{dt} 2^{-1} \left\langle \mathrm{i} \left(\partial_r + \frac{d-1}{2r} \right) u, u \right\rangle =$$
$$\left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, -\mathrm{i} \dot{u} \right\rangle = - \left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, -\triangle u + |u|^{p-1} u \right\rangle. \tag{5.4}$$ eq:virial_1

The equality (5.4) is crucial, indeed we will use it to prove

$$\frac{d}{dt} \langle \partial_r u, \mathrm{i} u \rangle \geq (d-1) \frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} dx, \tag{5.5}$$ eq:virial_1disp

which is crucial in our argument.

The first observation to obtain (5.5), is that the following is true,

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \mathrm{i}\dot{u} \right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle \partial_r u, \mathrm{i}u \right\rangle . \tag{5.6}$$ 7.6.9

Indeed, notice that

$$\begin{aligned} &\frac{1}{2}\partial_t \operatorname{Re}\left(\mathrm{i}u\overline{u}_r\right) + \frac{1}{2}\nabla\cdot\left(\frac{x}{r}\operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}\right)\right) \\ &= \frac{1}{2}\operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}_r\right) + \cancel{\frac{1}{2}\operatorname{Re}\left(\mathrm{i}u\dot{\overline{u}}_r\right)} + \frac{1}{2}\left(\nabla\cdot\frac{x}{r}\right)\operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}\right) + \cancel{\frac{1}{2}\operatorname{Re}\left(\mathrm{i}\dot{u}_r\overline{u}\right)} + \frac{1}{2}\operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}_r\right) \\ &= \operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}_r\right) + \frac{d-1}{2r}\operatorname{Re}\left(\mathrm{i}\dot{u}\overline{u}\right), \end{aligned}$$

so that integrating in x we obtain exactly (5.6).
The next step to prove (5.5), is the following inequality.

aim:7.6.10 **Claim 5.3.** Let $u \in H^2(\mathbb{R}^d, \mathbb{C})$. Then

$$\left\langle \left(\partial_r + \frac{d-1}{2r}\right) u, \triangle u \right\rangle \leq 0. \tag{5.7}$$ 7.6.10

Proof. The proof is based on the identity

$$\begin{aligned} &\nabla\cdot\operatorname{Re}\left\{\nabla u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \operatorname{Re}\left\{\triangle u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} - \nabla\cdot\left\{\frac{x}{2r}|\nabla u|^2\right\} \\ &+ \nabla\cdot\left(\frac{d-1}{4}\,\frac{x}{r^3}|u|^2\right) - \frac{1}{r}\left(|\nabla u|^2 - |u_r|^2\right) - \frac{(d-1)(d-3)}{4r^3}|u|^2, \end{aligned} \tag{5.8}$$ 7.6.10_1

which we check now. We have

$$\begin{aligned} &\nabla\cdot\operatorname{Re}\left\{\nabla u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} = \operatorname{Re}\left\{\triangle u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \operatorname{Re}\left\{\partial_j u\partial_j\left(\frac{x_k}{r}\partial_k\overline{u}\right)\right\} + \operatorname{Re}\left\{\partial_j u\partial_j\left(\frac{d-1}{2r}\overline{u}\right)\right\} \\ &= \operatorname{Re}\left\{\triangle u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \frac{x_k}{2r}\partial_k|\nabla u|^2 + \frac{1}{r}|\nabla u|^2 - \operatorname{Re}\left\{\frac{x_k x_j}{r^3}\partial_j u\partial_k\overline{u}\right\} + \frac{d-1}{2r}|\nabla u|^2 \\ &- \frac{d-1}{2}\frac{x_j}{r^3}\operatorname{Re}\left\{\partial_j u\overline{u}\right\} \\ &= \operatorname{Re}\left\{\triangle u\left(\overline{u}_r + \frac{d-1}{2r}\overline{u}\right)\right\} + \partial_k\left(\frac{x_k}{2r}|\nabla u|^2\right) - |\nabla u|^2\partial_k\left(\frac{x_k}{2r}\right) + \frac{|\nabla u|^2 - |u_r|^2}{r} + \frac{d-1}{2r}|\nabla u|^2 \\ &- \partial_j\left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right) - \frac{d-1}{4}|u|^2\partial_j\left(\frac{x_j}{r^3}\right). \end{aligned}$$

Now we use

$$\begin{aligned} \partial_k\left(\frac{x_k}{2r}\right) &= \frac{d-1}{2r} \\ \partial_j\left(\frac{x_j}{r^3}\right) &= \frac{d-3}{r^3}, \end{aligned}$$

to conclude

$$\begin{aligned}&\nabla\cdot\mathrm{Re}\left\{\nabla u\left(\overline{u}_r+\frac{d-1}{2r}\overline{u}\right)\right\}=\\&=\mathrm{Re}\left\{\triangle u\left(\overline{u}_r+\frac{d-1}{2r}\overline{u}\right)\right\}+\partial_k\left(\frac{x_k}{2r}|\nabla u|^2\right)+\frac{|\nabla u|^2-|u_r|^2}{r}\\&-\partial_j\left(\frac{d-1}{4}\frac{x_j}{r^3}|u|^2\right)-\frac{(d-1)(d-3)}{4r^3}|u|^2,\end{aligned}$$

which is (5.8). Now, applying the Divergence Theorem to (5.8) and Lemma 5.1, we have

$$\begin{aligned}&\left\langle\left(\partial_r+\frac{d-1}{2r}\right)u,\triangle u\right\rangle\leq-\int_{\mathbb{R}^d}\frac{1}{r}\left(|\nabla u|^2-|u_r|^2\right)dx-\frac{(d-1)(d-3)}{4}\lim_{a\to0^+}\int_{r\geq a}\frac{|u|^2}{r^3}dx\\&-\liminf_{a\to0^+}\int_{r=a}\left[\mathrm{Re}\left\{u_r\left(\overline{u}_r+\frac{d-1}{2a}\overline{u}\right)\right\}-\frac{|\nabla u|^2}{2}+\frac{d-1}{4}\frac{|u|^2}{a^2}\right]dS.\end{aligned}$$

Let us now suppose that $u\in C^\infty(\mathbb{R}^d,\mathbb{C})$. Then

$$\lim_{a\to0^+}\int_{\partial B(x,a)}|\nabla u|^2dS=0$$

Similarly, for $d>3$ and $u\in C^\infty(\mathbb{R}^d,\mathbb{C})$ we have

$$\lim_{a\to0^+}\int_{r=a}\frac{|u|^2}{a^2}dS=0$$

Hence, for $d>3$ and $u\in C^\infty(\mathbb{R}^d,\mathbb{C})$ we obtain 5.1, we have

$$\left\langle\left(\partial_r+\frac{d-1}{2r}\right)u,\triangle u\right\rangle\leq-\int_{\mathbb{R}^d}\frac{1}{r}\left(|\nabla u|^2-|u_r|^2\right)dx-\frac{(d-1)(d-3)}{4}\int_{\mathbb{R}^d}\frac{|u|^2}{r^3}dx\leq0. \tag{5.9}$$

eq:limit1

For $u\in H^2(\mathbb{R}^d,\mathbb{C})$ and, $u\notin C^\infty(\mathbb{R}^d,\mathbb{C})$ considered a sequence $u_n\xrightarrow{n\to\infty}u$ in $H^2(\mathbb{R}^d,\mathbb{C})$, we have

$$\left\langle\left(\partial_r+\frac{d-1}{2r}\right)u_n,\triangle u_n\right\rangle\leq-\int_{\mathbb{R}^d}\frac{1}{r}\left(|\nabla u_n|^2-|u_{nr}|^2\right)dx-\frac{(d-1)(d-3)}{4}\int_{\mathbb{R}^d}\frac{|u_n|^2}{r^3}dx$$

which in the limit converges to (5.9).
For $d=3$ then $u\in C^0(\mathbb{R}^3)$ and so

$$\lim_{a\to0^+}\int_{\partial B(0,a)}|u|^2\frac{dS}{a^2}=4\pi|u(0)|^2,$$

so that we obtain

$$\left\langle\left(\partial_r+\frac{d-1}{2r}\right)u,\triangle u\right\rangle=-\int_{\mathbb{R}^3}\frac{1}{r}\left(|\nabla u|^2-|u_r|^2\right)dx-2\pi|u(0)|^2.$$

□

The next step to prove inequality (5.5) is the following identity,

$$\left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, |u|^{p-1}u \right\rangle = \frac{d-1}{2}\frac{p-1}{p+1} \int \frac{|u|^{p+1}}{r}. \tag{5.10}$$ eq:virial_2

Indeed

$$\left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, |u|^{p-1}u \right\rangle = \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2} \int_{\mathbb{R}^d} (|u|^2)^{\frac{p-1}{2}} \partial_r |u|^2 dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} + \frac{1}{2}\frac{2}{p+1} \int_{\mathbb{R}^d} \partial_r (|u|^2)^{\frac{p+1}{2}} dx$$
$$= \frac{d-1}{2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} - \frac{d-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} = \frac{d-1}{2}\frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r}.$$

So now we can prove (5.5). Indeed, from (5.6), (5.4), (5.6) and (5.10), we obtain

$$-\frac{1}{2}\frac{d}{dt} \langle \partial_r u, \mathrm{i}u \rangle = \left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, -\mathrm{i}\dot{u} \right\rangle = -\left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, -\triangle u + |u|^{p-1}u \right\rangle$$
$$\leq -\left\langle \left(\partial_r + \frac{d-1}{2r} \right) u, |u|^{p-1}u \right\rangle = -\frac{d-1}{2}\frac{p-1}{p+1} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r},$$

which yields (5.5).

lem:mor4 **Lemma 5.4.** *We have*

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{r} \leq \frac{2}{d-1}\ \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} \|\nabla u\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))} \leq \frac{2^{\frac{3}{2}}}{d-1}\ \frac{p+1}{p-1} \|u_0\|_{L^2(\mathbb{R}^d)} E(u_0). \tag{5.11}$$ eq:mor41

furthermore, we have $u(t) \overset{t\to\infty}{\rightharpoonup} 0$ *in* $H^1(\mathbb{R}^d)$.

Proof. To get (5.11) it is enough to integrate. We skip the proof of the weak limit. □

We now start directly to prove Theorem 4.2.

lem:step1 **Lemma 5.5.** *We have*

$$\int_{|x|\geq t\log t} |u|^{p+1} dx \xrightarrow{t\to+\infty} 0. \tag{5.12}$$ eq:step11

Proof. We consider for $M > 0$

$$\theta_M(x) = \begin{cases} \frac{|x|}{M} \text{ for } |x| \leq M \\ 1 \text{ for } |x| \geq M \end{cases}$$

Then $\theta_M \in W^{1,\infty}(\mathbb{R}^d)$ with $\|\nabla\theta_M\|_{L^\infty} \le 1/M$. Now we have $u \in C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, H^{-1})$. Then, it can be proved, by a density argument, that $t \to 2^{-1}\langle\theta_M u(t), u(t)\rangle \in AC([-T,T])$ for any $T>0$ with

$$\frac{d}{dt}2^{-1}\langle\theta_M u(t), u(t)\rangle = \langle\theta_M u(t), \dot u(t)\rangle .$$

Since we have $\mathrm{i}\dot u(t) = -\triangle u + |u|^{p-1}u$ in $\mathcal{D}'(\mathbb{R}, H^{-1})$, we have

$$\left|\frac{d}{dt}2^{-1}\langle\theta_M u(t), u(t)\rangle\right| = \left|\left\langle\theta_M u(t), \mathrm{i}\triangle u - \mathrm{i}|u|^{p-1}u\right\rangle\right| = |\langle\theta_M u(t), \mathrm{i}\triangle u\rangle| \le \|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty}$$
$$\le \|\nabla u\|_{L^2}\|u\|_{L^2}\|\nabla\theta_M\|_{L^\infty} \le CM^{-1}.$$

Then it follows, for a C independent from M,

$$\langle\theta_M u(t), u(t)\rangle \le CM^{-1}t + \langle\theta_M u_0, u_0\rangle .$$

Setting $M = t\log t$, we obtain by dominated convergence

$$\begin{aligned}&\int_{|x|\ge t\log t}|u(t)|^2dx \le \langle\theta_{t\log t}u(t), u(t)\rangle\\ &\le \frac{C}{\log t} + \int_{|x|\le t\log t}\frac{|x|}{t\log t}|u_0|^2dx + \int_{|x|\ge t\log t}|u_0|^2dx \xrightarrow{t\to+\infty} 0.\end{aligned}$$

Finally

$$\begin{aligned}&\|u(t)\|_{L^{p+1}(|x|\ge t\log t)} \le \|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|u(t)\|^{1-\alpha}_{L^{d^*+1}(\mathbb{R}^d)}\\ &\le C\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)}\|\nabla u(t)\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \le C'\|u(t)\|^{\alpha}_{L^2(|x|\ge t\log t)} \xrightarrow{t\to+\infty} 0.\end{aligned}$$

□

lem:step2 **Lemma 5.6.** *For any $\epsilon>0$, $t>1$ and $\tau>0$ there exists $t_0>\max(t,2\tau)$ s.t.*

$$\int_{t_0-2\tau}^{t_0}\int_{|x|\le s\log s}|u|^{p+1}dxds \le \epsilon. \qquad (5.13)$$ eq:step21

Proof. The starting point is Lemma 5.4. We have

$$\begin{aligned}\infty &> \int_{\mathbb{R}}dt\int_{\mathbb{R}^d}\frac{|u|^{p+1}}{r} \ge \int_2^\infty\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx\\ &\ge \sum_{k=0}^\infty\int_{t+2k\tau}^{t+2(k+1)\tau}\frac{ds}{s\log s}\int_{|x|\le s\log s}|u|^{p+1}dx\\ &\ge \sum_{k=0}^\infty\frac{1}{(t+2(k+1)\tau)\log(t+2(k+1)\tau)}\int_{t+2k\tau}^{t+2(k+1)\tau}ds\int_{|x|\le s\log s}|u|^{p+1}dx.\end{aligned}$$

From this inequality we derive

$$\liminf_{k\to+\infty}\int_{t+2k\tau}^{t+2(k+1)\tau} ds\int_{|x|\le s\log s}|u|^{p+1}dx=0,$$

because otherwise the series would diverge. Hence for any $\epsilon>0$ there exists k_0 arbitrarily large with

$$\int_{t+2k_0\tau}^{t+2(k_0+1)\tau} ds\int_{|x|\le s\log s}|u|^{p+1}dx<\epsilon.$$

So for $t_0=t+2(k_0+1)\tau$ we obtain (5.13).

□

lem:step3 **Lemma 5.7.** *For any $\epsilon,a,b\in\mathbb{R}_+$ there exists $t_0>\max(a,b)$ s.t.*

$$\sup_{s\in[t_0-b,t_0]}\|u(s)\|_{L^{p+1}}\le\epsilon. \tag{5.14}$$ eq:step31

Proof. We have

$$\begin{aligned}
u(t)&=e^{\mathrm{i}t\triangle}u_0-\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds\\
&=e^{\mathrm{i}t\triangle}u_0\underbrace{-\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{w(t,\tau)}\underbrace{-\mathrm{i}\int_{t-\tau}^{1} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds}_{z(t,\tau)}\\
&=e^{\mathrm{i}t\triangle}u_0+w(t,\tau)+z(t,\tau).
\end{aligned}$$

Now we consider each of the last three terms.

cl:lim1 **Claim 5.8.** We have

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\xrightarrow{t\to+\infty}0. \tag{5.15}$$ eq:lim11

Proof. Indeed, if $u_0\in L^{\frac{p+1}{p}}$, then

$$\|e^{\mathrm{i}t\triangle}u_0\|_{L^{p+1}}\le Ct^{-d\left(\frac12-\frac{1}{p+1}\right)}\|u_0\|_{L^{\frac{p+1}{p}}}\xrightarrow{t\to+\infty}0.$$

The general case follows from the special one using the fact that $H^1\cap L^{\frac{p+1}{p}}$ is dense in H^1. □

cl:lim2 **Claim 5.9.** There is a constant C independent from t and τ s.t.

$$\|w(t,\tau)\|_{L^{p+1}}\le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}}. \tag{5.16}$$ eq:lim21

Proof. We define

$$q = \begin{cases} \infty \text{ if } p \geq 2 \\ \frac{2}{2-p} \text{ if } p < 2. \end{cases}$$

Then we have

$$\|w(t,\tau)\|_{L^q} \leq \int_0^{t-\tau} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|^p_{L^{pq'}} ds.$$

Here we claim

$$d\left(\frac{1}{2}-\frac{1}{q}\right) > 1. \tag{5.17}$$ eq:lim22

This is obvious by $d \geq 3$ if $q = \infty$. Otherwise, for $p < 2$

$$d\left(\frac{1}{2}-\frac{1}{q}\right) = d\left(\frac{1}{2}-\frac{2-p}{2}\right) = \frac{d}{2}(p-1) > 1 \Longleftrightarrow p > 1+\frac{2}{d},$$

where the last inequality follows from $p > 1 + \frac{4}{d}$. So we have

$$\|w(t,\tau)\|_{L^q} \leq C\tau^{-d\left(\frac{1}{2}-\frac{1}{q}\right)+1} \sup_s \|u(s)\|^p_{L^{pq'}}. \tag{5.18}$$ eq:lim23

We claim now that $2 \leq pq' \leq p+1$. Indeed, for $p \geq 2$ we have $q' = 1$ and the claim holds. If $p < 2$ then

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{2-p}{2} = \frac{p}{2}$$

so that $pq' = 2$. So in all cases we have $H^1 \hookrightarrow L^{pq'}$ and we can uniformly bound the last factor on the right in (5.18).

Next, we claim $\|w(t,\tau)\|_{L^2} \leq 2\|u_0\|_{L^2}$, which follows from

$$w(t,\tau) = -\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds = e^{\mathrm{i}\tau\triangle}\left(-\mathrm{i}\int_0^{t-\tau} e^{\mathrm{i}(t-\tau-s)\triangle}|u(s)|^{p-1}u(s)ds\right)$$
$$= e^{\mathrm{i}\tau\triangle}\left(u(t-\tau) - e^{\mathrm{i}(t-\tau)\triangle}u_0\right) = e^{\mathrm{i}\tau\triangle}u(t-\tau) - e^{\mathrm{i}t\triangle}u_0.$$

Finally, we claim $p+1 \leq q$. This is obviously the case if $q = \infty$. Otherwise $p < 2$, and then

$$q > p+1 \Longleftrightarrow \frac{2}{2-p} > p+1 \Longleftrightarrow 2 > (p+1)(2-p) = 2+p-p^2$$

where the last inequality follows from $p > 1$ and so from $p - p^2 < 0$. Finally by Hölder inequality

$$\|w(t,\tau)\|_{L^{p+1}} \leq \|w(t,\tau)\|^{1-\alpha}_{L^2}\|w(t,\tau)\|^{\alpha}_{L^q} \text{ where } \frac{1}{p+1} = \frac{1-\alpha}{2}+\frac{\alpha}{q}.$$

Notice that $\alpha = \dfrac{\frac{1}{2} - \frac{1}{p+1}}{\frac{1}{2} - \frac{1}{q}}$. So

$$\|w(t,\tau)\|_{L^{p+1}} \leq C\tau^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)+\frac{\frac{1}{2}-\frac{1}{p+1}}{\frac{1}{2}-\frac{1}{q}}}. \tag{5.19}$$

eq:lim24

We now examine the exponent in (5.19). If $q = \infty$ the exponent equals

$$-(d-2)\left(\frac{1}{2} - \frac{1}{p+1}\right) = -\frac{d(p-1) - 2(p-1)}{2(p+1)} = -\frac{d(p-1) - 2\max(1, p-1)}{2(p+1)}.$$

In the case $q < \infty$, then

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(-d + \frac{1}{\frac{1}{2} - \frac{1}{q}}\right) = -\frac{p-1}{2(p+1)}\left(d - \frac{2}{p-1}\right)$$
$$= -\frac{d(p-1) - 2}{2(p+1)} = -\frac{d(p-1) - 2\max(1, p-1)}{2(p+1)}.$$

So we have proved that the exponent in (5.19) is exactly the one in (5.16), which is then proved.

□

We now consider

$$z(t,\tau) = -\mathrm{i}\int_{t-\tau}^{t} e^{\mathrm{i}(t-s)\triangle}|u(s)|^{p-1}u(s)ds.$$

We have

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \int_{t-\tau}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u\|^{p}_{L^{p+1}}ds. \tag{5.20}$$

eq:dispers_z

Notice that $p < d^*$, that is $p + 1 < \frac{2d}{d-2}$ is equivalent to $d\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$. Indeed,

$$\frac{1}{p+1} > \frac{d-1}{2d} = \frac{1}{2} - \frac{1}{d}.$$

We now pick $q \in \left(1, \frac{2(p+1)}{d(p-1)}\right)$. Notice that this implies $qd\left(\frac{1}{2} - \frac{1}{p+1}\right) < 1$. Then

$$\|z(t,\tau)\|_{L^{p+1}} \lesssim \left(\int_{t-\tau}^{t} (t-s)^{-dq\left(\frac{1}{2}-\frac{1}{p+1}\right)}ds\right)^{1/q}\left(\int_{t-\tau}^{t}\|u\|^{pq'}_{L^{p+1}}ds\right)^{\frac{1}{q'}}$$
$$= C\tau^{\alpha}\left(\int_{t-\tau}^{t}\|u\|^{pq'}_{L^{p+1}}ds\right)^{\frac{1}{q'}}$$

for some $\alpha > 0$. Now we claim $q'p > p+1$ or, equivalently, $\frac{1}{q'} < \frac{p}{p+1}$ Indeed

$$\frac{1}{q} > \frac{d}{2} - \frac{d}{p+1} \Longleftrightarrow \frac{1}{q'} = 1 - \frac{1}{q} < 1 - \frac{d}{2} + \frac{d}{p+1} \Longleftrightarrow \frac{1}{q'} < \frac{2-d}{2} + \frac{d}{p+1}$$
$$= \frac{2(p+1) - (p+1)d + 2d}{2(p+1)} = \frac{p}{p+1} + \frac{2 - (p+1)d + 2d}{2(p+1)} < \frac{p}{p+1},$$

where the last inequality holds because

$$2 - (p+1)d + 2d = 2 - pd + d < 0 \Longleftrightarrow p > 1 + \frac{2}{d},$$

with the latter true because, in our case, $p > 1 + \frac{4}{d}$.
From $q'p > p+1$ it follows that

$$\|z(t,\tau)\|_{L^{p+1}} \le C\tau^{\alpha} \left(\int_{t-\tau}^{t} \|u\|_{L^{p+1}}^{p+1} ds \right)^{\mu}$$
$$= C\tau^{\alpha} \left(\int_{t-\tau}^{t} ds \int_{|x| \ge s \log s} |u|^{p+1} dx + \int_{t-\tau}^{t} ds \int_{|x| \le s \log s} |u|^{p+1} dx \right)^{\mu}$$
$$\le 2^{\mu} C\tau^{\delta+\mu} \left(\sup_{s \in [t-\tau,t]} \|u(s)\|_{L^{p+1}(|x| \ge s \log s)}^{p+1} \right)^{\mu} + 2^{\mu} C\tau^{\delta} \left(\int_{t-\tau}^{t} ds \int_{|x| \le s \log s} |u|^{p+1} dx \right)^{\mu}. \tag{5.21}$$

record1

Let us take now $\tau > b$ such that

$$\|w(t,\tau)\|_{L^{p+1}} \le C\tau^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} < \frac{\epsilon}{4}. \tag{5.22}$$

ineq1

Next, using Lemma 5.5 and Claim 5.8 let us take $t_1 > \max(a,b)$ such that for $t \ge t_1$

$$\|e^{\mathrm{i}t\triangle} u_0\|_{L^{p+1}} + 2^{\mu} C\tau^{\delta+\mu} \left(\sup_{s \in [t-\tau,t]} \|u(s)\|_{L^{p+1}(|x| \ge s \log s)}^{p+1} \right)^{\mu} < \frac{\epsilon}{4}. \tag{5.23}$$

ineq2

Using Lemma 5.6 there exists $t_2 > t_1 + 2\tau$ such that for $t \in [t_2, t_2 - \tau]$

$$2^{\mu} C\tau^{\delta} \left(\int_{t-\tau}^{t} ds \int_{|x| \le s \log s} |u|^{p+1} dx \right)^{\mu} \le 2^{\mu} C\tau^{\delta} \left(\int_{t_2-2\tau}^{t_2} ds \int_{|x| \le s \log s} |u|^{p+1} dx \right)^{\mu} < \frac{\epsilon}{4}. \tag{5.24}$$

ineq3

Making careful choices, we conclude the proof of Lemma 5.7.

□

We now move to complete the proof of Theorem 4.2.
Let us fix $\epsilon > 0$. Pick $t > \tau > 0$. Then, in view of $u(t) = e^{\mathrm{i}t\triangle} u_0 + w(t,\tau) + z(t,\tau)$, we have that by Claims 5.8–5.9 there exists $t_1 \ge 0$ and τ_ϵ with

$$\|u(t)\|_{L^{p+1}} \le \|e^{\mathrm{i}t\triangle} u_0\|_{L^{p+1}} + C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} + \|z(t,\tau)\|_{L^{p+1}} \le \frac{\epsilon}{2} + \|z(t,\tau_\epsilon)\|_{L^{p+1}},$$

where we chose $\|e^{it\triangle}u_0\|_{L^{p+1}} < \frac{\epsilon}{4}$ for $t > t_1$ and

$$C\tau_\epsilon^{-\frac{d(p-1)-2\max(1,p-1)}{2(p+1)}} = \frac{\epsilon}{4}. \tag{5.25}$$ choice_tau

In turn by (5.20)

$$\|z(t,\tau_\epsilon)\|_{L^{p+1}} \lesssim \int_{t-\tau_\epsilon}^{t} (t-s)^{-d\left(\frac{1}{2}-\frac{1}{p+1}\right)} \|u\|_{L^{p+1}}^p ds \leq C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t-\tau_\epsilon,t]} \|u(s)\|_{L^{p+1}}^p$$

From Lemma 5.7 we know that there exists $t_0 > \max(t_1, \tau_\epsilon)$ s.t.

$$\sup_{s\in[t_0-\tau_\epsilon,t_0]} \|u(s)\|_{L^{p+1}} \leq \frac{\epsilon}{4}. \tag{5.26}$$ induct_decay

Consider now

$$t_\epsilon = \sup\{t \geq t_0 : \|u(s)\|_{L^{p+1}} \leq \epsilon \text{ for all } s \in [t_0-\tau_\epsilon, t]\},$$

where (5.26) guarantees that the set on the right hand side is non empty.
If $t_\epsilon = +\infty$ we will have proved the desired result. So, let us suppose that $t_\epsilon < \infty$. Then, by $u \in C^0(\mathbb{R}, H^1)$, we have $\|u(t_\epsilon)\|_{L^{p+1}} = \epsilon$. Then we have

$$\epsilon < \frac{\epsilon}{2} + \|z(t_\epsilon,\tau_\epsilon)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \sup_{s\in[t_\epsilon-\tau_\epsilon,t_\epsilon]} \|u(s)\|_{L^{p+1}}^p,$$

so that we conclude

$$\epsilon < \frac{\epsilon}{2} + \left(C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1}\right)\epsilon.$$

We now need to check that it is possible to choose τ_ϵ such that both

$$C\tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} < \frac{1}{2} \tag{5.27}$$ choice_tau1

and (5.25) are true. This will lead to a contradiction. Suppose that for τ_ϵ which satisfies (5.25) inequality (5.27) is false. This implies

$$\frac{1}{2C} \leq \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}} \epsilon^{p-1} = C_1 4^{p-1} \tau_\epsilon^{1-\frac{d(p-1)}{2(p+1)}-\frac{d(p-1)^2-2(p-1)\max(1,p-1)}{2(p+1)}}, \tag{5.28}$$ contradict_1

where we substituted ϵ^{p-1} using the equality (5.25). We will show now that the exponent of τ_ϵ is negative, so that taking $\tau_\epsilon \gg 1$ formula (5.28) leads to a contradiction. Taking a unique fraction in the exponent and focusing on the numerator, we have

$$\begin{aligned} &2(p+1) - d(p-1) - d(p-1)^2 + 2(p-1)\max(1,p-1) \\ &= (p-1)\left(2\max(1,p-1) - d - d(p-1)\right) + 2(p+1) \\ &= (p-1)\left(2\max(1,p-1) + 2 - d(p-1)\right) - d(p-1) - 2(p-1) + 2(p+1) \\ &= (p-1)\left(2\max(1,p-1) + 2 - d(p-1)\right) - d(p-1) + 4. \end{aligned} \tag{5.29}$$ contradict_2

For $p-1 \leq 1$ the quantity in line (5.29) becomes

$$(p-1)\,(4-d(p-1)) - d(p-1) + 4 = p\,(4-d(p-1)) < 0$$

by $p > 1 + 4/d$ and this completes the proof for $p-1 \leq 1$.
For $p-1 > 1$ the quantity in line (5.29) becomes

$$\begin{aligned}&(p-1)\,(2(p-1)+2-d(p-1)) - d(p-1) + 4\\ &= (p-1)\,(2-(d-2)(p-1)) - d(p-1) + 4.\end{aligned}$$

For $d \geq 4$

$$\begin{aligned}&(p-1)\,(2-(d-2)(p-1)) - d(p-1) + 4\\ &\leq (p-1)\,(2-2(p-1)) - 4(p-1) + 4 = -2(p-1)p - 4(p-2) < 0.\end{aligned}$$

Finally, for $d = 3$ and $p-1 > 1$ the quantity in line (5.29) becomes, for $\alpha = p-1$,

$$\begin{aligned}&(p-1)\,(2(p-1)+2-3(p-1)) - 3(p-1) + 4\\ &= -\alpha^2 - \alpha + 4 =: -q(\alpha).\end{aligned}$$

Now, $q(\alpha) = 0$ for $\alpha_{\pm} = -1/2 \pm \frac{\sqrt{17}}{2}$. This means that $q(\alpha) < 0$ for $p-1 > \frac{\sqrt{17}-1}{2}$. The completion of the proof of Theorem 4.2 for the remaining cases, that is $d = 3$ and $2 < p \leq \frac{\sqrt{17}+1}{2}$ is not in caz [4].

□

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