# 1 Fourier transform

**Definition 1.1** (Fourier transform)**.** For $f \in L^1(\mathbb{R}^d, \mathbb{C})$ we call its Fourier transform the function defined by the following formula

$$\widehat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.1}$$

We use also the notation $\mathcal{F}f(\xi) = \widehat{f}(\xi)$.

*Example* 1.2. We have for any $\varepsilon > 0$

$$e^{-\varepsilon\frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}} dx. \tag{1.2}$$

We set also

$$\mathcal{F}^* f(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.3}$$

We have what follows.

**Theorem 1.3.** *The following facts hold.*

*(1) We have* $|\widehat{f}(\xi)| \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}$. *So in particular we have*

$$\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^d,\mathbb{C})} \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}. \tag{1.4}$$

*(2) (Riemann– Lebesgue Lemma) We have* $\lim_{\xi\to\infty} \widehat{f}(\xi) = 0$.

*(3) The bounded linear operator* $\mathcal{F} : L^1(\mathbb{R}^d, \mathbb{C}) \to L^\infty(\mathbb{R}^d, \mathbb{C})$ *has values in the following space* $C_0(\mathbb{R}^d, \mathbb{C}) \subset L^\infty(\mathbb{R}^d, \mathbb{C})$

$$C_0(\mathbb{R}^d, \mathbb{C}) := \{g \in C^0(\mathbb{R}^d, \mathbb{C}) : \lim_{x\to\infty} g(x) = 0\}. \tag{1.5}$$

*(4)* $\mathcal{F}$ *defines an isomorphism of the space of Schwartz functions* $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ *into itself.*

*(5)* $\mathcal{F}$ *defines an isomorphism of the space of tempered distributions* $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ *into itself. We have* $\mathcal{F}[\partial_{x_j} f] = -\mathrm{i}\xi_j \mathcal{F}f$.

*(6) For* $f, g \in L^1(\mathbb{R}^d, \mathbb{C})$ *we have*

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

□

**Theorem 1.4** (Fourier transform in $L^2$)**.** *The following facts hold.*

*(1) For a function $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ we have that $\widehat{f} \in L^2(\mathbb{R}^d, \mathbb{C})$ and $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. An operator*

$$\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{C}) \tag{1.6}$$

*remains defined. For $f \in L^2(\mathbb{R}^d, \mathbb{C})$ for any function $\varphi \in C_c(\mathbb{R}^d, \mathbb{C})$ with $\varphi = 1$ near 0 set*

$$\begin{aligned} \mathcal{F}f(\xi) &:= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi \cdot x} f(x) \varphi(x/\lambda) dx \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{|x| \le \lambda} e^{-\mathrm{i}\xi \cdot x} f(x) dx. \end{aligned} \tag{1.7}$$

*Then (1.7) defines an isometric isomorphism inside $L^2(\mathbb{R}^d, \mathbb{C})$, so in particular we have*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d, \mathbb{C})} = \|f\|_{L^2(\mathbb{R}^d, \mathbb{C})}. \tag{1.8}$$

*(2) The inverse map is defined by*

$$\begin{aligned} \mathcal{F}^* f(x) &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} f(\xi) \varphi(\xi/\lambda) d\xi \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{\frac{d}{2}} \int_{|\xi| \le \lambda} e^{\mathrm{i}\xi \cdot x} f(\xi) d\xi. \end{aligned} \tag{1.9}$$

*(3) For $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ the two definitions (1.1) and (1.7) of $\mathcal{F}$ coincide (by dominated convergence). Similarly, for $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ the two definitions (1.3) and (1.9) of $\mathcal{F}^*$ coincide.*

The above notions extend naturally to vector fields. So we have a Fourier transform $f \to \widehat{f}$ from $(L^1(\mathbb{R}^d))^d \to (C_0(\mathbb{R}^d))^d$, from $(L^2(\mathbb{R}^d))^d \to (L^2(\mathbb{R}^d))^d$ , from $(\mathcal{S}(\mathbb{R}^d))^d \to (\mathcal{S}(\mathbb{R}^d))^d$ and more generally from $(\mathcal{S}'(\mathbb{R}^d))^d \to (\mathcal{S}'(\mathbb{R}^d))^d$. Notice that all these maps except the 1st are isomorphisms, and all are one to one maps.

We have the following lemma.

We consider now for $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and for $f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the heat equation

$$u_t - \triangle u = 0\,, \quad u(0,x) = f(x). \tag{1.10}$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + |\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{f}(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$. Notice that since $\widehat{f} \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ and $e^{-t|\cdot|^2} \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ for any $t > 0$, the last product is well defined. Furthermore, we have $\widehat{u}(t,\cdot) \in C^0([0,+\infty), \mathcal{S}'(\mathbb{R}^d, \mathbb{C}))$ and, as a consequence, since $\mathcal{F}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ also $u(t,\cdot) \in C^0([0,+\infty), \mathcal{S}'(\mathbb{R}^n, \mathbb{C}))$. We have $e^{-t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. Then, from $\widehat{u}(t,\xi) = \widehat{G}(t,\xi)\widehat{f}(\xi)$ it follows $u(t,x) = (2\pi)^{-\frac{d}{2}} G(t,\cdot) * f(x)$. In particular, for $f \in L^p(\mathbb{R}^d, \mathbb{C})$, we have

$$u(t,x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) dy.$$

Notice that by (1.2) we have

$$(4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4t}} dx = 1.$$

We will write

$$e^{t\triangle} f(x) := (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{1.11}$$

Notice that for $p \geq 1$ we have $\|e^{t\triangle} f\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}$ and for $f \in L^1(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$

$$|e^{t\triangle} f(x)| \leq (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy \leq (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(y)| dy = (4\pi t)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d)}. \tag{1.12}$$

We set also $K_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. Then $e^{t\triangle} f = K_t * f$. $K_t(x-y)$ is the *Heath Kernel.*

**Lemma 1.5.** *For any $q \geq p \geq 1$ and $j \geq 1$ there exists $C_{jpq}$ s.t.*

$$\|\nabla^j e^{t\triangle} f\|_{L^q(\mathbb{R}^d)} \leq C_{jpq} t^{-\frac{j}{2} - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)} \text{ for any } f \in L^p(\mathbb{R}^d). \tag{1.13}$$

*Proof.* For brevity we consider only $j = 0$. Using Young's convolution inequality

$$\|K_t * f\|_{L^q(\mathbb{R}^d)} \leq \|K_t\|_{L^a(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \text{ where } \frac{1}{q} + 1 = \frac{1}{a} + \frac{1}{p},$$

where

$$\|K_t\|_{L^a(\mathbb{R}^d)} = (4\pi t)^{-\frac{d}{2}} \|e^{-\frac{|x|^2}{4t}}\|_{L^a(\mathbb{R}^d)} = C_{pq} t^{-\frac{d}{2} + \frac{d}{2a}} \text{ where} C_{pq} := (4\pi)^{-\frac{d}{2}} \|e^{-\frac{|x|^2}{4}}\|_{L^a(\mathbb{R}^d)}.$$

Now

$$t^{-\frac{d}{2} + \frac{d}{2a}} = t^{-\frac{d}{2}\left(1 - \frac{1}{a}\right)} = t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)},$$

and so this yields (1.13) for $j = 0$. The case $j \in \mathbb{N}$ is obtained in an elementary fashion by differentiating.

□

**Theorem 1.6.** *$\rho \in L^1(\mathbb{R}^d)$ be s.t. $\int \rho(x) dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-d} \rho(x/\epsilon)$. Consider $C_c(\mathbb{R}^d, \mathbb{C})$ and for each $p \in [1, \infty]$ let $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_p$ be the closure of $C_c(\mathbb{R}^d, \mathbb{C})$ in $L^p(\mathbb{R}^d, \mathbb{C})$, so that $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_p = L^p(\mathbb{R}^d, \mathbb{C})$ for $p < \infty$ and $\overline{C_c(\mathbb{R}^d, \mathbb{C})}_\infty = C_0(\mathbb{R}^d, \mathbb{C}) \subsetneqq L^\infty(\mathbb{R}^d, \mathbb{C})$. Then for any $f \in \overline{C_c(\mathbb{R}^d, \mathbb{C})}_p$ we have*

$$\lim_{\epsilon \searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}^d, \mathbb{C}). \tag{1.14}$$

*In particular we have*

$$\lim_{t \searrow 0} e^{t\triangle} f = f \text{ in } L^p(\mathbb{R}^d, \mathbb{C}). \tag{1.15}$$

*Proof.* Clearly, (1.15) is a special case of (1.14) setting $\epsilon = \sqrt{t}$ and $\rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$. To prove (1.14) we start with $f \in C_c(\mathbb{R}^d, \mathbb{C})$. In this case

$$\rho_\epsilon * f(x) - f(x) = \int_{\mathbb{R}^d} (f(x - \epsilon y) - f(x))\rho(y)dy$$

so that, by Minkowski inequality and for $\Delta(y) := \|f(\cdot - y) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(x) - f(x)\|_{L^p} \le \int |\rho(y)|\Delta(\epsilon\, y)dy.$$

Now we have $\lim_{y\to 0} \Delta(y) = 0$ and $\Delta(y) \le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon \searrow 0} \|\rho_\epsilon * f(x) - f(x)\|_{L^p} = \lim_{\epsilon \searrow 0} \int |\rho(y)|\Delta(\epsilon\, y)dy = 0.$$

So this proves (1.14) for $f \in C_c(\mathbb{R}^d, \mathbb{C})$. The general case is proved by a density argument. □

# 2 Some spaces of functions on $L^2$ based Sobolev Spaces

We will introduce the *homogeneous* Sobolev spaces $\dot{H}^k(\mathbb{R}^d)$ and we will generalize the standard Sobolev spaces $H^k(\mathbb{R}^d)$. For $\xi \in \mathbb{R}^d$ let $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ be the *Japanese bracket*. For a tempered distribution $u$ we denote by $\widehat{u}$ its Fourier transform. We consider for $s \in \mathbb{R}$ the space formed by the tempered distributions $u$

$$H^s(\mathbb{R}^d) \text{ with norm } \|u\|_{H^s(\mathbb{R}^d)} := \|\langle \xi \rangle^s \widehat{u}\|_{L^2(\mathbb{R}^d)} < \infty\ . \tag{2.1}$$

We consider for $s \in \mathbb{R}$ the space formed by the tempered distributions $u$ s.t. $\widehat{u} \in L^1_{loc}(\mathbb{R}^d)$

$$\dot{H}^s(\mathbb{R}^d) \text{ with norm } \|u\|_{\dot{H}^s(\mathbb{R}^d)} := \||\xi|^s \widehat{u}\|_{L^2(\mathbb{R}^d)} < \infty\ . \tag{2.2}$$

The following lemma is elementary.

**Lemma 2.1.** *The following statements are true.*

- $L^2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ *defined by* $f \to \mathcal{F}^*\left(\frac{\widehat{f}}{\langle \xi \rangle^s}\right)$ *is an isometric isomorphism and all the* $H^s(\mathbb{R}^d)$ *are Hilbert spaces with inner product* $\langle f, g \rangle_{H^s} = \langle \langle \xi \rangle^s \widehat{f}, \langle \xi \rangle^s \widehat{g} \rangle_{L^2}$ .
- *We have* $\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$ *if and only if* $s > -d/2$. *Furthermore, this embedding is dense.*
- *The* $\dot{H}^s(\mathbb{R}^d)$ *have an inner product defined by* $\langle f, g \rangle_{\dot{H}^s} = \langle |\xi|^s \widehat{f}, |\xi|^s \widehat{g} \rangle_{L^2}$.

We will use also the following.

**Lemma 2.2.** *Let* $\sigma > -d/2$. *Then* $C_c^\infty(\mathbb{R}^d)$ *is dense in* $\dot{H}^\sigma(\mathbb{R}^d)$.

*Proof.* It is immediate that $\mathcal{S}(\mathbb{R}^d)$ is dense in $\dot{H}^s(\mathbb{R}^d)$ (because $C_c^\infty(\mathbb{R}^d\backslash\{0\})$ is dense in $L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$). So it is enough to show that for any $\psi \in \mathcal{S}(\mathbb{R}^d)$ and for $\chi \in \mathbb{C}_c^\infty(\mathbb{R}^d, [0,1])$ a cutoff function with $\chi = 1$ near the origin, then $\chi\left(\frac{x}{n}\right)\psi \xrightarrow{n\to+\infty} \psi$ in $\dot{H}^\sigma(\mathbb{R}^d)$ for any $\sigma > -d/2$. Indeed recall

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}}\widehat{f}(\xi)\widehat{g}(\xi) \text{ so that}$$

$$\|\chi\left(\frac{x}{n}\right)\psi - \psi\|^2_{\dot{H}^\sigma} = \int d\xi|\xi|^{2\sigma}\left|\int (2\pi)^{-\frac{d}{2}}n^d\widehat{\chi}(n\eta)\widehat{\psi}(\xi-\eta)d\eta - \widehat{\psi}(\xi)\right|^2$$
$$= \int d\xi|\xi|^{2\sigma}\left|\int (2\pi)^{-\frac{d}{2}}\widehat{\chi}(\eta)\left(\widehat{\psi}\left(\xi - \frac{\eta}{n}\right) - \widehat{\psi}(\xi)\right)d\eta\right|^2.$$

So

$$\|\chi\left(\frac{x}{n}\right)\psi - \psi\|_{\dot{H}^\sigma} \le (2\pi)^{-\frac{d}{2}}\int d\eta|\widehat{\chi}(\eta)|\left(\int |\xi|^{2\sigma}\left|\int\left(\widehat{\psi}\left(\xi - \frac{\eta}{n}\right) - \widehat{\psi}(\xi)\right)d\xi\right|^2\right)^{\frac{1}{2}}.$$

We split in the right integrating in $|\eta| \le C$ and in $|\eta| \ge C$, where $C$ is arbitrary. In the integral in $|\eta| \le C$ we get a sequence that, essentially by dominated convergence, converges to 0. Next, we consider the integral in $|\eta| \ge C$. We can bound it from above by

$$(2\pi)^{-\frac{d}{2}}\int_{|\eta|\ge C} d\eta|\widehat{\chi}(\eta)|\left(\left(\int |\xi|^{2\sigma}\left|\int \widehat{\psi}\left(\xi - \frac{\eta}{n}\right)\right|^2 d\xi\right)^{\frac{1}{2}} + \|\psi\|_{\dot{H}^\sigma}\right). \tag{2.3}$$

Now we claim that for $c$ independent of $\eta$ we have

$$\int |\xi|^{2\sigma}\left|\int \widehat{\psi}(\xi - \eta)\right|^2 d\xi \le c + c|\eta|^{2\sigma}. \tag{2.4}$$

Indeed, we spit the integral into regions $|\eta| \ll |\xi|$, $|\eta| \sim |\xi|$ and $|\eta| \gg |\xi|$. We have

$$\int_{|\eta|\gg|\xi|} |\xi|^{2\sigma}\left|\widehat{\psi}(\xi-\eta)\right|^2 d\xi \lesssim \langle\eta\rangle^{-N}\int_{|\eta|\gg|\xi|}|\xi|^{2\sigma}d\xi \lesssim 1.$$

We have

$$\int_{|\eta|\ll|\xi|} |\xi|^{2\sigma}\left|\widehat{\psi}(\xi-\eta)\right|^2 d\xi \lesssim \int_{\mathbb{R}^d}|\xi-\eta|^{2\sigma}\left|\widehat{\psi}(\xi-\eta)\right|^2 d\xi = \|\psi\|^2_{\dot{H}^\sigma}.$$

Finally, for $|\eta| \sim |\xi|$

$$\int_{|\eta|\sim|\xi|} |\xi|^{2\sigma}\left|\widehat{\psi}(\xi-\eta)\right|^2 d\xi \le \int_{\mathbb{R}^d}|\xi-\eta|^{2\sigma}\left|\widehat{\psi}(\xi)\right|^2 d\xi \lesssim |\eta|^{2\sigma}\int_{\mathbb{R}^d}\left|\widehat{\psi}(\xi)\right|^2 d\xi + \|\psi\|^2_{\dot{H}^\sigma}.$$

So we proved (2.4). Inserting this in (2.3) and taking $C$ sufficiently large we obtain that (2.3) is arbitrarily small.

□

*Remark* 2.3. We will also consider the space $\dot{H}^{\sigma}(\mathbb{R}^d) \cap \dot{H}^{1+\sigma}(\mathbb{R}^d)$ for $\sigma > -d/2$. Then, by a similar argument, $C_c^{\infty}(\mathbb{R}^d)$ is dense in $\dot{H}^{\sigma}(\mathbb{R}^d) \cap \dot{H}^{1+\sigma}(\mathbb{R}^d)$.

While the $\dot{H}^s(\mathbb{R}^d)$ have an inner product, in general they are not complete topological vector spaces and the following will be important to us.

**Proposition 2.4.** *For $s < d/2$ the space $\dot{H}^s(\mathbb{R}^d)$ is complete and the Fourier transform establishes an isometric isomorphism $\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi)$.*

The above proposition is a consequence of the following lemma.

**Lemma 2.5.** *Let $s < \frac{d}{2}$. Then we have the following facts.*

- $L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi) \subset L^1_{loc}(\mathbb{R}^d, d\xi)$
- $L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$
- *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is s.t. $\mathcal{F}\left(\dot{H}^s(\mathbb{R}^d)\right) = L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ and establishes an isometry between these two spaces.*

*Proof.* Let $g \in L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi)$. Obviously $g \in L^1_{loc}(\mathbb{R}^d \backslash \{0\}, d\xi)$. Let now $B = \{\xi \in \mathbb{R}^d : |\xi| \le 1\}$. Then

$$\int_B |g(\xi)| d\xi \le \left( \int_B |\xi|^{2s} |g(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_B |\xi|^{-2s} d\xi \right)^{\frac{1}{2}}$$
$$\le \sqrt{\mathrm{vol}(S^{d-1})} ( \int_0^1 r^{d-1-2s} dr )^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)} = \sqrt{\frac{\mathrm{vol}(S^{d-1})}{d-2s}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)}.$$

Next, we check that $L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$. We split $g = \chi_B g + \chi_{B^c} g$. Then $\chi_B g \in L^1(\mathbb{R}^d, d\xi)$ implies $\chi_B g \in \mathcal{S}'(\mathbb{R}^d)$. On the other hand we have $\chi_{B^c} g \in L^2(\mathbb{R}^d, \langle \xi \rangle^{2s} d\xi)$. This in turn implies $\chi_{B^c} g \in \mathcal{S}'(\mathbb{R}^d)$, where the embedding $L^2(\mathbb{R}^d, \langle \xi \rangle^{2\sigma} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$ for any $\sigma \in \mathbb{R}$ follows from

$$\int_{\mathbb{R}^d} f(\xi) \varphi(\xi) d\xi = \int_{\mathbb{R}^d} \langle \xi \rangle^{\sigma} f(\xi) \langle \xi \rangle^{-\sigma} \varphi(\xi) d\xi \le \|f\|_{L^2(\mathbb{R}^d, \langle \xi \rangle^{2\sigma} d\xi)} ( \int_{\mathbb{R}^d} \langle \xi \rangle^{-2\sigma} \varphi(\xi) d\xi )^{\frac{1}{2}}$$
$$\le \|f\|_{L^2(\mathbb{R}^d, \langle \xi \rangle^{2\sigma} d\xi)} ( \int_{\mathbb{R}^d} \langle \xi \rangle^{-2\sigma - 2m} d\xi )^{\frac{1}{2}} \| \langle \xi \rangle^m \varphi \|_{L^{\infty}(\mathbb{R}^d)}$$

for $m$ chosen s.t. $2\sigma + 2m > d$. □

*Remark* 2.6. For $s \ge \frac{d}{2}$ the space $\dot{H}^s(\mathbb{R}^d)$ is not a complete space for the norm indicated. In particular, the Fourier trasform defines an embedding $\dot{H}^s(\mathbb{R}^d) \overset{\mathcal{F}}{\hookrightarrow} L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi)$ with image which is strictly contained and dense in $L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi)$. The fact that the image is dense can be seen observing that $C_c^{\infty}(\mathbb{R}^d \backslash \{0\})$ is dense in $L^2(\mathbb{R}^d \backslash \{0\}, |\xi|^{2s} d\xi)$ and we have $\mathcal{F} \dot{H}^s(\mathbb{R}^d) \supseteq C_c^{\infty}(\mathbb{R}^d \backslash \{0\})$.

For $s = \frac{d}{2} + \varepsilon_0$ with $\varepsilon_0 > 0$, if we pick $f \in C_c^\infty(\mathbb{R}^d)$ with $f(0) \neq 0$, then $\dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}}$ is a Borel function not contained in $L^1_{loc}(\mathbb{R}^d, d\xi)$. But $|\xi|^{2s} \left| \dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}} \right|^2 = \dfrac{|f(\xi)|^2}{|\xi|^{d-\varepsilon_0}} \in L^1(\mathbb{R}^d, d\xi)$ implies that $\dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}} \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$.

For $s = \frac{d}{2}$ consider $f(\xi) = \displaystyle\sum_{k=1}^{\infty} \frac{2^{kd}}{k} \chi_{[3/4,5/4]}(2^k|\xi|)$. Notice that for each $\xi$, at most one term of the sum is non zero, because $[2^{-k}3/4, 2^{-k}5/4] \cap [2^{-j}3/4, 2^{-j}5/4] = \emptyset$ for $j \neq k$. Indeed, if $j < k$ then

$$2^{-k}5/4 \leq 2^{-(j-1)}5/4 < 2^{-j}3/4 \text{ where the latter follows from } 5 < 6.$$

Then $|\xi|^{\frac{d}{2}}|f(\xi)| \in L^2(\mathbb{R}^d, d\xi)$ since

$$\int_{\mathbb{R}^d} |\xi|^d |f(\xi)|^2 d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} 2^{2kd} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(|\xi|) d\xi < \infty$$

but $f$, which is supported in the ball $B(0, 5/4)$, is not in $L^1(\mathbb{R}^d, d\xi)$ since otherwise we would have

$$\infty > \int_{\mathbb{R}^d} |f(\xi)| d\xi \geq \sum_{k=1}^{n} \frac{1}{k} 2^{kd} \int_{\mathbb{R}^d} \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{n} \frac{1}{k} \int_{\mathbb{R}^d} \chi_{[3/4,5/4]}(|\xi|) d\xi \overset{n\to\infty}{\to} \infty.$$

*Remark* 2.7. For $s \in (0,1)$ an equivalent definition of $\dot{H}^s(\mathbb{R}^d)$ and of its norm is that

$$u \in L_{loc}(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dx dy < \infty.$$

See [1, Proposition 1.37].

Later on we, when discussing the Navier Stokes Equation, we will deal with vector fields. Given a vector field $u = (u^j)_{j=1}^d \in (\mathcal{S}'(\mathbb{R}^d))^d$ its divergence is

$$\text{div} u = \nabla \cdot u := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} u^j .$$

Notice that $\widehat{\text{div} u} = -\mathrm{i} \sum_{j=1}^d \xi^j \widehat{u}^j$ so that a $u$ is divergence free, that is $\text{div} u = 0$, if and only if $\sum_{j=1}^d \xi^j \widehat{u}^j = 0$.

We have the following elementary representation in $d = 3$.

**Lemma 2.8.** *For any* $u \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ *we have*

$$\triangle u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u). \tag{2.5}$$

*Proof.* Obviously, summing on repeated indexes and for $\{\overrightarrow{e}_j\}_{j=1}^3$ the standard basis in $\mathbb{R}^3$, we have

$$\triangle u = \partial_i \partial_j u_j \overrightarrow{e}_i - (\partial_i \partial_j u_j \overrightarrow{e}_i - \partial_j \partial_j u_i \overrightarrow{e}_i) \tag{2.6}$$

Recalling the tensor $\varepsilon_{ijk}$, defined by $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = 1$ if $ijk$ is an even permutation of 123, $\varepsilon_{ijk} = -1$ if $ijk$ is an odd permutation of 123, $\varepsilon_{ijk} = 0$ if two indexes are equal, we have

$$\begin{aligned}\nabla \times (\nabla \times u) &= \varepsilon_{ijk} \partial_j (\nabla \times u)_k \overrightarrow{e}_i = \varepsilon_{ijk} \varepsilon_{ki'j'} \partial_j \partial_{i'} u_{j'} \overrightarrow{e}_i \\ &= \left(\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}\right) \partial_j \partial_{i'} u_{j'} \overrightarrow{e}_i = \partial_j \partial_i u_j \overrightarrow{e}_i - \partial_j \partial_j u_i \overrightarrow{e}_i,\end{aligned}$$

where we used the identity $\varepsilon_{ijk} \varepsilon_{ki'j'} = \varepsilon_{ijk} \varepsilon_{i'j'k} = \delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}$ with Kronecker's deltas. The last two displayed formulas prove that (2.5) is true.

□

A similar representation is true for $d = 2$.

**Lemma 2.9.** *For any* $u \in \mathcal{D}'(\mathbb{R}^2, \mathbb{R}^2)$ *we have*

$$\triangle u = \nabla(\nabla \cdot u) - \nabla^{\perp}(curl\ u), \tag{2.7}$$

*where* $curl\ u := \partial_1 u_2 - \partial_2 u_1$ *and* $\nabla^{\perp} V := (\partial_2 V, -\partial_1 V)$.

*Proof.* From (2.5) we have

$$\begin{aligned}\triangle u &= \nabla(\nabla \cdot u) - \begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ 0 & 0 & \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \\ &= \nabla(\nabla \cdot u) - (i \partial_2 (\partial_1 u_2 - \partial_2 u_1) - j \partial_1 (\partial_1 u_2 - \partial_2 u_1)) \,.\end{aligned}$$

This gives (2.7).

□

**Definition 2.10.** We call Leray's projector, the operator $\mathbb{P}$ defined by

$$(\mathcal{F}(\mathbb{P}u))^j = \widehat{u}^j - \frac{1}{|\xi|^2} \sum_{k=1}^d \xi_j \xi_k \widehat{u}^k. \tag{2.8}$$

We denote by $H(\mathbb{R}^d)$ the subspace of $L^2(\mathbb{R}^d, \mathbb{R}^d)$ formed by divergence free vector fields. We will also consider $V(\mathbb{R}^d) := H(\mathbb{R}^d) \cap H^1(\mathbb{R}^d, \mathbb{R}^d)$ and $C^\infty_{c\sigma}(\mathbb{R}^d, \mathbb{R}^d) := C^\infty_c(\mathbb{R}^d, \mathbb{R}^d) \cap H(\mathbb{R}^d)$.

A direct and elementary computation yields the following.

**Lemma 2.11.** *We have*

$$\mathbb{P}u = -\triangle^{-1}\nabla \times (\nabla \times u) \text{ for } d = 3 \text{ and} \tag{2.9}$$

$$\mathbb{P}u = -\triangle^{-1}\nabla^{\perp}(\text{curl } u) \text{ for } d = 2. \tag{2.10}$$

*Proof (sketch).* For example in the case $d = 3$, formally we have

$$\mathbb{P} = 1 - \frac{1}{\triangle}\nabla\text{div}$$

and from (2.5) we have

$$1 - \frac{1}{\triangle}\nabla\text{div} = -\frac{1}{\triangle}\nabla \times (\nabla \times \sqcup),$$

which yields (2.9). This was formal, but becomes rigorous taking Fourier Transform. $\square$

**Lemma 2.12.** $C^{\infty}_{c\sigma}(\mathbb{R}^d, \mathbb{R}^d)$ *is dense in* $H(\mathbb{R}^d)$ *for any* $d$.

*Proof.* Let us consider dimension $d = 3$. If $u \in H$ then $u = \nabla \times A$, for $A = -\triangle^{-1}\nabla \times u \in \dot{H}^1(\mathbb{R}^d, \mathbb{R}^d)$. Notice that from Lemma 2.2 we have that $C^{\infty}_{c}(\mathbb{R}^d, \mathbb{R}^d)$ is dense in $\dot{H}^1(\mathbb{R}^d, \mathbb{R}^d)$. Since $\dot{H}^1 \ni A \to \nabla \times A \in L^2$ is a bounded operator, the statement follows. For $d = 2$ the argument is similar and can be generalized to all $d$. $\square$

**Lemma 2.13.** $C^{\infty}_{c\sigma}(\mathbb{R}^d, \mathbb{R}^d)$ *is dense in* $V(\mathbb{R}^d)$ *for any* $d$.

*Proof.* The argument is similar to the previous one. Let us consider dimension $d = 3$. For $u \in V(\mathbb{R}^d)$ we have $u = \nabla \times A$, for $A = -\triangle^{-1}\nabla \times u \in \dot{H}^1 \cap \dot{H}^2$. But from Remark 2.3 we have that $C^{\infty}_{c}(\mathbb{R}^d, \mathbb{R}^d)$ is dense in $\dot{H}^1(\mathbb{R}^d, \mathbb{R}^d) \cap \dot{H}^2(\mathbb{R}^d, \mathbb{R}^d)$. Since $\dot{H}^1 \cap \dot{H}^2 \ni A \to \nabla \times A \in H^1$ is a bounded operator, the statement follows. We will use this lemma only for $d = 3$. $\square$

For $u \in \dot{H}^k(\mathbb{R}^d)$ and $\lambda > 0$ let us set $\mathbf{P}_\lambda u := \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}u)$. Notice that this map sends $L^2(\mathbb{R}^d)$ into itself since

$$\|\mathbf{P}_\lambda u\|_{\dot{H}^k(\mathbb{R}^d)} = \||\xi|^k\chi_{|\xi|\le\lambda}\mathcal{F}u\|_{L^2(\mathbb{R}^d)} \le \||\xi|^k\mathcal{F}u\|_{L^2(\mathbb{R}^d)} = \|u\|_{\dot{H}^k(\mathbb{R}^d)}.$$

Notice that $\mathbf{P}_\lambda$ is a projection, that is $\mathbf{P}^2_\lambda = \mathbf{P}_\lambda$, by

$$\mathbf{P}^2_\lambda u = \mathbf{P}_\lambda \circ \mathbf{P}_\lambda u = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}\mathbf{P}_\lambda u) = \mathcal{F}^*(\chi^2_{|\xi|\le\lambda}\mathcal{F}u) = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}u) = \mathbf{P}_\lambda u.$$

If $\text{div}u = 0$ then also $\text{div}\mathbf{P}_\lambda u = 0$. Indeed

$$(\text{div}u = 0 \Leftrightarrow \sum_{j=1}^{d} \xi^j\ \widehat{u}^j = 0) \Rightarrow \mathcal{F}(\text{div}\mathbf{P}_\lambda u) = \sum_{j=1}^{d} \xi^j\chi_{|\xi|\le\lambda}\widehat{u}^j = \chi_{|\xi|\le\lambda}\sum_{j=1}^{d} \xi^j\ \widehat{u}^j = 0,$$

which in turn implies $\text{div}\mathbf{P}_\lambda u = 0$.

## 2.1 $L^p$ based Sobolev Spaces

The following spaces, for $p \in (1, \infty)$ are formed by tempered distributions $u$ s.t. for $s \in \mathbb{R}$:

$$\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d) \text{ requiring } \widehat{u} \text{ in } L^1_{loc}(\mathbb{R}^d) \text{ and with } \|u\|_{\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d)} := \|(|\xi|^s \widehat{u})^\vee\|_{L^p(\mathbb{R}^d)} \ ; \tag{2.11}$$

$$\mathcal{W}^{s,p}(\mathbb{R}^d) \text{ defined with } \|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)} := \|(\langle \xi \rangle^s \widehat{u})^\vee\|_{L^p(\mathbb{R}^d)} \ . \tag{2.12}$$

We will not use the above spaces except for $p = 2$. The following is true.

**Theorem 2.14.** *We have*

$$W^{k,p}(\mathbb{R}^d) = \mathcal{W}^{k,p}(\mathbb{R}^d) \text{ for all } p \in (1, \infty) \text{ and all } k \in \mathbb{N}. \tag{2.13}$$

*Proof.* For this we need the theory of Calderon and Zygmund operators, see later in Sect. 3. □

For $p = 1$ and $p = \infty$ (2.13) is not true, see [18].

## 2.2 A generalization of Young's convolution inequality

We will use Young's convolution inequality

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \text{ for } \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q} \tag{2.14}$$

but also a generalization.

**Definition 2.15** (Distribution functions)**.** The distribution function of a measurable function $g : \mathbb{R}^d \to \mathbb{R}$ is

$$d_g(\alpha) := \text{vol}(\{x \in \mathbb{R}^d : |g(x)| > \alpha\}).$$

Notice that $d_g : [0, \infty) \to [0, \infty]$ is decreasing. This implies that $d_g$ is measurable. Notice that for a function $g \in L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |g(x)|^p dx = \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\alpha^{p-1} d\alpha = \int_0^\infty d\alpha p\alpha^{p-1} \int_{\{x \in \mathbb{R}^d : |g(x)| > \alpha\}} dx \\ &= \int_0^\infty p\alpha^{p-1} d_g(\alpha) d\alpha \end{aligned} \tag{2.15}$$

where the 1st equality is elementary, the last follows immediately by the definition of $d_g(\alpha)$, and the 2nd follows from Tonelli's Theorem applied to the positive measurable function $F(x, \alpha) := |\alpha|^{p-1} \chi_{\mathbb{R}_+}(|g(x)| - \alpha) \chi_{\mathbb{R}_+}(\alpha)$.

**Definition 2.16.** The Lorentz space $L^{p,\infty}(\mathbb{R}^d)$ are defined by

$$L^{p,\infty}(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) : \|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{\alpha > 0} \alpha d_f^{\frac{1}{p}}(\alpha). \tag{2.16}$$

*Example* 2.17. Let $f \in L^p(\mathbb{R}^d)$. Then $f \in L^{p,\infty}(\mathbb{R}^d)$ with

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.17}$$

Indeed by Chebyshev's inequality we have the following, which proves (2.17),

$$d_f(\alpha) = \mathrm{vol}(\{x : |f(x)| > \alpha\}) \leq \frac{|f|^p_{L^p(\mathbb{R}^d)}}{\alpha^p} \text{ for any } \alpha > 0 \tag{2.18}$$

Inequality (2.18) follows immediately from.

$$\begin{aligned}|f|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |f(y)|^p dy \geq \int_{\{x:|f(x)|>\alpha\}} |f(y)|^p dy \geq \int_{\{x:|f(x)|>\alpha\}} \alpha^p dy \\ &= \alpha^p \mathrm{vol}(\{x : |f(x)| > \alpha\})\alpha^p d_f(\alpha)\end{aligned}$$

So from (2.18) we have $\alpha d_f^{\frac{1}{p}}(\alpha) \leq |f|_{L^p(\mathbb{R}^d)}$.

*Example* 2.18. We have the following.

1. For $a \in (0, 1)$ we have $t^{-a}\chi_{\mathbb{R}_+} \in L^{p,\infty}(\mathbb{R})$ if and only if $ap = 1$.
2. For $T \in \mathbb{R}_+$ and $a \in (0, 1)$ we have $t^{-a}\chi_{[0,T]} \in L^{p,\infty}(\mathbb{R})$ if and only if $ap \leq 1$.

Notice that for $t > 0$ and $\alpha >$ we have $t^{-a} > \alpha \iff t < \alpha^{-1/a}$, so

$$\alpha \left|\{t \in \mathbb{R}_+ : t^{-a} > \alpha\}\right|^{\frac{1}{p}} = \alpha \left(\alpha^{-1/a}\right)^{\frac{1}{p}} = \alpha^{1-\frac{1}{ap}}$$

which is bounded in $\alpha \in \mathbb{R}_+$ exactly when the last exponent is 0, that is if and only if $\alpha p = 1$. We have

$$\begin{aligned}&\alpha \left|\{t \in \mathbb{R}_+ : t^{-a}\chi_{[0,T]} > \alpha\}\right|^{\frac{1}{p}} = \alpha \left|\{t \in (0, T] : t^{-a}\chi_{[0,T]} > \alpha\}\right|^{\frac{1}{p}} \\ &= \alpha \left(\min\left(T, \alpha^{-1/a}\right)\right)^{\frac{1}{p}} = \begin{cases}\alpha T^{\frac{1}{p}} \leq T^{\frac{1}{p}-a} \text{ if } \alpha \leq T^{-a} \\ \alpha^{1-\frac{1}{ap}} \text{ if } \alpha \geq T^{-a}\end{cases}\end{aligned}$$

where $\alpha^{1-\frac{1}{ap}}$ is unbounded if $ap > 1$ and is $\leq T^{-a\left(1-\frac{1}{ap}\right)}$ if $ap \leq 1$.

□

*Example* 2.19. For $a \in (0, d)$ we have $|x|^{-a} \in L^{p,\infty}(\mathbb{R}^d)$ if and only if $ap = d$.
Indeed

$$\alpha \left|\{x \in \mathbb{R}^d : |x|^{-a} > \alpha\}\right|^{\frac{1}{p}} = \alpha \left(c_d \alpha^{-d/a}\right)^{\frac{1}{p}} = c_d^{\frac{1}{p}} \alpha^{1-\frac{d}{ap}}$$

which is bounded in $\alpha \in \mathbb{R}_+$ exactly when the last exponent is 0, that is if and only if $ap = d$.

□

**Theorem 2.20** (Refined Young's convolution inequality)**.** *For* $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ *with* $1<p,q,r<\infty$ *and some constant* $C$,

$$\|g*f\|_{L^r}\le C\|g\|_{L^{q,\infty}}\|f\|_{L^p}. \tag{2.19}$$

Before discussing the proof of Theorem 2.20, we derive as a consequence the following classical fractional integration theorem, which as we will see, is related to Sobolev's Embedding Theorem.

**Theorem 2.21** (Hardy-Littlewood-Sobolev inequality)**.** *For any*

$$\gamma\in(0,d) \text{ and } 1<p<q<\infty \text{ with } \frac{1}{p}=\frac{1}{q}+\frac{d-\gamma}{d} \tag{2.20}$$

*there exists a constant* $C$ *s.t.*

$$\|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy\|_{L^q(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.21}$$

*Proof.* By Example 2.19 we have $|x|^{-\gamma}\in L^{\frac{d}{\gamma},\infty}\left(\mathbb{R}^d\right)$. Moreover $1+\frac{1}{q}=\frac{\gamma}{d}+\frac{1}{p}$ is exactly the condition in (2.20). So (2.21) follows from (2.19) and

$$\|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy\|_{L^q(\mathbb{R}^d)}=\|f*|x|^{-\gamma}\|_{L^q(\mathbb{R}^d)}\le C\||x|^{-\gamma}\|_{L^{\frac{d}{\gamma},\infty}\left(\mathbb{R}^d\right)}\|f\|_{L^p(\mathbb{R}^d)}.$$

□

Before giving the proof of Theorem 2.20 it is interesting to recall the proof of the standard Young's convolution inequality (2.14). It is enough to prove that for there exists a constant $C_{q,p}$ such that, if $h\in L^{r'}(\mathbb{R}^d)$, $f\ge0$, $g\ge0$, $h\ge0$, $\|g\|_{L^q}=\|f\|_{L^p}=\|h\|_{L^{r'}}=1$ and

$$I(f,g,h)=\int f(y)g(x-y)h(x)dxdy, \text{ we have } I(f,g,h)\le C_{q,p}.$$

The condition $\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$ is the same as $2=\frac{1}{r'}+\frac{1}{p}+\frac{1}{q}$. So we have

$$\left(2-\frac{1}{p}-\frac{1}{q}\right)r'=1 \quad , \quad \left(2-\frac{1}{p}-\frac{1}{r'}\right)q=1,$$
$$\left(2-\frac{1}{r'}-\frac{1}{q}\right)p=1,$$

which obviously is the same of as

$$\left(1-\frac{1}{p}\right)r'+\left(1-\frac{1}{q}\right)r'=1$$
$$\left(1-\frac{1}{p}\right)q+\left(1-\frac{1}{r'}\right)q=1$$
$$\left(1-\frac{1}{r'}\right)p+\left(1-\frac{1}{q}\right)p=1.$$

Hence

$$I(f,g,h) = \int (f^p(y)g^q(x-y))^{1-\frac{1}{r'}} \left(f^p(y)h^{r'}(x)\right)^{1-\frac{1}{q}} \left(g^q(x-y)h^{r'}(x)\right)^{1-\frac{1}{p}} dxdy.$$

Using $\frac{1}{r}+\frac{1}{p'}+\frac{1}{q'}=1$, by Hölder inequality we obtain

$$I(f,g,h) \le \left(\int f^p(y)g^q(x-y)dxdy\right)^{\frac{1}{r}} \left(\int f^p(y)h^{r'}(x)dxdy\right)^{\frac{1}{q'}} \left(\int g^q(x-y)h^{r'}(x)dxdy\right)^{\frac{1}{p'}}.$$

From this we obtain the standard Young's convolution inequality (2.14).

The proof of Theorem 2.20 is more subtle. A rather direct proof of Theorem 2.20 is in [1]. It seems to be directly inspired by the classical proof of Strichartz estimates by Keel and Tao [7]. It is based on the following *atomic decomposition* of elements in $L^p(\mathbb{R}^d)$.

**Lemma 2.22** (Atomic decomposition). *Let $p \in (0,\infty)$. Then any $f \in L^p_x$ can be written as*

$$f = \sum_{k\in\mathbb{Z}} c_k \chi_k$$

*where $meas(supp\chi_k) \le 2\ 2^k$, $|\chi_k| \le 2^{-\frac{k}{p}}$ and $2^{-\frac{1}{p}}\|f\|_{L^p} \ge \|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.*

*Proof.* Consider the distribution function $d_f(\alpha) = \text{meas}(\{|f(x)| > \alpha\})$. Then for each $k$ consider

$$\alpha_k := \inf_{d_f(\alpha)<2^k} \alpha\,, \quad c_k := 2^{\frac{k}{p}}\alpha_k\,, \quad \chi_k := \frac{1}{c_k}\chi_{(\alpha_{k+1},\alpha_k]}(|f|)f.$$

Notice that $\{\alpha_k\}_{k\in\mathbb{Z}}$ is decreasing in $k$ (since, the larger $k$, the larger is the set $\{\alpha : d_f(\alpha) < 2^k\}$).

We show the desired properties. We have

$$\text{supp}\chi_k \subseteq \{x : \alpha_{k+1} < |f(x)| \le \alpha_k\} \subseteq \{x : |f(x)| > \alpha_{k+1}\}.$$

Then we get the 1st inequality:

$$\text{meas}(\text{supp}\chi_k) \le \text{meas}(\{x : |f(x)| > \alpha_{k+1}\}) = \lim_{\alpha\to\alpha_{k+1}^+} d_f(\alpha) = \sup\{d_f(\alpha) : \alpha > \alpha_{k+1}\}) \le 2^{k+1}.$$

Next, by $|f(x)| \le \alpha_k$ in $\text{supp}\chi_k$, we have

$$|\chi_k(x)| \le 2^{-\frac{k}{p}}\frac{|f(x)|}{\alpha_k} \le 2^{-\frac{k}{p}}.$$

Let now $\lim_{k\to+\infty}\alpha_k = \inf_{k\in\mathbb{Z}}\alpha_k = \underline{\alpha}$ and $\lim_{k\to-\infty}\alpha_k = \sup_{k\in\mathbb{Z}}\alpha_k = \overline{\alpha}$. Then we claim that $\underline{\alpha} = 0$ and that $|f(x)| \le \overline{\alpha}$ a.e. Indeed, suppose that $|f(x)| > \overline{\alpha}$ on a set of positive measure. There there is $\alpha > \overline{\alpha}$ with $d_f(\alpha) > 2^k$ for some $k \in \mathbb{Z}$. Then $\alpha_k \ge \alpha > \overline{\alpha}$, which is a contradiction. On the other hand, suppose we have $0 < \alpha < \underline{\alpha}$. Then $d_f(\alpha) = \infty$, since otherwise $d_f(\alpha) < 2^k$ for a $k$, and then $\alpha \geq \alpha_k \geq \underline{\alpha} > \alpha$, getting a contradiction. But by Chebyshev's inequality,

$$\infty > \|f\|_{L^p}^p \geq \alpha^p d_f(\alpha),$$

hence getting a contradiction. The above claim and the obvious fact that for any $x$ we have $|f(x)| \in (\alpha_{k+1}, \alpha_k]$ for at most one $k$, prove $f = \sum_{k\in\mathbb{Z}} c_k \chi_k$ (the claim guarantees the existence of one such $k$).

We have $\|f\|_{L^p} \leq 2^{\frac{1}{p}} \|c_k\|_{\ell^p}$ by

$$\|f\|_{L^p}^p = \int |f|^p dx = \int \sum_{k\in\mathbb{Z}} |c_k|^p |\chi_k|^p dx = \sum_{k\in\mathbb{Z}} |c_k|^p \int |\chi_k|^p dx \leq \sum_{k\in\mathbb{Z}} |c_k|^p 2^{-k} \text{meas}(\text{supp}\chi_k)$$
$$\leq 2 \sum_{k\in\mathbb{Z}} |c_k|^p$$

Next we have

$$\sum_{k\in\mathbb{Z}} |c_k|^p = \sum_{k\in\mathbb{Z}} 2^k \alpha_k^p = \int_{\mathbb{R}_+} \alpha^p \left(\sum 2^k \delta(\alpha - \alpha_k)\right) d\alpha = \int_{\mathbb{R}_+} \alpha^p (-F'(\alpha)) d\alpha$$

where

$$F(\alpha) := \sum_{k\in\mathbb{Z}} 2^k H(\alpha_k - \alpha) = \sum_{\alpha_k > \alpha} 2^k \leq \sum_{2^k \leq d_g(\alpha)} 2^k \leq 2 d_f(\alpha).$$

Then, integrating by parts and using (2.15),

$$\sum_{k\in\mathbb{Z}} |c_k|^p = p \int_{\mathbb{R}_+} \alpha^{p-1} F(\alpha) d\alpha \leq 2p \int_{\mathbb{R}_+} \alpha^{p-1} d_f(\alpha) d\alpha = 2\|f\|_{L^p}^p,$$

so that $\|c_k\|_{\ell^p} \leq 2^{\frac{1}{p}} \|f\|_{L^p}$.

□

*Proof of Theorem 2.20.* It is enough to show that there exists a constant $C_{q,p}$ such that, if $h \in L^{r'}(\mathbb{R}^d)$, $f \geq 0$, $g \geq 0$, $h \geq 0$, $\|g\|_{L^{q,\infty}} = \|f\|_{L^p} = \|h\|_{L^{r'}} = 1$ and

$$I(f, g, h) = \int f(y) g(x - y) h(x) dx dy,$$

then

$$I(f, g, h) \leq C_{q,p}. \tag{2.22}$$

Now, for

$$C_j := \{x : 2^j < g(x) \leq 2^{j+1}\}$$

we have

$$I(f,g,h) \leq 2^{j+1} I_j(f,h) \text{ where } I_j(f,h) = \int f(y)\chi_{C_j}(x-y)h(x)dxdy.$$

Using the atomic decomposition and Hölder inequality and (the standard) Young's inequality for convolutions, we have

$$\begin{aligned} I_j(f,h) &= \sum_{k,\ell} c_k d_\ell I_j(f_k,h_\ell) \leq \sum_{k,\ell} c_k d_\ell \|f_k * \chi_{C_j}\|_{L^{b'}} \|h_\ell\|_{L^b} \\ &\leq \sum_{k,\ell} c_k d_\ell \|\chi_{C_j}\|_{L^{c'}} \|f_k\|_{L^a} \|h_\ell\|_{L^b} \text{ where } \frac{1}{b'} + 1 = \frac{1}{c'} + \frac{1}{a}, \end{aligned}$$

where the latter is the same as $\frac{1}{c} = \frac{1}{a} + \frac{1}{b} - 1$ and requires $b' > a$ and hence $b < a'$. Now we have

$$\begin{aligned} &\|\chi_{C_j}\|_{L^{c'}} = \|\chi_{C_j}\|_{L^1}^{1/c'} \leq \left(d_g(2^j)\right)^{1/c'} \leq 2^{-\frac{qj}{c'}} = 2^{qj\left(\frac{1}{a}+\frac{1}{b}-2\right)}, \\ &\|f_k\|_{L^a} \leq \|f_k\|_{L^\infty} |\text{supp } f_k|^{\frac{1}{a}} \leq 2^{-\frac{k}{p}} 2^{\frac{k+1}{a}} \text{ and} \\ &\|h_\ell\|_{L^b} \leq \|h_\ell\|_{L^\infty} |\text{supp } h_\ell|^{\frac{1}{b}} \leq 2^{-\frac{\ell}{r'}} 2^{\frac{\ell+1}{b}}. \end{aligned}$$

Hence

$$\begin{aligned} 2^j I_j(f,h) &\leq 4 \sum_{k,\ell} c_k d_\ell 2^{jq\frac{1}{q}} 2^{qj\left(\frac{1}{a}+\frac{1}{b}-2\right)} 2^{k\left(\frac{1}{a}-\frac{1}{p}\right)} 2^{\ell\left(\frac{1}{b}-\frac{1}{r'}\right)} \text{ and, by } \frac{1}{q} = 2 - \frac{1}{r'} - \frac{1}{p}, \\ &= 4 \sum_{k,\ell} c_k d_\ell 2^{qj\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{r'}-\frac{1}{p}\right)} 2^{k\left(\frac{1}{a}-\frac{1}{p}\right)} 2^{\ell\left(\frac{1}{b}-\frac{1}{r'}\right)} \\ &= 4 \sum_{k,\ell} c_k d_\ell 2^{(qj+k)\left(\frac{1}{a}-\frac{1}{p}\right)} 2^{(qj+\ell)\left(\frac{1}{b}-\frac{1}{r'}\right)}. \end{aligned}$$

Now let $\varepsilon > 0$ small enough and set

$$\begin{aligned} \frac{1}{a} &:= \frac{1}{p} - \varepsilon \text{sign}\,(qj+k) \\ \frac{1}{b} &:= \frac{1}{r'} - \varepsilon \text{sign}\,(qj+\ell). \end{aligned}$$

Notice that $1 < p,q,r < \infty$ imply $1 < a,b < \infty$. Furthermore $1 < p,q,r < \infty$ and $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ imply $p < r \Leftrightarrow p' > r'$ and for $\varepsilon > 0$ small enough this yields $b < a'$.
Then we have

$$\sum_j 2^j I_j(f,h) \leq 4 \sum_{k,\widetilde{k}} c_k d_{\widetilde{k}} \sum_j 2^{-\varepsilon|qj+k|} 2^{-\varepsilon|qj+\widetilde{k}|}.$$

We claim that for a fixed $C = C(q, \varepsilon)$

$$\sum_j 2^{-\varepsilon|k+jq|-\varepsilon|\widetilde{k}+jq|} = \sum_j 2^{-\varepsilon|k-jq|-\varepsilon|\widetilde{k}-jq|} \le C 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|), \tag{2.23}$$

where the equality is obvious. To prove the inequality, it is not restrictive to assume $k \le \widetilde{k}$. Then the summation on the left can be rewritten as

$$\sum_{jq\le k} 2^{2\varepsilon jq-\varepsilon(k+\widetilde{k})} + \sum_{k<jq\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} + \sum_{\widetilde{k}<jq} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon jq}.$$

Then (here $[t] \in \mathbb{Z}$ is the integer part of $t \in \mathbb{R}$, defined by $[t] \le t < [t]+1$)

$$\sum_{jq\le k} 2^{2\varepsilon jq-\varepsilon(k+\widetilde{k})} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j\le\left[\frac{k}{\sigma}\right]} 2^{2\varepsilon jq} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{2\varepsilon\sigma\left(\left[\frac{k}{\sigma}\right]-j\right)} = C_{\varepsilon q}\ 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon q\left[\frac{k}{q}\right]}$$

$$\le C_{\varepsilon q} 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon q\frac{k}{\sigma}} = C_{\varepsilon q} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon q} 2^{-\varepsilon|k-\widetilde{k}|} \text{ where } C_{\varepsilon q} = \frac{1}{1-2^{-2\varepsilon q}}.$$

We have

$$\sum_{\widetilde{k}<jq} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon jq} \le 2^{\varepsilon(k+\widetilde{k})} \sum_{j\ge\left[\frac{\widetilde{k}}{q}\right]+1} 2^{-2\varepsilon jq} = 2^{\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{-2\varepsilon q\left(\left[\frac{\widetilde{k}}{q}\right]+1+j\right)} = C_{\varepsilon q} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon q\left(\left[\frac{\widetilde{k}}{q}\right]+1\right)}$$

$$\le C_{\varepsilon q} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon q\frac{\widetilde{k}}{q}} = C_{\varepsilon q} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon q} 2^{-\varepsilon|k-\widetilde{k}|}.$$

Finally

$$\sum_{k<jq\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} = 2^{-\varepsilon(\widetilde{k}-k)} \sum_{\left[\frac{k}{q}\right]+1\le j\sigma\le\left[\frac{\widetilde{k}}{q}\right]} 1 = 2^{-\varepsilon(\widetilde{k}-k)} \left(\left[\frac{\widetilde{k}}{q}\right] - \left[\frac{k}{q}\right] - 1\right) \le q^{-1} 2^{-\varepsilon(\widetilde{k}-k)}(\widetilde{k}-k)$$

Hence (2.23) is proved.
Then we have

$$\begin{aligned}
\sum_j 2^j I_j(f,h) &\le 4C \sum_k c_k \sum_{\widetilde{k}} d_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|) \\
&\le 4C \|c_k\|_{\ell^p(\mathbb{Z})} \| \sum_{\widetilde{k}} d_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|_{\ell^{p'}(\mathbb{Z})} \\
&\le 4C \|2^{-\varepsilon|j|}(1+|j|)\|_{\ell^1(\mathbb{Z})} \|c_k\|_{\ell^p(\mathbb{Z})} \|d_{\widetilde{k}}\|_{\ell^{p'}(\mathbb{Z})} \le C' \|c_k\|_{\ell^p(\mathbb{Z})} \|d_{\widetilde{k}}\|_{\ell^{p'}(\mathbb{Z})} \\
&\le C'' \|c_k\|_{\ell^p(\mathbb{Z})} \|d_{\widetilde{k}}\|_{\ell^{r'}(\mathbb{Z})} \le 4C'' \|f\|_{L^p(\mathbb{R}^d)} \|h\|_{L^{r'}(\mathbb{R}^d)} = 4C''.
\end{aligned}$$

So we have found (2.22) with $C_{q,p} = 4C(q,\varepsilon)\|2^{-\varepsilon|j|}(1+|j|)\|_{\ell^1(\mathbb{Z})}$ with $C(q,\varepsilon)$ the constant in (2.23). $\square$

## 2.3 Back to Sobolev Embedding

We go back to the Hardy, Littlewood and Sobolev inequality. The following, which will be used soon but also later on, is proved like in [18] p.73..

**Lemma 2.23.** *For any $\gamma \in (0, d)$ there exists $c_\gamma > 0$ s.t.*

$$\mathcal{F}(|\cdot|^{-\gamma})(\xi) = c_\gamma |\xi|^{\gamma-d}. \tag{2.24}$$

*Proof.* It is enough to show that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |x|^{-\gamma}\phi(x)dx = c_\gamma \int_{\mathbb{R}^d} |\xi|^{\gamma-d}\widehat{\phi}(\xi)d\xi. \tag{2.25}$$

Starting from (1.2) and Plancherel we have

$$\int_{\mathbb{R}^n} \varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}\phi(x)dx = \int_{\mathbb{R}^d} e^{-\varepsilon\frac{|\xi|^2}{2}}\widehat{\phi}(\xi)d\xi.$$

Now we apply to both sides $\int_0^\infty \frac{d\varepsilon}{\varepsilon}\varepsilon^{\frac{d-\gamma}{2}}$ and commuting order of integration we obtain

$$\int_{\mathbb{R}^d} dx\phi(x) \underbrace{\int_0^\infty \varepsilon^{-\frac{\gamma}{2}} e^{-\frac{|x|^2}{2\varepsilon}}\frac{d\varepsilon}{\varepsilon}}_{a_\gamma|x|^{-\gamma}} = \int_{\mathbb{R}^d} d\xi\widehat{\phi}(\xi) \underbrace{\int_0^\infty \varepsilon^{\frac{d-\gamma}{2}} e^{-\varepsilon\frac{|\xi|^2}{2}}\frac{d\varepsilon}{\varepsilon}}_{b_\gamma|\xi|^{\gamma-d}}$$

for appropriate constants $a_\gamma$ and $b_\gamma$. In fact $a_\gamma = 2^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}+1\right)$, $b_\gamma = 2^{\frac{d-\gamma}{2}}\Gamma\left(\frac{d-\gamma}{2}\right)$ and $c_\gamma = \frac{2^{\frac{d-\gamma}{2}}\Gamma\left(\frac{d-\gamma}{2}\right)}{2^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}+1\right)}$.

□

We have the following version of Sobolev's Embedding Theorem.

**Theorem 2.24** (Sobolev Embedding Theorem with fractional derivatives)**.** *Let $p \in (1, \infty)$, $0 < s < \frac{d}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. Then there exists a $C$ s.t. we have*

$$\|f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d)} \text{ for any } f \in \mathcal{S}(\mathbb{R}^d). \tag{2.26}$$

*Proof.* For $f \in \mathcal{S}(\mathbb{R}^d)$ we have for some fixed $c$

$$f(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}|\xi|^{-s}\left(|\xi|^s\widehat{f}(\xi)\right)d\xi = c\int_{\mathbb{R}^d}|x-y|^{s-d}g(y)dy \text{ where } \widehat{g}(\xi) = |\xi|^s\widehat{f}(\xi)$$

where we used $\widehat{\varphi * T} = (2\pi)^{\frac{d}{2}}\widehat{\varphi}\,\widehat{T}$ which holds for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$.
Since $g \in L^p(\mathbb{R}^d)$, by the Hardy-Littlewood-Sobolev Theorem we have that $f \in L^q(\mathbb{R}^d)$ for

$$\frac{1}{q} = \frac{1}{p} - \frac{d-(d-s)}{d} = \frac{1}{p} - \frac{s}{d}$$

□

Notice that for $0 < s < \frac{d}{2}$ we know that $\dot{H}^s(\mathbb{R}^d)$ contains $\mathcal{S}(\mathbb{R}^d)$ as a dense subspace, so (2.26) with $p = 2$ extends to all $f \in \dot{H}^s(\mathbb{R}^d)$.

### 2.4 Assorted inequalities

**Lemma 2.25** (Interpolation of Sobolev norms)**.** *For any $s \in [0,1]$ and any $k = sk_1 + (1-s)k_2$ we have*

$$\|f\|_{\dot{H}^k(\mathbb{R}^d)} \le \|f\|^s_{\dot{H}^{k_1}(\mathbb{R}^d)} \|f\|^{1-s}_{\dot{H}^{k_2}(\mathbb{R}^d)} \text{ for any } f \in \dot{H}^{k_1}(\mathbb{R}^d) \cap \dot{H}^{k_2}(\mathbb{R}^d). \tag{2.27}$$

*In particular, for $s \in [0,1]$ and any $f \in H^1(\mathbb{R}^d)$*

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \le \|f\|^{1-s}_{L^2(\mathbb{R}^d)} \|f\|^s_{\dot{H}^1(\mathbb{R}^d)} \tag{2.28}$$

*Proof.* (2.28) follows from (2.27) for $k_1 = 1$ and $k_2 = 0$. So let us turn to (2.27). Obviously there is nothing to prove for $s = 0, 1$, so we can assume $s \in (0,1)$. Notice that for $p = \frac{1}{s}$ we have $p' := \frac{p}{p-1} = \frac{1}{1-s}$. Now, we have

$$\begin{aligned}
\|f\|^2_{\dot{H}^k(\mathbb{R}^d)} &= \int \left(|\xi|^{2sk_1} |\widehat{f}(\xi)|^{2s}\right) \left(|\xi|^{2(1-s)k_2} |\widehat{f}(\xi)|^{2(1-s)}\right) d\xi \\
&\le \||\xi|^{2sk_1} |\widehat{f}(\xi)|^{2s}\|_{L^{\frac{1}{s}}(\mathbb{R}^d)} \||\xi|^{2(1-s)k_2} |\widehat{f}(\xi)|^{2(1-s)}\|_{L^{\frac{1}{1-s}}(\mathbb{R}^d)} \\
&= \||\xi|^{k_1} |\widehat{f}(\xi)|\|^{2s}_{L^2(\mathbb{R}^d)} \||\xi|^{k_1} \widehat{f}(\xi)\|^{2(1-s)}_{L^2(\mathbb{R}^d)} = \|f\|^{2s}_{\dot{H}^{k_1}(\mathbb{R}^d)} \|f\|^{2(1-s)}_{\dot{H}^{k_2}(\mathbb{R}^d)}.
\end{aligned}$$

□

**Lemma 2.26** (Agmon's inequality)**.** *Given a pair $0 < r < d/2 < s$ we have*

$$\|u\|_{L^\infty(\mathbb{R}^d)} \le \|u\|^{\frac{s-\frac{d}{2}}{s-r}}_{\dot{H}^r(\mathbb{R}^d)} \|u\|^{\frac{\frac{d}{2}-r}{s-r}}_{\dot{H}^s(\mathbb{R}^d)}. \tag{2.29}$$

*Example* 2.27. For instance,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \le \|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla^2 u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}, \tag{2.30}$$

where notice that here we are assuming $\widehat{u} \in L^1_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, which excludes additive constants. It is well known that $H^{\frac{d}{2}}(\mathbb{R}^d) \not\subset L^\infty(\mathbb{R}^d)$. Indeed, for $\widehat{u}(\xi) := \dfrac{\langle \xi \rangle^{-d}}{1 + \log \langle \xi \rangle}$ we have $u \in H^{\frac{d}{2}}(\mathbb{R}^d)$. On the other hand we have $\widehat{u} \notin L^1(\mathbb{R}^d)$ . We show that $u \notin L^\infty(\mathbb{R}^d)$. Suppose by contradiction that $u \in L^\infty(\mathbb{R}^d)$. Then for $\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$ with $\chi(0) = 1$, radial and decreasing as $|\xi|$ grows,

$$\int_{\mathbb{R}^d} \chi(\xi/k) \widehat{u}(\xi) d\xi = \int_{\mathbb{R}^d} k^d \widehat{\chi}(kx) u(x) dx \le \|\widehat{\chi}\|_{L^1(\mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^d)}$$

where the first equality follows from Plancherel and the 2nd from Hölder.
But then, since $\chi(\xi/k)\widehat{u}(\xi)$ is an increasing sequence of functions, we have $\chi(\cdot/k)\widehat{u} \xrightarrow{k\to\infty} \widehat{u}$ in $L^1(\mathbb{R}^d)$ with $\|\widehat{u}\|_{L^1(\mathbb{R}^d)} \le \|\widehat{\chi}\|_{L^1(\mathbb{R}^d)} \|u\|_{L^\infty(\mathbb{R}^d)}$. This is a contradiction.

*Proof of Lemma 2.26.* For $R > 0$ we have

$$\begin{aligned}
|u(x)| &\leq (2\pi)^{-\frac{d}{2}} \int_{|\xi|<R} |\widehat{u}(\xi)|\ |\xi|^{r}\ |\xi|^{-r} d\xi + (2\pi)^{-\frac{d}{2}} \int_{|\xi|>R} |\widehat{u}(\xi)|\ |\xi|^{s}\ |\xi|^{-s} d\xi \\
&\leq (2\pi)^{-\frac{d}{2}} \|u\|_{\dot{H}^{r}(\mathbb{R}^d)} \left( \int_{|\xi|<R} |\xi|^{-2r} d\xi \right)^{\frac{1}{2}} + (2\pi)^{-\frac{d}{2}} \|u\|_{\dot{H}^{s}(\mathbb{R}^d)} \left( \int_{|\xi|>R} |\xi|^{-2s} d\xi \right)^{\frac{1}{2}} \\
&\lesssim \|u\|_{\dot{H}^{r}(\mathbb{R}^d)} R^{\frac{d}{2}-r} + \|u\|_{\dot{H}^{s}(\mathbb{R}^d)} R^{\frac{d}{2}-s}.
\end{aligned}$$

We choose $R$ so that the two terms are equal, which yields

$$R = \frac{\|u\|_{\dot{H}^{s}(\mathbb{R}^d)}^{\frac{1}{s-r}}}{\|u\|_{\dot{H}^{r}(\mathbb{R}^d)}^{\frac{1}{s-r}}},$$

so that $|u(x)| \leq C_d \|u\|_{\dot{H}^{r}(\mathbb{R}^d)}^{1-\left(\frac{d}{2}-r\right)\frac{1}{s-r}} \|u\|_{\dot{H}^{s}(\mathbb{R}^d)}^{\left(\frac{d}{2}-r\right)\frac{1}{s-r}}$.

□

Later in Sect. 12 we will use the following modification of Lemma 2.26.

**Lemma 2.28.** *Let $U \subset \mathbb{R}^3$ be a bounded open subspace with $\partial U$ a smooth submanifold of $\mathbb{R}^3$ and suppose $f \in H^k(U)$ with $k \geq 2$. Then for any $r \in [1, 2]$ we have*

$$\|f\|_{L^\infty(U)} \leq C_{k,r} \|f\|_{L^r(U)}^{\theta} \|f\|_{H^k(U)}^{1-\theta} \ \text{with}\ \theta = \frac{r\left(k - \frac{3}{2}\right)}{kr + \frac{3}{2}(2-r)}. \tag{2.31}$$

*Proof.* We know that there is an appropriate extension operator $H^k(U) \ni f \to Ef \in H^k(\mathbb{R}^d)$ with $Ef|_U = f$. Then we use

$$\|Ef\|_{H^1(\mathbb{R}^3)} \leq \|Ef\|_{L^2(\mathbb{R}^3)}^{1-\frac{1}{k}} \|Ef\|_{H^k(\mathbb{R}^3)}^{\frac{1}{k}} \text{ and } \|Ef\|_{H^2(\mathbb{R}^3)} \leq \|Ef\|_{L^2(\mathbb{R}^3)}^{1-\frac{2}{k}} \|Ef\|_{H^k(\mathbb{R}^3)}^{\frac{2}{k}}$$

and Agmon's

$$\|Ef\|_{L^\infty(\mathbb{R}^3)} \leq \|Ef\|_{H^1(\mathbb{R}^3)}^{\frac{1}{2}} \|Ef\|_{H^2(\mathbb{R}^3)}^{\frac{1}{2}} \leq \|Ef\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2k}} \|Ef\|_{H^k(\mathbb{R}^3)}^{\frac{3}{2k}}$$

which yields

$$\|f\|_{L^\infty(U)} \leq c \|f\|_{L^2(U)}^{1-\frac{3}{2k}} \|f\|_{H^k(U)}^{\frac{3}{2k}}.$$

Substitute by Hölder $\|f\|_{L^2(U)} \leq \|f\|_{L^r(U)}^{\frac{r}{2}} \|f\|_{L^\infty(U)}^{1-\frac{r}{2}}$ and then we get

$$\|f\|_{L^\infty(U)} \leq c \|f\|_{L^r(U)}^{\frac{r}{2}\left(1-\frac{3}{2k}\right)} \|f\|_{L^\infty(U)}^{\left(1-\frac{r}{2}\right)\left(1-\frac{3}{2k}\right)} \|f\|_{H^k(U)}^{\frac{3}{2k}}.$$

Solving with respect to $\|f\|_{L^\infty(U)}$ we obtain

$$\|f\|_{L^\infty(U)}^{1-\left(1-\frac{r}{2}\right)\left(1-\frac{3}{2k}\right)} = \|f\|_{L^\infty(U)}^{\frac{r}{2}+\frac{3}{2k}-\frac{3r}{4k}} \le c\|f\|_{L^r(U)}^{\frac{r}{2}\left(1-\frac{3}{2k}\right)}\|f\|_{H^k(U)}^{\frac{3}{2k}}.$$

So we get the following, which is the desired result:

$$\|f\|_{L^\infty(U)}^{\frac{r}{2}+\frac{3}{2k}-\frac{3r}{4k}} \le c_1\|f\|_{L^r(U)}^{\frac{\frac{r}{2}\left(1-\frac{3}{2k}\right)}{\frac{r}{2}+\frac{3}{2k}-\frac{3r}{4k}}}\|f\|_{H^k(U)}^{\frac{\frac{3}{2k}}{\frac{r}{2}+\frac{3}{2k}-\frac{3r}{4k}}} = c_1\|f\|_{L^r(U)}^{\frac{r\left(k-\frac{3}{2}\right)}{kr+3-\frac{3r}{2}}}\|f\|_{H^k(U)}^{\frac{3}{kr+3-\frac{3r}{2}}}.$$

□

**Theorem 2.29** (Gagliardo–Nirenberg)**.** *If $p \in [2, \infty)$ is s.t. $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$ then there exists $C$ s.t.*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}^{1-s}\|f\|_{\dot{H}^1(\mathbb{R}^d)}^{s} \text{ where } s = d\left(\frac{1}{2} - \frac{1}{p}\right). \tag{2.32}$$

*Proof.* By Sobolev, for $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$ we have

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Here $s$ is like in the statement. Also $s = d\left(\frac{1}{2} - \frac{1}{p}\right) < 1 \Leftrightarrow \frac{1}{2} - \frac{1}{p} < \frac{1}{d}$. Finally, apply (2.28). □

*Remark* 2.30. For $p = 4$ and $d = 2, 3$ we have $s = d/4$ and $\|f\|_{L^4(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}\|f\|_{\dot{H}^1(\mathbb{R}^d)}^{d/4}$.

**Lemma 2.31** (Gronwall's inequality)**.** *Let $T > 0$, $\lambda$ and $\varphi$ two functions in $L^1(0,T)$, both $\ge 0$ a.e., and $C_1$, $C_2$ two non negative constants. Let $\lambda\varphi \in L^1(0,T)$ and let*

$$\varphi(t) \le C_1 + C_2\int_0^t \lambda(s)\ \varphi(s)ds \text{ for a.e. } t \in (0,T).$$

*Then we have*

$$\varphi(t) \le C_1 e^{C_2\int_0^t \lambda(s)ds} \text{ for a.e. } t \in (0,T).$$

*Proof.* Set

$$\psi(t) := C_1 + C_2\int_0^t \lambda(s)\ \varphi(s)ds.$$

Then $\psi(t)$ is absolutely continuous and so it is differentiable almost everywhere and we have

$$\psi'(t) = C_2\lambda(t)\ \varphi(t) \le C_2\lambda(t)\ \psi(t) \text{ for a.e. } t \in (0,T).$$

Also, the function $\psi(t)e^{-C_2\int_0^t \lambda(s)ds}$ is absolutely continuous with

$$\frac{d}{dt}\left(\psi(t)e^{-C_2\int_0^t \lambda(s)ds}\right) \le 0 \text{ for a.e. } t \in (0,T).$$

Then we have

$$\psi(t) \le e^{C_2\int_0^t \lambda(s)ds}\psi(0) = C_1 e^{C_2\int_0^t \lambda(s)ds} \text{ for all } t \in (0,T).$$

Since $\varphi(t) \le \psi(t)$ a.e., the result follows. □

# 3 The Calderon–Zygmund theory

## 3.1 Hardy Littlewood maximal function

Let $f \in L^1_{loc}(\mathbb{R}^d)$ and consider (for $B(x,r)$ the ball of center $x$ and radius $r$ in $\mathbb{R}^d$) averages

$$A_r f(x) = \frac{1}{\mathrm{vol}(B(x,r))} \int_{B(x,r)} f(y)dy.$$

Notice that for any $r > 0$ the function $x \to A_r f(x)$ is continuous. Indeed, fix $\delta_0 > 0$ and consider $\delta x \in B(0,\delta_0)$. Then by the triangular inequality $B(x+\delta x, r) \subset B(x, r+\delta_0)$. So, for $\delta x \in B(0,\delta_0)$

$$A_r f(x) - A_r f(x+\delta x) = \frac{1}{\mathrm{vol}(B(0,1))r^d} \int_{B(x,r+\delta_0)} \left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) f(y)dy$$

with for any $y$

$$\left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) \chi_{B(x,r+\delta_0)}(y) f(y) \overset{|\delta x|\to 0}{\to} 0.$$

By dominated convergence $A_r f(x) - A_r f(x+\delta x) \to 0$. We define

$$Mf(x) = \sup_{r>0} A_r |f|(x). \tag{3.1}$$

From the definition we conclude that $Mf$ is lower semi continuous that is $\{x : Mf(x) > a\}$ is open for any $a$. It also obvious that $M$ is sub additive:

$$M(f+g)(x) \le Mf(x) + Mg(x).$$

We have the following obvious estimate

$$|Mf(x)| \le |f|_{L^\infty(\mathbb{R}^d)}. \tag{3.2}$$

One important fact is that it is not true that $M$ maps $L^1(\mathbb{R}^d)$ into itself. Indeed if say $K \subset \mathbb{R}^d$ is any compact set and if $B(0,c_0) \supset K$, then since for $|x| > c_0$ we have $B(x, 2|x|) \supset B(0,|x|) \supset K$, we have computing at $r = 2|x|$

$$M\chi_K(x) = \sup_{r>0} \frac{\mathrm{vol}(B(x,r)\cap K)}{\mathrm{vol}(B(0,1))r^d} \ge \frac{\mathrm{vol}(K)}{\mathrm{vol}(B(0,1))2^d|x|^d}$$

which shows that $M\chi_K \notin L^1(\mathbb{R}^d)$.
Notice that each $g \in L^1(\mathbb{R}^d)$ satisfies Chebyshev's inequality:

$$\mathrm{vol}(\{x : |g(x)| > \alpha\}) \le \frac{|g|_{L^1(\mathbb{R}^d)}}{\alpha} \text{ for any } \alpha > 0 \tag{3.3}$$

Indeed (3.3) follows immediately from.

$$|g|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} \alpha dy = \alpha \text{vol}(\{x : |g(x)| > \alpha\})$$

If $T : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ satisfies $\|Tf\|_{L^1(\mathbb{R}^d)} \leq A\|f\|_{L^1(\mathbb{R}^d)}$ for all $f \in L^1(\mathbb{R}^d)$ and for a fixed constant $A$, from (3.3) it is easy to conclude that

$$\text{vol}(\{x : |Tf(x)| > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \text{ and any } f \in L^1(\mathbb{R}^d).$$

Unfortunately we have seen that $M$ does not map $L^1(\mathbb{R}^d)$ into itself. However we will show that it satisfies the last property. Indeed we will prove now that $M$ *is weak* $(1,1)$ *bounded*, that is there exists a constant $A > 0$ (in fact we will prove $A = 3^d$) s.t.

$$\text{vol}(\{x : Mf(x) > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \ . \tag{3.4}$$

To prove this we consider the set $\{x : Mf(x) > \alpha\}$. Then, for any $x$ in this set, there is a ball with center in $x$, which we denote by $B_x$, with $\int_{B_x} |f| > \alpha \text{vol}(B_x)$. Pick any compact subset $K$ of the above set, and cover it with such balls $B_x$. Extract now a finite cover, corresponding to finitely many points $x_1$, $...x_N$. We have the following covering result, which we state without proof.

**Theorem 3.1** (Vitali's lemma)**.** *Let* $B_{x_1}$,...,$B_{x_N}$ *be a finite number of balls in* $\mathbb{R}^d$*. There exists a subset of balls*

$$\{B_1, ..., B_m\} \subseteq \{B_{x_1}, ..., B_{x_N}\} \tag{3.5}$$

*with the* $B_1 ... B_m$ *pairwise disjoint, s.t.*

$$vol(B_{x_1} \cup \cdots \cup B_{x_N}) \leq 3^d \sum_{j=1}^{m} vol(B_j). \tag{3.6}$$

We consider balls $B_1 ... B_m$ as in (3.5) and from

$$K \subset B_{x_1} \cup \cdots \cup B_{x_N} \Rightarrow \text{vol}(K) < \text{vol}(B_{x_1} \cup \cdots \cup B_{x_N}),$$

from (3.6) and from the definition of the $B_{x_j}$ we get

$$3^{-d}\text{vol}(K) \leq \sum_{j=1}^{m} \text{vol}(B_j) < \sum_{j=1}^{m} \frac{1}{\alpha} \int_{B_j} |f| \leq \frac{|f|_1}{\alpha}. \tag{3.7}$$

(3.7) implies $\text{vol}(K) \leq 3^d \alpha^{-1}|f|_1$. By $\text{vol}(\{x : |Mf(x)| > \alpha\}) = \sup_{K \subset \{x:|Mf(x)|>\alpha\}} \text{vol}(K)$ for compact sets $K$, then (3.7) implies (3.4).

(3.2) and (3.4) imply by the Marcinkiewicz Interpolation Theorem 3.2, proved below,

$$\|Mf\|_{L^p(\mathbb{R}^d)} < A_p\|f\|_{L^p(\mathbb{R}^d)} \text{ for all } p \in (1, \infty] \ . \tag{3.8}$$

We will use this result in the proof of the Hardy-Littlewood-Sobolev Theorem, and of Sobolev's estimates.

**Theorem 3.2** (Marcinkiewicz Interpolation)**.** *Let* $T : L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$ *be a sublinear operator s.t. for two constants* $A_1$ *and* $A_\infty$ *and for all* $f$

$$\|Tf\|_{L^\infty(\mathbb{R}^d)} \le A_\infty \|f\|_{L^\infty(\mathbb{R}^d)} \tag{3.9}$$

$$|\{x : |Tf(x)| > \alpha\}| \le \frac{A_1}{\alpha} |f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \ . \tag{3.10}$$

*Then for any* $p \in (1, \infty)$ *there is a constant* $A_p$ *such that for any* $f \in L^p(\mathbb{R}^d)$ *we have*

$$\|Tf\|_{L^p(\mathbb{R}^d)} \le A_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.11}$$

*Proof.* Dividing $T$ by a constant, we can assume $A_\infty = 1$. Fix $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$. For $\alpha > 0$ arbitrary set

$$f_1(x) = \begin{cases} f(x) \text{ if } |f(x)| \ge \frac{\alpha}{2} \\ 0 \text{ otherwise.} \end{cases}$$

Notice that $f_1 \in L^1(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} |f_1(x)| dx = \int_{\{x:|f(x)|\ge \frac{\alpha}{2}\}} |f(x)| dx \le \frac{2^{p-1}}{\alpha^{p-1}} \int_{\mathbb{R}^d} |f(x)|^p dx.$$

Using (3.9), we get $|Tf(x)| \le |Tf_1(x)| + \frac{\alpha}{2}$, since $\|f - f_1\|_{L^\infty(\mathbb{R}^d)} \le \frac{\alpha}{2}$. Then

$$\{x : |Tf(x)| > \alpha\} \subseteq \{x : |Tf_1(x)| > \frac{\alpha}{2}\}.$$

We have, using (3.10),

$$\mathrm{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) \le A_1 \frac{2}{\alpha} \int_{\mathbb{R}^d} |f_1(x)| dx = A_1 \frac{2}{\alpha} \int_{\{x:|f(x)|\ge \frac{\alpha}{2}\}} |f(x)| dx.$$

Substituting $g = Tf$ in (2.15)

$$\begin{aligned} \int_{\mathbb{R}^d} |Tf(x)|^p dx &= \int_0^\infty p\alpha^{p-1} \mathrm{vol}(\{x : |Tf(x)| > \alpha\}) d\alpha \\ &\le \int_0^\infty p\alpha^{p-1} \mathrm{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) d\alpha \le 2A_1 \int_0^\infty p\alpha^{p-2} \int_{\{x:|f(x)|\ge \frac{\alpha}{2}\}} |f(x)| dx \\ &= 2pA_1 \int_{\mathbb{R}^d} dx |f(x)| \underbrace{\int_0^{2|f(x)|} \alpha^{p-2} d\alpha}_{\frac{2^{p-1}|f(x)|^{p-1}}{p-1}} = \frac{2^p p}{p-1} A_1 \int_{\mathbb{R}^d} |f(x)|^p dx. \end{aligned}$$

$\square$

### 3.2 Calderon–Zygmund kernels

We consider now Calderon–Zygmund (CZ) kernels. We will use the following definition.

**Definition 3.3.** In these notes, we will say that a function $K : \mathbb{R}^d \times \mathbb{R}^d \backslash \Delta \to \mathbb{C}$ with $\Delta$ the diagonal $\{(x,x) : x \in \mathbb{R}^d\}$, is CZ if there exists a fixed constant $C$ s.t. the following conditions hold:

(C–Z1) we have

$$\begin{aligned} |K(x,y)| &\le \frac{C}{|x-y|^d} \text{ for any } x \neq y \text{ and} \\ |\nabla_{x,y} K(x,y)| &\le \frac{C}{|x-y|^{d+1}} \text{ for any } x \neq y. \end{aligned} \tag{3.12}$$

(C–Z2) there is an operator $T$, which satisfies

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy \text{ for } x \notin \operatorname{supp} f \tag{3.13}$$

and which defines a bounded operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with norm bounded by $C$.

There are many examples.

*Example* 3.4. 1. Let us consider the operator $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ which is a well defined bounded operator in $L^2(\mathbb{R}^d)$ since

$$\widehat{R_j f}(\xi) = -\mathrm{i} \frac{\xi_j}{|\xi|} \widehat{f}(\xi).$$

Notice that for $K = \mathcal{F}^* \left( -\mathrm{i} \frac{\xi_j}{|\xi|} \right)$, we have $R_j f(x) = (2\pi)^{-\frac{d}{2}} K * f(x)$ where for $\varphi \in C_c^\infty(\mathbb{R}^d, [0,1])$ any function with $\varphi = 1$ in $B(0,a)$ and $\varphi = 0$ outside $B(0,b)$, for some $0 < a < b$, we have

$$K(x) = -\mathrm{i} \lim_{R \to +\infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi/R) d\xi.$$

It is easy to see that for any $x \neq 0$ the above limit converges and that $K(x-y)$ satisfies the inequalities (3.12) for a fixed $C$. For example, the 1st inequality follows splitting

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi|x|) \varphi(\xi/R) d\xi + \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi/R)(1 - \varphi(\xi|x|)) d\xi$$

where we bound the absolute value of the 1st integral by

$$\int_{|\xi| \le \frac{b}{|x|}} d\xi = \frac{b^d \mathrm{vol}(S^{d-1})}{d} \frac{1}{|x|^d}$$

and the absolute value of the 2nd integral by means of an integration by parts using $Le^{\mathrm{i}\xi\cdot x} = e^{\mathrm{i}\xi\cdot x}$ with $L := \frac{x\cdot\nabla_\xi}{\mathrm{i}|x|^2}$, and writing it as

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}(L^*)^N \left[\frac{\xi_j}{|\xi|}\varphi(\xi/R)(1-\varphi(\xi|x|))\right] d\xi.$$

It is now easy to see that

$$\left|(L^*)^N \left[\frac{\xi_j}{|\xi|}\varphi(\xi/R)(1-\varphi(\xi|x|))\right]\right| \le C_N \frac{1}{|x|^N|\xi|^N}.$$

Hence the absolute of the 2nd integral is bounded by

$$C_N \frac{1}{|x|^N}\int_{|\xi|\ge\frac{a}{|x|}} \frac{1}{|\xi|^N} d\xi \le C'_N \frac{|x|^{N-d}}{a^{N-d}|x|^N} = \frac{C'_N}{a^{N-d}}\frac{1}{|x|^d}.$$

The 2nd inequality in (3.12) can be obtained noticing that

$$\partial_k K(x) = -\mathrm{i}\lim_{R\to+\infty}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x}\xi_k\frac{\xi_j}{|\xi|}\varphi(\xi/R)d\xi.$$

When one considers the above inequalities with an additional factor $\xi_k$ inside the integral, one gets the upper bound of the 2nd inequality in (3.12).

The operators $R_j$ are called Riesz transforms.

2. The above discussion works out similarly with operators $\frac{\partial_j}{\sqrt{1-\triangle}}$ and $\frac{\partial^\alpha}{(1-\triangle)^{\frac{k}{2}}}$ with $\alpha$ any multi–index with $|\alpha|\le k$. In particular, $\frac{\partial_j}{\sqrt{1-\triangle}}$ has symbol $\frac{-\mathrm{i}\xi_j}{\langle\xi\rangle}$.

3. Notice that $(\mathbb{P}u)_j = u_j - R_jR_ku_k$, and so in particular it is a CZ operator.

4. Let us consider in $\mathbb{R}$ the Hilbert transform

$$Hf(x) := -\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy = -\frac{1}{\pi}(P.V\frac{1}{x}) * f \tag{3.14}$$

with $P.V\frac{1}{x}$ the tempered distribution that acts on a $\phi\in\mathcal{S}(\mathbb{R})$ as $\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx$.

Notice that using the Residue theorem we have

$$\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon} e^{-\mathrm{i}\xi x}\frac{dx}{x} = -\mathrm{i}\pi\,\mathrm{sign}(\xi)$$

so that

$$\frac{1}{\pi}\mathcal{F}(P.V.\frac{1}{x}) = -\mathrm{i}(2\pi)^{-\frac{1}{2}}\mathrm{sign}(\xi).$$

Then

$$\mathcal{F}(Hf)(\xi) = -\mathrm{i}\,\mathrm{sign}(\xi)\widehat{f}(\xi).$$

which implies that (C–Z2) is true. Since (C–Z1) is obvious, we conclude that the Hilbert transform meets the conditions of Definition 3.3.

*Remark* 3.5. Consider the operator $T_{\mathbb{R}_+}f := \mathcal{F}^*\left[\chi_{\mathbb{R}_+}\widehat{f}\right]$. Then $\chi_{\mathbb{R}_+} = 2^{-1}\mathrm{i}(-\mathrm{isign} - \mathrm{i})$ implies $T_{\mathbb{R}_+} = 2^{-1}(I + \mathrm{i}H)$. Analogously $T_{\mathbb{R}_-} = 2^{-1}(I - \mathrm{i}H)$. Next,

$$T_{(a,+\infty)} = 2^{-1}(I + \mathrm{i}e^{\mathrm{i}ax}He^{-\mathrm{i}ax}) \text{ and } T_{(-\infty,b)} = 2^{-1}(I - \mathrm{i}e^{\mathrm{i}bx}He^{-\mathrm{i}bx}).$$

Finally

$$T_{(a,b)} = 2^{-1}(T_{(a,+\infty)} - T_{(b,+\infty)}) = 4^{-1}\mathrm{i}(e^{\mathrm{i}ax}He^{-\mathrm{i}ax} - e^{\mathrm{i}bx}He^{-\mathrm{i}bx}).$$

Next, if in $\mathbb{R}^d$ we consider the half–plane $x_1 > 0$, then

$$\mathcal{F}^*\left[\chi_{\{x_1>0\}}\widehat{f}\right] = 2^{-1}(I + \mathrm{i}H_1)f \text{ where}$$
$$(H_1f)(x_1, x_2, ...., x_d) := H(f(\cdot, x_2, ...., x_d))(x_1).$$

In general, any operator of the form $\mathcal{F}^*\left[\chi_P\widehat{f}\right]$ with $P$ a polygon in $\mathbb{R}^d$ can be expressed in terms of the Hilbert transform.

*Remark* 3.6. Let $p \in (1, \infty)$ and let $L^p(\mathbb{R}, \mathbb{C}) \ni f = \lim_{y\to 0^+} F(\cdot + \mathrm{i}y)$ where $F : \{x + \mathrm{i}y : x \in \mathbb{R},\ y > 0\} \to \mathbb{C}$ is a holomorphic function with $\sup_{y>0}\int_{\mathbb{R}} |F(x + \mathrm{i}y)|^p dx < \infty$. Then, if $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, we have $v = Hu$ (and, by $H^2 = -1$, $u = -Hv$). We give a brief impressionistic and non–rigorous discussion of how this comes about. Notice that if $f$ is the boundary value in $\mathbb{R}$ of $F$ by Cauchy integral formula we have

$$F(x + \mathrm{i}y) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{1}{t - x - \mathrm{i}y} f(t)dt = \frac{1}{2\pi\mathrm{i}}\ (\frac{1}{\cdot - \mathrm{i}y} * f)(x)$$

where here we assume $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then for $y \to 0^+$ by the Sokhotski–Plemelj theorem we get

$$\lim_{y\to 0^+} \frac{1}{t - \mathrm{i}y} = P.V.\frac{1}{t} + \mathrm{i}\pi\delta(t) \text{ in } \mathcal{S}'(\mathbb{R}, \mathbb{C}). \tag{3.15}$$

This implies, assuming here $F \in C^0(\mathbb{R} \times [0, \infty))$, that by $f(x) = \lim_{y\to 0^+} F(x + \mathrm{i}y)$ we have

$$f(x) = \frac{1}{2\pi\mathrm{i}}\left(\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon} \frac{f(x)}{x} dx + \mathrm{i}\pi f(x)\right),$$

that is $f = \mathrm{i}Hf$, which is the desired result.

As for (3.15), for $f \in \mathcal{S}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \frac{f(t)}{t - \mathrm{i}y} dt = \int_{\mathbb{R}} \frac{t}{t^2 + y^2} f(t)dt + \mathrm{i}\int_{\mathbb{R}} \frac{y}{t^2 + y^2} f(t)dt.$$

By a change of variables, by dominated convergence and by the continuity of $f$ in 0 we have

$$\int_{\mathbb{R}} \frac{y}{t^2 + y^2} f(t)dt = \int_{\mathbb{R}} \frac{1}{t^2 + 1} f(ty)dt \overset{y\to 0}{\to} \pi f(0).$$

Next we write

$$\int_{\mathbb{R}} \frac{t}{t^2+y^2} f(t)dt = \int_{|t|\le y} \frac{t}{t^2+y^2} f(t)dt + \int_{|t|\ge y} \frac{t}{t^2+y^2} f(t)dt.$$

We have

$$\left| \int_{|t|\le y} \frac{t}{t^2+y^2} f(t)dt \right| = \left| \int_{|t|\le y} \frac{t}{t^2+y^2} (f(t)-f(0))dt \right| \stackrel{y\to 0}{\to} 0.$$

Next we write

$$\int_{|t|\ge y} \frac{t}{t^2+y^2} f(t)dt = \int_{|t|\ge y} \left( \frac{t}{t^2+y^2} - \frac{1}{t} \right) f(t)dt + \int_{|t|\ge y} \frac{f(t)}{t} dt$$

and observe that, changing variable,

$$\begin{aligned} &\int_{|t|\ge y} \left( \frac{t}{t^2+y^2} - \frac{1}{t} \right) f(t)dt = \int_{|t|\ge y} \frac{-y^2}{t(t^2+y^2)} f(t)dt = \int_{|s|\ge 1} \frac{-1}{s(s^2+1)} f(sy)dt \\ &\stackrel{y\to 0}{\to} -f(0) \int_{|s|\ge 1} \frac{1}{s(s^2+1)} dt \end{aligned}$$

by dominated convergence. But the last integral is null. This proves (3.15).

**Theorem 3.7.** *Consider an operator $T$ as in Definition 3.3. Then for any $p \in (1,\infty)$ the operator $T$, initially defined in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, extends into a bounded operator $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ with operator norm that depends only on $p$ and $C$.*

Before proving Theorem 3.7 we need the Calderon–Zygmund decomposition lemma.

**Theorem 3.8** (C–Z Decomposition)**.** *For any $f \in L^1(\mathbb{R}^d)$ and any $\alpha > 0$ there exist families of balls $B_j$, disjoint sets $Q_j$ with $B_j \subseteq Q_j \subseteq 3B_j$ with $\cup_j Q_j = \cup_j 3B_j$ (here $3B_j$ has same center and trice the radius of $B_j$) functions $g$ and $b_j$ s.t.*

1. $f = g + \sum_j b_j$.
2. $|g(x)| \le 3^d\alpha$ *for a.a.* $x$, $\|g\|_{L^1(\mathbb{R}^d)} \le \left(1+3^{2d}\right)\|f\|_{L^1(\mathbb{R}^d)}$.
3. *supp* $b_j \subseteq Q_j$, $\int_{\mathbb{R}^d} b_j(x)dx = 0$ *and* $\sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \le \left(1+3^{2d}\right)\|f\|_{L^1(\mathbb{R}^d)}$.
4. $\sum_j vol(B_j) \le \frac{3^d}{\alpha}\|f\|_{L^1(\mathbb{R}^d)}$.

*Remark* 3.9. Notice that in the Calderon–Zygmund decomposition $g$ is the *good* part of $f$ and $b_j$ form the *bad* part of $f$.

*Proof.* Define $\Omega = \{x \in \mathbb{R}^d : Mf(x) > \alpha\}$. Here notice that if $\Omega = \emptyset$ then just set $g = f$. For any $x \in \Omega$ there exists a maximal $r_x$ s.t.

$$A_{r_x}|f|(x) := \frac{1}{\mathrm{vol}(B(x,r_x))} \int_{B(x,r_x)} |f(y)|dy = \alpha.$$

Let us consider the family of balls $\{B(x,r_x)\}_{x\in\Omega}$. It contains, by a generalization of Vitali's Lemma, see Theorem 3.1, a maximal family of pairwise disjoint balls $\{B_j\}$ s.t.

$$\Omega \subseteq \cup_{x\in\Omega} B(x,r_x) \subseteq \cup_j (3B_j).$$

Notice that this implies

$$\mathrm{vol}(\cup_{x\in\Omega} B(x,r_x)) \leq \sum_j \mathrm{vol}(3B_j) \leq 3^d \sum_j \mathrm{vol}(B_j) \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

It is possible to choose disjoint sets $Q_j$ s.t. $B_j \subseteq Q_j \subseteq 3B_j$ and $\cup_j Q_j = \cup_j (3B_j)$. One way is to choose

$$Q_k = 3B_k \cap C\left(\cup_{j<k} Q_j\right) \cap C\left(\cup_{j>k} B_j\right) \tag{3.16}$$

with $CX$ the complement of $X$. Notice indeed that obviously for $k > \ell$ we have

$$Q_k \cap Q_\ell \subseteq C\left(\cup_{j<k} Q_j\right) \cap Q_\ell = \left(\cap_{j<k} CQ_j\right) \cap Q_\ell \subseteq CQ_\ell \cap Q_\ell = \emptyset.$$

Obviously $Q_k \subseteq 3B_k$.
We have $B_k \cap (\cup_{j>k} B_j) = \emptyset$ and so $B_k \subseteq C\left(\cup_{j>k} B_j\right)$. We have $B_k \cap (\cup_{j<k} Q_j) = \emptyset$ because, by (3.16), we have $B_k \cap Q_j = \emptyset$ for any $j < k$. Hence we conclude $B_k \subseteq Q_k$.
Finally we show $\cup_k Q_k = \cup_k 3B_k$. Obviously we have $\cup_k Q_k \subseteq \cup_k 3B_k$. Suppose there exists $x \in \cup_k 3B_k$ with $x \notin \cup_k Q_k$. The latter implies $x \notin \cup_k B_k$, and so $x \in C\left(\cup_{j>k} B_j\right)$ for all $k$, as well as $x \in C\left(\cup_{j<k} Q_j\right)$ for all $k$. But then, since $x \in 3B_\ell$ for some $\ell$, it follows that $x \in Q_\ell$. And so we get a contradiction. Hence $\cup_k Q_k = \cup_k 3B_k$.
Now define

$$b_j(x) := \left(f(x) - \mathrm{average}_{Q_j} f\right) \chi_{Q_j}(x)$$

$$g(x) := \begin{cases} \mathrm{average}_{Q_j} f \text{ for } x \in Q_j, \\ f(x) \text{ for } x \notin \cup_j Q_j \end{cases}$$

Then we claim that the statement of the theorem is satisfied. First of all for any $x \in \mathbb{R}^d$ either $x \notin Q_j$ for all $j$, and so $f(x) = g(x)$ with $b_j(x) = 0$ for all $j$, or $x \in Q_{j_0}$ for exactly one $j_0$, and so $f(x) = g(x) + b_{j_0}(x)$ with $b_j(x) = 0$ for all $j \neq j_0$. This proves the 1st claim.

For $x \notin \cup_j Q_j \supseteq \Omega$ we have $Mf(x) \leq \alpha$. Then, since for a.e. $x$ we have

$$|f(x)| = \lim_{r\to 0^+} |A_r f(x)| \leq Mf(x)$$

we get $|g(x)| = |f(x)| \leq \alpha$ a.e. in the complement of $\cup_j Q_j$. For $x \in Q_j$ we have

$$|g(x)| = |\mathrm{average}_{Q_j} f| \leq \frac{1}{\mathrm{vol}(Q_j)} \int_{Q_j} |f(y)|dy \leq \frac{1}{\mathrm{vol}(B_j)} \int_{3B_j} |f(y)|dy = \frac{3^d}{\mathrm{vol}(3B_j)} \int_{3B_j} |f(y)|dy < 3^d \alpha.$$

Furthermore we have

$$\|g\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d\setminus\cup_j Q_j} |f(x)|dx + \sum_j \int_{Q_j} |g(x)|dx \le \|f\|_{L^1(\mathbb{R}^d)} + 3^d\alpha \sum_j \mathrm{vol}(3B_j)$$
$$\le \left(1+3^{2d}\right)\|f\|_{L^1(\mathbb{R}^d)}.$$

The fact that supp $b_j \subseteq Q_j$, $\int_{\mathbb{R}^d} b_j(x)dx = 0$ follows immediately by the definition of $b_j$. We have

$$\sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)} + \sum_j \mathrm{vol}(Q_j)|\mathrm{average}_{Q_j} f| \le \|f\|_{L^1(\mathbb{R}^d)} + 3^d\alpha \sum_j \mathrm{vol}(Q_j)$$
$$\le \left(1+3^{2d}\right)\|f\|_{L^1(\mathbb{R}^d)}.$$

□

*Proof of Theorem 3.7.* By duality it is enough to consider only $p \in (1,2]$. Furthermore, since by hypothesis (C–Z2) we know that the case $p = 2$ is true, by Marcinkiewicz Interpolation the statement of Theorem 3.7 results from proving that $T$ is weak–type $(1,1)$. We need to prove that there exists an $A > 0$ s.t.

$$\mathrm{vol}(\{x : |Tf(x)| > \alpha\}) \le \frac{A}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \text{ and any } f \in L^1(\mathbb{R}^d). \tag{3.17}$$

For fixed $\alpha > 0$ and any $f \in L^1(\mathbb{R}^d)$ consider the C–Z decomposition $f = g + \sum_j b_j$. Notice that $|g(x)| \le 3^d\alpha$ a.e. and $\|g\|_{L^1(\mathbb{R}^d)} \le \left(1+3^{2d}\right)\|f\|_{L^1(\mathbb{R}^d)}$ imply $g \in L^2(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} |g|^2dx \le C_d\alpha \int_{\mathbb{R}^d} |f|dx \text{ for } C_d = 3^d\left(1+3^{2d}\right)$$

and so by Hypothesis (C–Z2) we have $\|Tg\|^2_{L^2(\mathbb{R}^d)} \le C\alpha\|f\|_{L^1(\mathbb{R}^d)}$.
Then by Chebyshev's inequality (3.3) we have

$$\mathrm{vol}(\{x : |(Tg)(x)| > \alpha/2\}) \le \frac{4\|Tg\|^2_{L^2(\mathbb{R}^d)}}{\alpha^2} \le 4C\frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}.$$

We next consider $b_j$ and consider for $x \notin 3B_j$ and for $y_j$ the center of $B_j$,

$$Tb_j(x) = \int_{Q_j} K(x,y)b_j(y)dy = \int_{Q_j} \left(K(x,y) - K(x,y_j)\right)b_j(y)dy$$

were we used $\mathrm{average}_{Q_j} b_j = 0$. Then by (3.12) we have

$$|Tb_j(x)| \le \frac{C}{|x-y_j|^{d+1}}\int_{Q_j} |y-y_j|\ |b_j(y)|dy.$$

Then for $r = \text{radius}(B_j)$

$$\int_{\mathbb{R}^d \setminus 3B_j} |Tb_j(x)| dx \leq \int_{|x-y_j|\geq 3r} dx \frac{C}{|x-y_j|^{d+1}} \int_{|y-y_j|\leq 3r} |y-y_j|\ |b_j(y)| dy$$
$$\leq c_d \frac{C}{3r} \int_{|y-y_j|\leq 3r} |y-y_j|\ |b_j(y)| dy \leq c_d C \|b_j\|_{L^1(\mathbb{R}^d)}.$$

Let now $E = \cup_j (3B_j)$. Then for $b = \sum_j b_j$ we have

$$\int_{\mathbb{R}^d \setminus E} |Tb| \leq \sum_j \int_{\mathbb{R}^d \setminus 3B_j} |Tb_j| \leq c_d C \sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \leq c_d C (1+3^{2d}) \|f\|_{L^1(\mathbb{R}^d)}.$$

Hence

$$\text{vol}(\{x \notin E : |(Tb)(x)| > \alpha/2\}) \leq \frac{2\|Tb\|_{L^1(\mathbb{R}^d)}}{\alpha} \leq c_d C (1+3^{2d}) \frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}.$$

So since

$$\text{vol}(\{x \notin E : |Tf(x)| > \alpha\}) \leq \text{vol}(\{x \notin E : |Tg(x)| > \alpha/2\} + \text{vol}(\{x \notin E : |(Tb)(x)| > \alpha/2\})$$
$$\leq \left[4C + c_d C (1+3^{2d})\right] \frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}$$

and

$$\text{vol}(E) \leq \sum_j \text{vol}(3B_j) \leq 3^d \sum_j \text{vol}(B_j) \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$$

we conclude that (3.17) as been proved with $A = 3^d + 4C + c_d C (1+3^{2d})$.

□

Now we consider the Proof of Theorem 2.14. We follow [18] from p. 136. Preliminarily, we state the following lemma.

**Lemma 3.10.** *Suppose $1 < p < \infty$ and $s \geq 1$. Then $f \in \mathcal{W}^{s,p}(\mathbb{R}^d)$ if and only if $f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ and $\partial_{x^j} f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ for all $j = 1, ..., d$ and furthermore the norms $\|f\|_{\mathcal{W}^{s,p}}$ and $\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^d \|\partial_{x^j} f\|_{\mathcal{W}^{s-1,p}}$ are equivalent.*

*Proof of Theorem 2.14 assuming Lemma 3.10.* Obviously for $k = 0$ we have $\mathcal{W}^{0,p} = W^{0,p} = L^p$.
It is obvious that $f \in W^{k,p}(\mathbb{R}^d)$ if and only if $f \in W^{k-1,p}(\mathbb{R}^d)$ and $\partial_{x^j} f \in W^{k-1,p}(\mathbb{R}^d)$ and that the the norms $\|f\|_{W^{k,p}}$ and $\|f\|_{W^{k-1,p}} + \sum_{j=1}^d \|\partial_{x^j} f\|_{W^{k-1,p}}$ are equivalent. But then Lemma 3.10 guarantees that $\mathcal{W}^{1,p} = W^{1,p}$ with equivalent norms, and so on for all $k \in \mathbb{N}$.

□

*Proof of Lemma 3.10.* Let us start assuming that $f \in \mathcal{W}^{s,p}(\mathbb{R}^d)$. Then setting $\widehat{g}(\xi) := \langle \xi \rangle^s \widehat{f}(\xi)$ we have $g \in L^p(\mathbb{R}^d)$ by definition of $\mathcal{W}^{s,p}(\mathbb{R}^d)$. Then notice that

$$(\langle \xi \rangle^{s-1} \widehat{f})^\vee = (\langle \xi \rangle^{-1} \widehat{g})^\vee = (2\pi)^{-\frac{d}{2}} \mathcal{J}_{-1} * g$$

where $\mathcal{J}_{-s} = (\langle\xi\rangle^{-1})^{\vee}$ is easily seen to be an $L^1(\mathbb{R}^d)$ function: this can be seen by an integration by parts argument like in the discussion of the Riesz transforms above. Hence we have

$$\|f\|_{\mathcal{W}^{s-1,p}} \leq (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|g\|_{L^p} = (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|f\|_{\mathcal{W}^{s,p}}.$$

Next we consider

$$\langle\xi\rangle^{s-1}\widehat{\partial_j f}(\xi) = -\mathrm{i}\,\langle\xi\rangle^{s-1}\,\xi_j\widehat{f}(\xi) = -\mathrm{i}\frac{\xi_j}{\langle\xi\rangle}\widehat{g}(\xi) = \widehat{\widetilde{R}_j g}(\xi),$$

where $\widetilde{R}_j$ is a variant of the Riesz transform considered considered in the list in Example 3.4. But then, since the Riesz transforms are CZ operators, it follows that

$$\|\partial_j f\|_{W^{k-1,p}} \leq \|\widetilde{R}_j\|_{L^p\to L^p}\|g\|_{L^p} = \|\widetilde{R}_j\|_{L^p\to L^p}\|g\|_{L^p}\|f\|_{\mathcal{W}^{s,p}}.$$

Summing up, we obtained

$$\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^{d}\|\partial_{x^j} f\|_{\mathcal{W}^{s-1,p}} \leq \Big((2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1} + d\|\widetilde{R}_1\|_{L^p\to L^p}\Big)\|f\|_{\mathcal{W}^{s,p}},$$

where we used the fact, easy to show, that $\|\widetilde{R}_j\|_{L^p\to L^p}$ is constant in $j$, so that one implication is proved.

Now we consider the opposite implication, assuming $f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ and $\partial_{x^j} f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ for all $j = 1, ..., d$. Then $\widehat{g}(\xi) := \langle\xi\rangle^{s-1}\widehat{f}(\xi)$ is $g \in L^p(\mathbb{R}^d)$ and, from $\widehat{\partial_{x_j} g}(\xi) = \langle\xi\rangle^{s-1}\widehat{\partial_{x_j} f}(\xi)$, $\partial_{x_j} g \in L^p(\mathbb{R}^d)$ for any $j$. Now we have

$$\langle\xi\rangle^{s}\widehat{f} = \langle\xi\rangle\,\widehat{g} = \langle\xi\rangle^2\frac{1}{\langle\xi\rangle}\widehat{g} = \frac{1}{\langle\xi\rangle}\widehat{g} - \sum_{j=1}^{d}\frac{-\mathrm{i}\xi_j}{\langle\xi\rangle}(-\mathrm{i}\xi_j)\widehat{g}.$$

This means that

$$(\langle\xi\rangle^{s}\widehat{f})^{\vee} = (2\pi)^{-\frac{d}{2}}\mathcal{J}_{-1} * g - \sum_{j=1}^{d}\widetilde{R}_j\partial_{x_j} g$$

and so

$$\begin{aligned}\|f\|_{\mathcal{W}^{s,p}} &\leq (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|g\|_{L^p} + \sum_{j=1}^{d}\|\widetilde{R}_j\|_{L^p\to L^p}\|\partial_{x_j} g\|_{L^p}\\ &= (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^{d}\|\widetilde{R}_j\|_{L^p\to L^p}\|\partial_{x_j} f\|_{\mathcal{W}^{s-1,p}},\end{aligned}$$

which obviously proves the opposite implication and completes the proof of Lemma 3.10.

$\square$

# 4 Linear heat equation, take 1

For Section 4 see [4].

Let $T \in \mathbb{R}_+$ and $f : [0,T] \to \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d)$, for $d = 2, 3$, be an external force s.t. $f = \mathbb{P}f$ and consider the following heat equation:

$$\begin{cases} u_t - \triangle u = f \\ \nabla \cdot u = 0 \qquad (t,x) \in [0,T] \times \mathbb{R}^d \\ u(0) = u_0 \in \mathbb{P}\dot{H}^s(\mathbb{R}^d, \mathbb{R}^d) \end{cases} \tag{4.1}$$

**Definition 4.1.** For a fixed $s < d/2$ let $f \in L^2([0,T], \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d))$ with $f = \mathbb{P}f$. Then $u$ is a solution of (4.1) if

$$u \in L^\infty([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d)) \ , \ \nabla u \in L^2([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d)), \tag{4.2}$$

if

$$u \text{ is weakly continuous from } [0,T] \text{ into } \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d) \tag{4.3}$$

(that is, if for any $\psi \in \dot{H}^{-s}(\mathbb{R}^d, \mathbb{R}^d)$ the function $t \to \langle u(t), \psi \rangle$, which is a well defined function in $L^\infty([0,T], \mathbb{R})$, is in fact in $C^0([0,T], \mathbb{R})$ )
and if for any $\Psi \in C_c^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ we have

$$\langle u(t), \Psi(t) \rangle_{L^2} = \int_0^t \left( \langle u(t'), \triangle \Psi(t') \rangle_{L^2} + \langle u(t'), \partial_t \Psi(t') \rangle_{L^2} + \langle f(t'), \Psi(t') \rangle_{L^2} \right) dt' + \langle u_0, \Psi(0) \rangle_{L^2}. \tag{4.4}$$

The following theorem yields existence, uniqueness and energy estimate for (4.1).

**Theorem 4.2.** *Problem* (4.1) *admits exactly one solution in the sense of the above definition. For any $t$ the following energy estimate is satisfied:*

$$\|u(t)\|_{\dot{H}^s}^2 + 2\int_0^t \|\nabla u(t')\|_{\dot{H}^s}^2 dt' = \|u_0\|_{\dot{H}^s}^2 + 2\int_0^t \langle f(t'), u(t') \rangle_{\dot{H}^s} dt'. \tag{4.5}$$

*Furthermore we have*

$$u \in C^0([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d)) \tag{4.6}$$

*and the formula*

$$\widehat{u}(t,\xi) = e^{-t|\xi|^2} \widehat{u}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2} \widehat{f}(t', \xi) dt'. \tag{4.7}$$

*Proof.* (**Uniqueness**). It is enough to show that the only solution of the case $u_0 = 0$ and $f = 0$ is $u = 0$. Let $u$ be such a solution. Then

$$\langle u(t), \Psi(t) \rangle_{L^2} = \int_0^t \left( \langle u(t'), \triangle \Psi(t') \rangle_{L^2} + \langle u(t'), \partial_t \Psi(t') \rangle_{L^2} \right) dt'.$$

Let $\Psi(t,x)=\psi(x)$ with $\psi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$. Then the above equality reduces to

$$\langle u(t),\psi\rangle_{L^2}=\int_0^t\langle u(t'),\triangle\psi\rangle_{L^2}. \tag{4.8}$$

We claim that this identity holds for all $\psi\in\dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d)\cap\dot{H}^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$. From Lemma 2.2 we know that $C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ is dense in $\dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d)$ (here we use $s<d/2$) and in $\dot{H}^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$ and so the claim follows by density and the fact that

$$\langle\quad,\quad\rangle_{L^2}:\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d)\times\dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d)\longrightarrow\mathbb{R}$$
$$L^2([0,T],\dot{H}^{s+1}(\mathbb{R}^d,\mathbb{R}^d))\times\dot{H}^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)\ni(u(t),\psi)\longrightarrow\int_0^T\langle u(t'),\triangle\psi\rangle_{L^2}dt'\in\mathbb{R}$$

are both continuous bilinear forms.
Hence we can conclude that (4.8) is true for all $\psi\in\dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d)\cap\dot{H}^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$. In particular we can replace $\psi$ by $\mathbf{P}_n\psi$ and get

$$\begin{aligned}&\langle\mathbf{P}_nu(t),\mathbf{P}_n\psi\rangle_{L^2}=\int_0^t\langle\mathbf{P}_nu(t'),\triangle\mathbf{P}_n\psi\rangle_{L^2}\le\nu\|\triangle\mathbf{P}_n\psi\|_{\dot{H}^{-s}}\int_0^t\|\mathbf{P}_nu(t')\|_{\dot{H}^s}dt'\\&\le n^2\|\psi\|_{\dot{H}^{-s}}\int_0^t\|\mathbf{P}_nu(t')\|_{\dot{H}^s}dt'\end{aligned}$$

where the integral $\int_0^t\|\mathbf{P}_nu(t')\|_{\dot{H}^s}dt'$ is well defined by $\mathbf{P}_nu\in L^\infty([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$.
So, we obtained

$$|\langle\mathbf{P}_nu(t),\psi\rangle_{L^2}|\le n^2\|\psi\|_{\dot{H}^{-s}}\int_0^t\|\mathbf{P}_nu(t')\|_{\dot{H}^s}dt'\text{ for all }\psi\in\dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d).$$

This implies

$$\|\mathbf{P}_nu(t)\|_{\dot{H}^s}\le n^2\int_0^t\|\mathbf{P}_nu(t')\|_{\dot{H}^s}dt'$$

and hence $\|\mathbf{P}_nu(t)\|_{\dot{H}^s}=0$ by the Gronwall inequality. This implies $u(t)=0$ for $t\in[0,T]$.

(**Existence**). First of all, there exists a sequence $(f_n)$ in $C^0([0,T],\dot{H}^{s-1}(\mathbb{R}^d,\mathbb{R}^d))$ s.t. $f_n\xrightarrow{n\to+\infty}f$ in $L^2([0,T],\dot{H}^{s-1}(\mathbb{R}^d,\mathbb{R}^d))$. This follows from the density of $C_c^\infty(I,X)$ in $L^p(I,X)$ for $p<\infty$ for $I$ an interval and $X$ a Banach space, see Appendix A.
Applying $\mathbf{P}_n$ to (4.1) and replacing $f$ by $f_n$ we obtain the equation

$$\begin{cases}(u_n)_t-\mathbf{P}_n\triangle u_n=\mathbf{P}_nf_n\\ \quad u_n(0)=\mathbf{P}_nu_0\end{cases} \tag{4.9}$$

Notice that $\mathbf{P}_nf_n\in C^0([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$. Since (4.9) is a standard linear equation it admits a solution $u_n\in C^1([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$. Notice furthermore that $u_n=\mathbf{P}_nu_n$ and so in particular $u_n\in C^0([0,T],\dot{H}^r(\mathbb{R}^d,\mathbb{R}^d))$ for all $r\ge s$.

Furthermore, applying $\langle \cdot, u_n \rangle_{\dot{H}^s}$ to (4.9) and using

$$\langle \mathbf{P}_n \triangle u_n, u_n \rangle_{\dot{H}^s} = -\sum_{k=1}^{d} \int_{B(0,n)} |\xi|^{2s} \xi_k^2 |\widehat{u}_n(t,\xi)|^2 d\xi = -\sum_{k=1}^{d} \langle \xi_k \widehat{u}_n, \xi_k \widehat{u}_n \rangle_{L^2(B(0,n),|\xi|^{2s}d\xi)}$$

$$= \sum_{k=1}^{d} \langle \xi_k \widehat{u}_n, \xi_k \widehat{u}_n \rangle_{L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} = \|\nabla u_n\|^2_{\dot{H}^s},$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{\dot{H}^s} + \|\nabla u_n\|^2_{\dot{H}^s} = \langle \mathbf{P}_n f_n, u_n \rangle_{\dot{H}^s}$$

s.t., after integration, we obtain

$$\frac{1}{2}\|u_n(t)\|^2_{\dot{H}^s} + \int_0^t \|\nabla u_n(t')\|^2_{\dot{H}^s} dt' = \frac{1}{2}\|\mathbf{P}_n u_0\|^2_{\dot{H}^s} + \int_0^t \langle \mathbf{P}_n f_n(t'), u_n(t') \rangle_{\dot{H}^s} dt'. \tag{4.10}$$

The difference $u_n - u_{n+\ell}$ solves

$$\begin{cases} (u_n - u_{n+\ell})_t - \mathbf{P}_{n+\ell} \triangle (u_n - u_{n+\ell}) = \mathbf{P}_n f_n - \mathbf{P}_{n+\ell} f_{n+\ell} \\ u_n(0) - u_{n+\ell}(0) = (\mathbf{P}_n - \mathbf{P}_{n+\ell}) u_0 \end{cases}$$

Then, like for (4.10) we get

$$\frac{1}{2}\|u_n(t) - u_{n+\ell}(t)\|^2_{\dot{H}^s} + \cancel{2}\frac{1}{2}\int_0^t \|\nabla(u_n - u_{n+\ell})(t')\|^2_{\dot{H}^s} dt' =$$

$$= \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \int_0^t \langle \mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t'), (u_n - u_{n+\ell})(t') \rangle_{\dot{H}^s} dt'$$

$$\leq \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \int_0^t \|\mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t')\|_{\dot{H}^{s-1}} \|\nabla(u_n - u_{n+\ell})(t')\|_{\dot{H}^s} dt'$$

$$\leq \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \frac{1}{2}\int_0^t \|\mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t')\|^2_{\dot{H}^{s-1}} dt' + \cancel{\frac{1}{2}\int_0^t \|\nabla(u_n - u_{n+\ell})(t')\|^2_{\dot{H}^s} dt'}.$$

Hence

$$\|u_n(t) - u_{n+\ell}(t)\|^2_{\dot{H}^s} + \int_0^t \|\nabla(u_n - u_{n+\ell})(s)\|^2_{\dot{H}^s} ds$$

$$\leq \|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \int_0^t \|\mathbf{P}_n f_n(s) - \mathbf{P}_{n+\ell} f_{n+\ell}(s)\|^2_{\dot{H}^{s-1}} ds.$$

Since $f_n \xrightarrow{n \to +\infty} f$ in $L^2([0,T], \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d))$ implies also $\mathbf{P}_n f_n \xrightarrow{n \to +\infty} f$ therein, the last inequality implies that $(u_n)$ is Cauchy in $C([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d))$ and $(\nabla u_n)$ is Cauchy in $L^2([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d))$. Let $u$ be the limit. Notice that $u$ satisfies (4.2) and (4.6), and so obviously also (4.3).
Taking the limit in (4.10) we see that $u$ satisfies the energy equality (4.5).

Next, we check that $u$ is a weak solution of (4.1) in the sense of Def. 4.1. We apply $\langle ., \Psi(t)\rangle_{L^2}$ to (4.9) with $\Psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$. Then we have

$$\frac{d}{dt}\langle u_n, \Psi\rangle_{L^2} = \langle \triangle u_n, \Psi\rangle_{L^2} + \langle \mathbf{P}_n f_n, \Psi\rangle_{L^2} + \langle u_n, \partial_t \Psi\rangle_{L^2}.$$

Integrating we have

$$\langle u_n(t), \Psi(t)\rangle_{L^2} = \langle \mathbf{P}_n u_0, \Psi(0)\rangle_{L^2} - \int_0^t \langle u_n(t'), \triangle\Psi(t')\rangle_{L^2} dt' \\ + \int_0^t \langle \mathbf{P}_n f_n(t'), \Psi(t')\rangle_{L^2} dt' + \int_0^t \langle u_n(t'), \partial_t\Psi(t')\rangle_{L^2} dt'.$$

Taking the limit for $n\to\infty$ we get

$$\langle u(t), \Psi(t)\rangle_{L^2} = \langle u_0, \Psi(0)\rangle_{L^2} - \int_0^t \langle u(t'), \triangle\Psi(t')\rangle_{L^2} dt' + \int_0^t \langle f(t'), \Psi(t')\rangle_{L^2} dt' + \int_0^t \langle u(t'), \partial_t\Psi(t')\rangle_{L^2} dt'.$$

which yields (4.4). Hence $u$ is a weak solution of (4.1) in the sense of Def. 4.1.
Next, we prove the Duhamel formula (4.7). Applying the Fourier transform to (4.9)

$$\begin{cases} \partial_t \widehat{u}_n(t,\xi) + \chi_{|\xi|\le n}|\xi|^2 \widehat{u}_n(t,\xi) = \chi_{|\xi|\le n}\widehat{f}_n(t,\xi) \\ \widehat{u_n}(0,\xi) = \chi_{|\xi|\le n}\widehat{u}_0(\xi) \end{cases} \tag{4.11}$$

Notice that $\mathrm{supp}\widehat{u}_n(t,\cdot) \subseteq \{|\xi|\le n\}$ so that $\chi_{|\xi|\le n}|\xi|^2\widehat{u}_n(t,\xi) = |\xi|^2\widehat{u}_n(t,\xi)$. Then, by the variation of parameters formula

$$\widehat{u}_n(t,\xi) = e^{-t|\xi|^2}\chi_{|\xi|\le n}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2}\chi_{|\xi|\le n}\widehat{f}_n(t',\xi)dt'. \tag{4.12}$$

Now we know

$$\widehat{u}_n(t,\xi) \overset{n\to\infty}{\to} \widehat{u}(t,\xi) \text{ in } C([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi))$$
$$\chi_{|\xi|\le n}\widehat{u}_0(\xi) \overset{n\to\infty}{\to} \widehat{u}_0(\xi) \text{ in } L^2(\mathbb{R}^d, |\xi|^{2s}d\xi),$$
$$\chi_{|\xi|\le n}\widehat{f}_n(t',\xi) \overset{n\to\infty}{\to} \widehat{f}(t',\xi) \text{ in } L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$$

Notice that

$$\mathbf{T}g(t,\xi) := \int_0^t e^{-(t-t')\nu|\xi|^2} g(t',\xi)dt'$$

is a bounded operator from $L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$ into $L^\infty([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$. Indeed for $t\in[0,T]$ and fixed $\xi\in\mathbb{R}^d$ and for $g\in C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$

$$|\mathbf{T}g(t,\xi)| \le (\int_0^t e^{-2(t-t')|\xi|^2}dt')^{\frac{1}{2}}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}} \le \frac{1}{\sqrt{2}|\xi|}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}}$$

and so

$$\int_{\mathbb{R}^d} |\xi|^{2s} |\mathbf{T}g(t,\xi)|^2 d\xi \le \frac{1}{2} \int_{[0,T]\times\mathbb{R}^d} |\xi|^{2(s-1)} |g(t',\xi)|^2 dt' d\xi.$$

This implies

$$\|\mathbf{T}g\|_{L^\infty([0,T],L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} \le \sqrt{1/2} \|g\|_{L^2([0,T]\times\mathbb{R}^d,|\xi|^{2(s-1)}dtd\xi)}.$$

Since $C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$ is dense in $L^2([0,T]\times\mathbb{R}^d,|\xi|^{2(s-1)}dtd\xi)$ a well defined bounded operator remains defined. Taking the limit for $n\to\infty$ in (4.12) all terms converge in $L^\infty([0,T],L^2(\mathbb{R}^d,|\xi|^{2s}d\xi))$ to the corresponding terms of

$$\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')|\xi|^2} \widehat{f}(t',\xi)dt'.$$

□

*Remark* 4.3. Notice that applying the Fourier transform to (4.7) we get

$$u(t) = e^{t\triangle}u_0 + \int_0^t e^{(t-t')\triangle} f(t')dt'. \tag{4.13}$$

The following theorem yields additional estimates.

**Theorem 4.4.** *Let $f$ be like in Theorem 4.2 and consider the corresponding solution*

$$u\in C([0,T],\dot{H}^s), \quad \nabla u\in L^2([0,T],\dot{H}^s).$$

*Then, additionally, we have*

$$\|u(t)\|_{\dot{H}^{s+\frac{2}{p}}} \in L^p([0,T],\mathbb{R}) \text{ for any } p\ge 2. \tag{4.14}$$

*Moreover we have*

$$\begin{aligned} &V(t) := \left( \int_{\mathbb{R}^d} |\xi|^{2s} \left( \sup_{0\le t'\le t} |\widehat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}} \le \|u_0\|_{\dot{H}^s} + \frac{1}{2^{\frac{1}{2}}} \|f\|_{L^2([0,t],\dot{H}^{s-1})} ; \\ &\| \|u\|_{\dot{H}^{s+\frac{2}{p}}} \|_{L^p(0,T)} \le \left( \|u_0\|_{\dot{H}^s} + \|f\|_{L^2([0,T],\dot{H}^{s-1})} \right) . \end{aligned} \tag{4.15}$$

*Proof.* From the Duhamel formula (4.7) and the previous computation

$$|\widehat{u}(t,\xi)| \le e^{-t|\xi|^2}|\widehat{u}_0(\xi)| + \frac{1}{\sqrt{2}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

so that

$$|\xi|^s \sup_{0\le t'\le t} |\widehat{u}(t',\xi)| \le |\xi|^s|\widehat{u}_0(\xi)| + |\xi|^s \frac{1}{\sqrt{2}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

Taking the $L^2(\mathbb{R}^d, d\xi)$ norm we get

$$V(t) \leq \|u_0(\xi)\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)} + \frac{1}{\sqrt{2}} \|\widehat{f}\|_{L^2((0,t), L^2(\mathbb{R}^d, |\xi|^{2(s-1)} d\xi))}.$$

and this yields the 1st line in (4.15).
To get the 2nd line in (4.15), from the energy estimate (4.5) we obtain

$$\begin{aligned}
&\|u(t)\|^2_{\dot{H}^s} + \cancel{2} \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' \leq \|u_0\|^2_{\dot{H}^s} + 2 \int_0^t \frac{1}{\sqrt{\nu}} \|f(t')\|_{\dot{H}^{s-1}} \sqrt{\nu} \|\nabla u(t')\|_{\dot{H}^s} dt' \\
&\leq \|u_0\|^2_{\dot{H}^s} + \cancel{\int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt'} + \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'.
\end{aligned}$$

This yields

$$\|u(t)\|^2_{\dot{H}^s} + \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' \leq \|u_0\|^2_{\dot{H}^s} + \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'.$$

and hence

$$\begin{aligned}
&\|u\|_{L^\infty([0,T], \dot{H}^s)} \leq \|u_0\|_{\dot{H}^s} + \|f\|_{L^2([0,T], \dot{H}^s)} \\
&\| \|u\|_{\dot{H}^{s+1}} \|_{L^2(0,T)} \leq \|u_0\|_{\dot{H}^s} + \|f\|_{L^2([0,T], \dot{H}^s)}.
\end{aligned}$$

So by the interpolation of Sobolev norms Lemma 2.25 for $2 < p < \infty$

$$\begin{aligned}
&\| \|u\|_{\dot{H}^{s+\frac{2}{p}}} \|_{L^p(0,T)} \leq \| \|u\|^{1-\frac{2}{p}}_{\dot{H}^s} \|\nabla u\|^{\frac{2}{p}}_{\dot{H}^s} \|_{L^p(0,T)} \leq \|u\|^{1-\frac{2}{p}}_{L^\infty([0,T], \dot{H}^s)} \| \|\nabla u\|^{\frac{2}{p}}_{\dot{H}^s} \|_{L^p(0,T)} \\
&= \|u\|^{1-\frac{2}{p}}_{L^\infty([0,T], \dot{H}^s)} \|\nabla u\|^{\frac{2}{p}}_{L^2([0,T], \dot{H}^s)} \leq \|u_0\|_{\dot{H}^s} + \|f\|_{L^2([0,T], \dot{H}^s)}.
\end{aligned}$$

□

# 5 The heat equation, take 2

For this section see [14]. In this section pairs like $(q', q)$ of indexes will not be dual to each other.

**Proposition 5.1.** *Assume that*

$$\begin{cases} 1 \leq l \leq r \leq \infty, \quad 1 \leq l' \leq r' < \infty \\ \frac{d}{l} + \frac{2}{l'} \leq \frac{d}{r} + \frac{2}{r'} + 2 \end{cases} \tag{5.1}$$

*for $r' \neq l'$, or*

$$\begin{cases} 1 \leq l \leq r \leq \infty, \quad 1 \leq l' \leq r' < \infty \\ \frac{d}{l} < \frac{d}{r} + 2 \end{cases} \tag{5.2}$$

*for $l' = r'$ or, finally and if $r' = \infty$,*

$$\begin{cases} 1 \leq l \leq r \leq \infty, \quad 1 \leq l' \leq r' = \infty \\ \frac{d}{l} + \frac{2}{l'} < \frac{d}{r} + 2. \end{cases} \tag{5.3}$$

*Then there exists a fixed constant* $c(a,b,d,l,l',r,r')$ *s.t.*

$$\| \int_a^t e^{\triangle(t-t')} f dt' \|_{L^{r'} L^r((a,b)\times\mathbb{R}^d)} < c(a,b,d,l,l',r,r') \|f\|_{L^{l'} L^l((a,b)\times\mathbb{R}^d)}. \tag{5.4}$$

*Proof.* First of, by translation invariance we can always assume $[a,b]=[0,T]$.
For $r'<\infty$ and $l'<r'$ we have

$$\begin{aligned}
&\| \int_0^t e^{\triangle(t-t')} f dt' \|_{L_t^{r'} L_x^r} \le \| \int_0^t \|e^{\triangle(t-t')} f\|_{L_x^r} dt' \|_{L_t^{r'}} \\
&\| \lesssim \int_0^t (t-t')^{-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)} \|f\|_{L_x^l} dt' \|_{L_t^{r'}} \\
&\lesssim \|\chi_{[0,T]} t^{-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)}\|_{L^{\alpha,\infty}} \|f\|_{L^{l'}L^l} \text{ where } 1+\frac{1}{r'}=\frac{1}{\alpha}+\frac{1}{l'}
\end{aligned}$$

where we need $\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)\alpha \le 1$ for the above to hold. This is equivalent to

$$\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right) \le \frac{1}{\alpha} = 1+\frac{1}{r'}-\frac{1}{l'}$$

which in turn is equivalent to

$$\frac{d}{2l}+\frac{1}{l'} \le 1+\frac{1}{r'}+\frac{d}{2r},$$

equivalent to the condition in (5.1).
For $r'<\infty$ and $l'=r'$ we have by Young's convolution inequalities

$$\begin{aligned}
&\| \int_0^t e^{\triangle(t-t')} f dt' \|_{L_t^{r'} L_x^r} \lesssim \| \int_0^t \int (t-t')^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4(t-t')}} f(t',y) dy dt' \|_{L_t^{r'} L_x^r} \\
&\le \|t^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}\|_{L_t^1 L_x^\alpha} \|f\|_{L^{l'}L^l}
\end{aligned}$$

where $1+\frac{1}{r}=\frac{1}{\alpha}+\frac{1}{l}$. Now proceeding

$$\|t^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}\|_{L_t^1 L_x^\alpha} \|f\|_{L^{l'}L^l} \lesssim \|t^{-\frac{d}{2}+\frac{d}{2\alpha}}\|_{L_t^1[0,T]} \|f\|_{L^{l'}L^l} \lesssim \|f\|_{L^{l'}L^l}$$

where the latter makes sense exactly if $\frac{d}{2}-\frac{d}{2\alpha}<1$ which is equivalent

$$1-\frac{1}{\alpha} = 1-\left(1+\frac{1}{r}-\frac{1}{l}\right) < \frac{2}{d},$$

which gives (5.2). Finally, for $r'=\infty$, and for $t\in[0,T]$,

$$\begin{aligned}
&\| \int_0^t e^{\triangle(t-t')} f dt' \|_{L_x^r} \lesssim \int_0^t \|e^{\triangle(t-t')} f\|_{L_x^r} dt' \lesssim \int_0^t (t-t')^{-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)} \|f\|_{L_x^l} dt' \\
&\lesssim \|t^{-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)}\|_{L^{\frac{l'}{l'-1}}[0,T]} \|f\|_{L^{l'}L^l}
\end{aligned}$$

by the Hölder inequality, where the latter makes sense exactly if $\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)\frac{l'}{l'-1}<1$, that is

$$\frac{d}{l}-\frac{d}{r}<2\left(1-\frac{1}{l'}\right),$$

which coincides with (5.3). □

We will use an analogous version involving the gradient of $f$.

**Proposition 5.2.** *Assume that*

$$\begin{cases}1\le l\le r\le\infty, \quad 1\le l'\le r'<\infty\\ \frac{d}{l}+\frac{2}{l'}\le\frac{d}{r}+\frac{2}{r'}+1\end{cases} \tag{5.5}$$

*for $r'\neq l'$, or*

$$\begin{cases}1\le l\le r\le\infty, \quad 1\le l'\le r'<\infty\\ \frac{d}{l}<\frac{d}{r}+1\end{cases} \tag{5.6}$$

*for $l'=r'$ or, finally and if $r'=\infty$,*

$$\begin{cases}1\le l\le r\le\infty, \quad 1\le l'\le r'=\infty\\ \frac{d}{l}+\frac{2}{l'}<\frac{d}{r}+1.\end{cases} \tag{5.7}$$

*Then there exists a fixed constant $c(a,b,d,l,l',r,r')$ s.t.*

$$\|\int_a^t e^{\triangle(t-t')}\nabla f dt'\|_{L^{r'}L^r((a,b)\times\mathbb{R}^d)}<c(a,b,d,l,l',r,r')\|f\|_{L^{l'}L^l((a,b)\times\mathbb{R}^d)} \text{ for any } f\in L^l((a,b)\times\mathbb{R}^d). \tag{5.8}$$

*Proof.* Again, by translation invariance we can always assume $[a,b]=[0,T]$. For $r'<\infty$ and $l'<r'$ we have by Corollary 1.5

$$\begin{aligned}&\|\int_0^t e^{\triangle(t-t')}\nabla f dt'\|_{L_t^{r'}L_x^r}\le\|\int_0^t\|e^{\triangle(t-t')}\nabla f\|_{L_x^r}dt'\|_{L_t^{r'}}\\ &\|\lesssim\int_0^t(t-t')^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)}\|f\|_{L_x^l}dt'\|_{L_t^{r'}}\\ &\lesssim\|\chi_{[0,T]}t^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)}\|_{L^{\alpha,\infty}}\|f\|_{L^{l'}L^l} \text{ where } 1+\frac{1}{r'}=\frac{1}{\alpha}+\frac{1}{l'}\end{aligned}$$

where we need $\frac{\alpha}{2}+\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)\alpha\le1$ for the above to hold. This is equivalent to

$$\frac{1}{2}+\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)\le\frac{1}{\alpha}=1+\frac{1}{r'}-\frac{1}{l'}$$

which in turn is equivalent to

$$\frac{d}{l}+\frac{2}{l'}\le1+\frac{2}{r'}+\frac{d}{r},$$

that is the condition in (5.5).
For $r' < \infty$ and $l' = r'$ we have by Young's convolution inequalities

$$
\begin{aligned}
&\| \int_0^t e^{\triangle(t-t')} \nabla f dt' \|_{L_t^{r'} L_x^r} \lesssim \| \int_0^t \int (t-t')^{-\frac{d}{2}-\frac{1}{2}} \frac{x-y}{\sqrt{t-t'}} e^{-\frac{|x-y|^2}{4(t-t')}} f(t',y) dy dt' \|_{L_t^{r'} L_x^r} \\
&\leq \| t^{-\frac{d}{2}-\frac{1}{2}} \frac{x}{\sqrt{t}} e^{-\frac{|x|^2}{4t}} \|_{L_t^1 L_x^\alpha} \| f \|_{L^{l'} L^l}
\end{aligned}
$$

where $1 + \frac{1}{r} = \frac{1}{\alpha} + \frac{1}{l}$. Now

$$
\| t^{-\frac{d}{2}-\frac{1}{2}} \frac{x}{\sqrt{t}} e^{-\frac{|x|^2}{4t}} \|_{L_t^1 L_x^\alpha} \| f \|_{L^{l'} L^l} \lesssim \| t^{-\frac{d}{2}-\frac{1}{2}+\frac{d}{2\alpha}} \|_{L_t^1 [0,T]} \| f \|_{L^{l'} L^l} \lesssim \| f \|_{L^{l'} L^l}
$$

where the latter makes sense exactly if $\frac{d}{2} + \frac{1}{2} - \frac{d}{2\alpha} < 1$ which is equivalent

$$
d - d\left(1 + \frac{1}{r} - \frac{1}{l}\right) < 1,
$$

which gives (5.6). Finally, for $r' = \infty$, and for $t \in [0,T]$,

$$
\begin{aligned}
&\| \int_0^t e^{\triangle(t-t')} \nabla f dt' \|_{L_x^r} \lesssim \int_0^t \| e^{\triangle(t-t')} \nabla f \|_{L_x^r} dt' \lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)} \| f \|_{L_x^l} dt' \\
&\lesssim \| t^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{l}-\frac{1}{r}\right)} \|_{L^{\frac{l'}{l'-1}}[0,T]} \| f \|_{L^{l'} L^l}
\end{aligned}
$$

by the Hölder inequality, where the latter makes sense exactly if $\left[\frac{d}{2}\left(\frac{1}{l} - \frac{1}{r}\right) + \frac{1}{2}\right] \frac{l'}{l'-1} < 1$, that is

$$
\frac{d}{l} - \frac{d}{r} + 1 < 2\left(1 - \frac{1}{l'}\right),
$$

which coincides with (5.7). □

Later we will consider parabolic cylinders for $d = 3$.

**Definition 5.3** (Parabolic cylinders)**.** Given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$ for any $R > 0$ we will denote by $Q_R^*(t_0, x_0)$ the set

$$
Q_R^*(t_0, x_0) = \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}\right) \times B_R(x_0)
$$

and with $Q_R(t_0, x_0)$ the set

$$
Q_R(t_0, x_0) = \left(t_0 - R^2, t_0\right) \times B_R(x_0).
$$

Notice the relation

$$
Q_R^*(t_0, x_0) = Q_R\left(t_0 + \frac{R^2}{2}, x_0\right). \tag{5.9}
$$

We will focus especially on the cylinders $Q_R(t_0, x_0)$. We will write $Q_R = Q_R(0,0)$.

**Proposition 5.4.** *Let $f \in L^2(Q_R(t_0,x_0))$ such that such $f$ vanishes outside $\overline{Q}_{\rho_s R}(t_0,x_0)_{\rho_s}$ for an $\rho_s \in (0,1)$ Consider the equation $W_t - \triangle W = \partial_j f$ in $(t_0 - R^2, 0) \times \mathbb{R}^3$ and its restriction in $Q_R(t_0,x_0)$. Assume that $W(-1,x) \equiv 0$. Then, for any $\rho_i \in (0,\rho_s)$:*

1. $f \in L^\infty_t L^\infty_x(Q_R(t_0,x_0)) \Rightarrow W \in L^\infty_t C^{0,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ *for any* $\alpha \in (0,1)$;
2. $f \in L^\infty_t W^{k,\infty}_x(Q_R(t_0,x_0))) \Rightarrow W \in L^\infty_t C^{k,\alpha}_x(\overline{Q}_{\rho_i R}(t_0,x_0))$ *for any* $\alpha \in (0,1)$;
3. $f \in L^\infty_t C^{0,\alpha}_x(Q_R(t_0,x_0))$ *for an* $\alpha \in (0,1) \Rightarrow \nabla W \in L^\infty(\overline{Q}_{\rho_i R}(t_0,x_0))$;
4. $f \in L^\infty_t C^{k,\alpha}_x(Q_R(t_0,x_0)))$ *for an* $\alpha \in (0,1) \Rightarrow \nabla^{k+1} W \in L^\infty(\overline{Q}_{\rho_i R}(t_0,x_0))$.

*Proof.* By scaling and translation we reduce to the case $Q_R(t_0,x_0) = Q_1$.
Notice that Theorem 4.2 guarantees that the existence and uniqueness of a solution $W \in L^\infty((-1,0), L^2(\mathbb{R}^3)) \cap L^2((-1,0), \dot{H}^1(\mathbb{R}^3))$.
Next, it is enough to prove the 1st and 3rd claim. Let us start with the 1st claim. First of all

$$\begin{aligned}
W(t,x) &= \int_{-1}^t \nabla e^{(t-s)\triangle} f ds \\
&= (4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} (t-s)^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \partial_j f(s,y) dy \\
&= (4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{x_j - y_j}{2(t-s)^{1+\frac{3}{2}}} f(s,y) dy.
\end{aligned}$$

Now, by Corollary 1.5 we have

$$\begin{aligned}
&\|W\|_{L^\infty(Q_1)} \leq \| \int_{-1}^t \nabla e^{(t-s)\triangle} f ds \|_{L^\infty((-1,0)\times\mathbb{R}^3)} \\
&\leq \int_{-1}^t \|\nabla e^{(t-s)\triangle} f\|_{L^\infty((-1,0)\times\mathbb{R}^3)} ds \lesssim \int_{-1}^t (t-s)^{-1/2} ds \|f\|_{L^\infty(Q_1)} \lesssim \|f\|_{L^\infty(Q_1)}.
\end{aligned}$$

Next, we write

$$W(t,x) - W(t,z) = (4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} \left[ e^{-\frac{|x-y|^2}{4(t-s)}} \frac{x_j - y_j}{2(t-s)^{\frac{5}{2}}} - e^{-\frac{|z-y|^2}{4(t-s)}} \frac{z_j - y_j}{2(t-s)^{\frac{5}{2}}} \right] f(s,y) dy.$$

Introducing

$$\xi = \frac{x}{(t-s)^{\frac{1}{2}}}, \quad \eta = \frac{y}{(t-s)^{\frac{1}{2}}}, \quad \rho = \frac{z}{(t-s)^{\frac{1}{2}}}$$

we have

$$\begin{aligned}
&W(t,x) - W(t,z) \\
&= 2^{-1}(4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} (t-s)^{-1/2} \left[ e^{-\frac{|\xi-\eta|^2}{4}} (\xi_j - \eta_j) - e^{-\frac{|\rho-\eta|^2}{4}} (\rho_j - \eta_j) \right] f(s,(t-s)^{1/2}\eta) d\eta.
\end{aligned}$$

Then

$$\frac{W(t,x)-W(t,z)}{|x-z|^\alpha}$$

$$= 2^{-1}(4\pi)^{-\frac{3}{2}}\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\frac{e^{-\frac{|\xi-\eta|^2}{4}}(\xi_j-\eta_j)-e^{-\frac{|\rho-\eta|^2}{4}}(\rho_j-\eta_j)}{|\xi-\rho|^\alpha}f(s,(t-s)^{1/2}\eta)d\eta.$$

Now split the domain of integration in two parts. In the first part $|\xi-\rho|\geq 1$. When this holds we bound the integral by

$$\begin{aligned}&C\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\left(e^{-\frac{|\xi-\eta|^2}{4}}|\xi-\eta|+e^{-\frac{|\rho-\eta|^2}{4}}|\rho-\eta|\right)d\eta\|f\|_{L^\infty_{t,x}}\\ &\leq 2C\int_{-1}^{t} ds(t-s)^{-1/2-\alpha/2}\int_{\mathbb{R}^3}e^{-\frac{|\eta|^2}{4}}|\eta|d\eta\|f\|_{L^\infty_{t,x}}.\end{aligned}$$

In the region where $|\xi-\rho|<1$ we bound from above the integral by

$$\begin{aligned}&\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\frac{\left|e^{-\frac{|\xi-\eta|^2}{4}}(\xi_j-\eta_j)-e^{-\frac{|\rho-\eta|^2}{4}}(\rho_j-\eta_j)\right|}{|\xi-\rho|}|f(s,(t-s)^{1/2}\eta)|d\eta\\ &\leq\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\sup_{\tau\in[0,1]}\left|\partial_\tau\left(e^{-\frac{|\rho+\tau(\xi-\rho)-\eta|^2}{4}}(\rho_j+\tau(\xi_j-\rho_j)-\eta_j)\right)\right|d\eta\|f\|_{L^\infty_{t,x}}\\ &\leq\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\sup_{\tau\in[0,1]}\left(e^{-\frac{|\tau(\xi-\rho)-\eta|^2}{4}}+e^{-\frac{|\tau(\xi-\rho)-\eta|^2}{4}}|\tau(\xi-\rho)-\eta|^2\right)d\eta\|f\|_{L^\infty_{t,x}}\\ &\leq\int_{-1}^{t} ds\int_{|\eta|\leq 2}(t-s)^{-1/2-\alpha/2}(1+9)\,d\eta\|f\|_{L^\infty_{t,x}}\\ &+\int_{-1}^{t} ds\int_{\mathbb{R}^3}(t-s)^{-1/2-\alpha/2}\left(e^{-\frac{|\eta|^2}{8}}+2e^{-\frac{|\eta|^2}{8}}|\eta|^2\right)d\eta\|f\|_{L^\infty_{t,x}}\leq C\|f\|_{L^\infty_{t,x}}.\end{aligned}$$

We now consider the 3rd statement. For $\epsilon>0$ we consider

$$W_\epsilon(t,x)=\int_{-1}^{t-\epsilon} ds\int_{\mathbb{R}^3}e^{-\frac{|x-y|^2}{4(t-s)}}\frac{x_j-y_j}{(t-s)^{1+\frac{3}{2}}}f(s,y)dy.$$

Then

$$\begin{aligned}\partial_k W_\epsilon(t,x)&=\int_{-1}^{t-\epsilon} ds\int_{\mathbb{R}^3}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\frac{\delta_{jk}}{(t-s)^{1+\frac{3}{2}}}-\frac{(x_j-y_j)(x_k-y_k)}{2(t-s)^{2+\frac{3}{2}}}\right]f(s,y)dy\\ &=\int_{-1}^{t-\epsilon} ds\int_{\mathbb{R}^3}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\frac{\delta_{jk}}{(t-s)^{1+\frac{3}{2}}}-\frac{(x_j-y_j)(x_k-y_k)}{2(t-s)^{2+\frac{3}{2}}}\right]|x-y|^\alpha\frac{f(s,y)-f(s,x)}{|x-y|^\alpha}dy.\end{aligned}$$

So, using

$$\xi=\frac{x}{(t-s)^{\frac{1}{2}}}\,,\qquad \eta=\frac{y}{(t-s)^{\frac{1}{2}}}\,,$$

$$|\partial_k W_\epsilon(t,x)| \le \|f\|_{L^\infty((-1,0),C_x^{0,\alpha})} \int_{-1}^{t-\epsilon} ds \int_{\mathbb{R}^3} e^{-\frac{|\xi-\eta|^2}{4}} \left|\delta_{jk} - 2^{-1}(\xi_j-\eta_j)(\xi_k-\eta_k)\right| \frac{|\xi-\eta|^\alpha}{(t-s)^{1-\alpha/2}} d\eta$$
$$\le C\|f\|_{L^\infty((-1,0),C_x^{0,\alpha})} \text{ for } C = \sup_{t\in(-1,0)} \int_{-1}^{t} ds \int_{\mathbb{R}^3} e^{-\frac{|\eta|^2}{4}} \left(1+|\eta|^2\right) \frac{|\eta|^\alpha}{(t-s)^{1-\alpha/2}} d\eta.$$

□

**Proposition 5.5.** *Assume that $W_t - \triangle W = f$ in $Q_R(t_0,x_0)$ and $W(t_0-R^2)\equiv 0$. Assume $f$ vanishes outside $\overline{Q}_{\rho_s R}(t_0,x_0)$ for an $\rho_s\in(0,1)$. Then, for any $\rho_i\in(0,\rho_s)$:*

1. *$f\in L^\infty L^\infty(Q_R(t_0,x_0)) \Rightarrow W\in L_t^\infty C_x^{1,\alpha}(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha\in(0,1)$;*
2. *$f\in L^\infty W^{k,\infty}(Q_R(t_0,x_0)) \Rightarrow W\in L_t^\infty C_x^{k+1,\alpha}(\overline{Q}_{\rho_i R}(t_0,x_0))$ for any $\alpha\in(0,1)$;*
3. *$f\in L^\infty C^{0,\alpha}(Q_R(t_0,x_0))$ for an $\alpha\in(0,1) \Rightarrow \nabla^2 W\in L_t^\infty L_x^\infty(\overline{Q}_{\rho_i R}(t_0,x_0))$;*
4. *$f\in L^\infty C^{k,\alpha}(Q_R(t_0,x_0))$ for an $\alpha\in(0,1) \Rightarrow \nabla^{k+2} W\in L_t^\infty L_x^\infty(\overline{Q}_{\rho_i R}(t_0,x_0))$.*

*Proof.* The proof is similar to the previous one. It is enough to prove the 1st and 3rd claim. Let us start with the 1st claim. First of all, by Corollary 1.5

$$\|W\|_{L^\infty(Q_1)} \le \|\int_{-1}^t e^{(t-s)\triangle} f ds\|_{L^\infty((-1,0)\times\mathbb{R}^3)}$$
$$\le \int_{-1}^t \|e^{(t-s)\triangle} f\|_{L^\infty((-1,0)\times\mathbb{R}^3)} ds \lesssim \int_{-1}^t ds \|f\|_{L^\infty(Q_1)} = \|f\|_{L^\infty(Q_1)}.$$

Next

$$W(t,x)-W(t,z) = (4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} (t-s)^{-\frac{3}{2}} \left[e^{-\frac{|x-y|^2}{4(t-s)}} - e^{-\frac{|z-y|^2}{4(t-s)}}\right] f(s,y) dy$$

and, differentiating,

$$\partial_j W(t,x)-\partial_j W(t,z) = (4\pi)^{-\frac{3}{2}} \int_{-1}^t ds \int_{\mathbb{R}^3} \left[e^{-\frac{|x-y|^2}{4(t-s)}} \frac{x_j-y_j}{2(t-s)^{\frac{5}{2}}} - e^{-\frac{|z-y|^2}{4(t-s)}} \frac{z_j-y_j}{2(t-s)^{\frac{5}{2}}}\right] f(s,y) dy,$$

so that we get in the r.h.s. the exact same quantity discussed in the 1st claim of Proposition 5.4 and the same exact proof holds yielding the 1st claim.

Next, we consider like before

$$W_\epsilon(t,x) = \int_{-1}^{t-\epsilon} ds \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{x_j-y_j}{(t-s)^{\frac{3}{2}}} f(s,y) dy.$$

Then

$$\partial_j\partial_k W_\epsilon(t,x) = \int_{-1}^{t-\epsilon} ds \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \frac{\delta_{jk}}{(t-s)^{1+\frac{3}{2}}} - \frac{(x_j-y_j)(x_k-y_k)}{2(t-s)^{2+\frac{3}{2}}} \right] f(s,y)dy$$

$$= \int_{-1}^{t-\epsilon} ds \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \frac{\delta_{jk}}{(t-s)^{1+\frac{3}{2}}} - \frac{(x_j-y_j)(x_k-y_k)}{2(t-s)^{2+\frac{3}{2}}} \right] |x-y|^\alpha \frac{f(s,y)-f(s,x)}{|x-y|^\alpha} dy$$

is exactly the formula used in the proof of the 3rd claim of Proposition 5.4.

□

# 6 The Navier Stokes equation

We will only deal with the Incompressible Navier Stokes (NS) equation:

$$\begin{cases} u_t + u\cdot\nabla u - \triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad\qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{6.1}$$

where $u:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ with $u=\sum_{j=1}^d u^j e_j$ with $e_j$ the standard basis of $\mathbb{R}^d$,

$$\triangle := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}\,,\ \nabla\cdot u = \sum_{j=1}^d \frac{\partial}{\partial x_j} u^j\,,\ u\cdot\nabla v = \sum_{j=1}^d u_j \frac{\partial}{\partial x_j} v.$$

$p$ is the pressure and it is simply serves the purpose to absorb the divergence part of the l.h.s. of (6.1).

We can write

$$u\cdot\nabla u = \mathrm{div}(u\otimes u) \text{ for } \mathrm{div}(u\otimes v)^j := \sum_{k=1}^d \partial_k(u^k v^j) \text{ since} \tag{6.2}$$

$$\mathrm{div}(u\otimes u)^j = \sum_{k=1}^d \partial_k(u^k u^j) = \sum_{k=1}^d u^k\partial_k u^j + u^j \underbrace{\mathrm{div} u}_{0} = u\cdot\nabla u^j$$

So we rewrite (6.1) and

$$\begin{cases} u_t + \mathrm{div}(u\otimes u) - \triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad\qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{6.3}$$

**Definition 6.1** (Weak solutions)**.** Let $u_0$ be in $L^2(\mathbb{R}^d)$. A vector field $u\in L^2_{loc}([0,\infty)\times\mathbb{R}^d)$ which is weakly continuous as a function from $[0,\infty)$ to $L^2(\mathbb{R}^d,\mathbb{R}^d)$ (we will write $u\in C^0_w([0,\infty),L^2(\mathbb{R}^d,\mathbb{R}^d))$, and what we mean is that $t\longrightarrow\langle u(t),\phi\rangle_{L^2}\in C^0([0,\infty),\mathbb{R})$

for any $\phi \in L^2(\mathbb{R}^d, \mathbb{R}^d)$) and s.t. $\text{div} u(t) = 0$ for every $t$, is a weak solution of (6.3) if for $\Psi \in C_c^\infty([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$ with $\text{div}\Psi = 0$ we have

$$\begin{aligned}\langle u(t), \Psi(t)\rangle_{L^2} = \int_0^t \big(&\langle u(t'), \triangle \Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t \Psi(t')\rangle_{L^2} \\ &-\langle \text{div}(u \otimes u)(t'), \Psi(t')\rangle_{L^2}\big)\, dt' + \langle u_0, \Psi(0)\rangle_{L^2}.\end{aligned} \tag{6.4}$$

*Remark* 6.2. Notice that in Definition 6.1 we could replace the half–line $[0,\infty)$ with a half–line $[t_0,\infty)$ with $t_0 \in \mathbb{R}$. In this sense, observe that any solution in Definition 6.1 solves weekly the NS equation in $[t_0,\infty)$ for $t_0 > 0$ and initial value $u(t_0)$, that is to say, for any for $\Psi \in C_c^\infty([t_0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$ with $\text{div}\Psi = 0$ we have

$$\begin{aligned}\langle u(t), \Psi(t)\rangle_{L^2} = \int_{t_0}^t \big(&\langle u(t'), \triangle \Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t \Psi(t')\rangle_{L^2} \\ &-\langle \text{div}(u \otimes u)(t'), \Psi(t')\rangle_{L^2}\big)\, dt' + \langle u(t_0), \Psi(t_0)\rangle_{L^2}.\end{aligned} \tag{6.5}$$

Indeed, we can extend any such test function into a $\Psi \in C_c^\infty([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$ with $\text{div}\Psi = 0$. Then taking the difference of (6.4) and

$$\begin{aligned}\langle u(t_0), \Psi(t_0)\rangle_{L^2} = \int_0^{t_0} \big(&\langle u(t'), \triangle \Psi(t')\rangle_{L^2} + \langle u(t'), \partial_t \Psi(t')\rangle_{L^2} \\ &-\langle \text{div}(u \otimes u)(t'), \Psi(t')\rangle_{L^2}\big)\, dt' + \langle u_0, \Psi(0)\rangle_{L^2},\end{aligned}$$

we obtain exactly (6.5).

Let us now formally take the inner product of the first line of (6.1) with $u$ and integrate in $\mathbb{R}^d$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \langle u \cdot \nabla u, u\rangle_{L^2} - \langle \triangle u, u\rangle_{L^2} = -\langle \nabla p, u\rangle_{L^2}$$

We have, summing on repeated indexes,

$$\langle u \cdot \nabla u, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j u^k \partial_j u^k dx = 2^{-1} \int_{\mathbb{R}^d} u^j \partial_j (u^k u^k)^2 dx = -2^{-1} \int_{\mathbb{R}^d} |u|^2 \text{div} u \, dx = 0 \text{ and}$$

$$\langle \nabla p, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j \partial_j p dx = -\int_{\mathbb{R}^d} p \text{div} u \, dx = 0.$$

So, formally (rigorously if $u$ is regular and we can integrate by parts), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = 0$$

This in particular yields the following *energy equality*

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' = \|u_0\|_{L^2(\mathbb{R}^d)}^2. \tag{6.6}$$

**Theorem 6.3** (Leray)**.** *Let $u_0 \in L^2(\mathbb{R}^d)$ for $d = 2, 3$ be divergence free. Then* (6.3) *admits a weak solution with $u(t) \in L^\infty(\mathbb{R}_+, H) \cap L^2_{loc}(\mathbb{R}_+, V)$ such that the following energy inequality holds:*

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{6.7}$$

The proof of Theorem 6.3 is long and will be considered later.

*Remark* 6.4. Theorem 6.3, along with other results on more regular solutions, was published originally by Leray [9], in 1934, before the appearance of the notions of distribution [16] and Sobolev space [17]. A presentation in a modern framework is in Ozanski–Poonen [11], which is freely available in https://arxiv.org/abs/1708.09787 .

Now we consider the following.

**Theorem 6.5** (Case $d = 2$)**.** *When $d = 2$ the solution in Theorem 6.3 is unique, it satisfies* (6.6) *and $u(t) \in C^0([0, \infty), L^2)$.*

Theorem 6.5 depends on Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$. Furthermore, we will use the following lemma.

**Lemma 6.6.** *There exists a constant $C = C_T$ such that for any $u \in L^2((0, T), H^1(\mathbb{R}^d)) \cap H^1((0, T), H^{-1}(\mathbb{R}^d))$ we have $u \in C^0([0, T], L^2(\mathbb{R}^d))$ with*

$$\|u\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \le C\left(\|u\|_{L^2((0,T),H^1(\mathbb{R}^d))} + \|\dot{u}\|_{L^2((0,T),H^{-1}(\mathbb{R}^d))}\right). \tag{6.8}$$

*Furthermore we have $\|u(t)\|^2_{L^2} \in AC([0, T])$ with*

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\langle u(t), \dot{u}(t)\rangle. \tag{6.9}$$

*Proof.* Let us assume additionally that $u \in C^1([0, T], L^2(\mathbb{R}^d))$. Then for any fixed $t_0 \in [0, T]$ we have

$$\begin{aligned} \|u(t)\|^2_{L^2} &= \|u(t_0)\|^2_{L^2} + 2\int_{t_0}^t \langle u(s), \dot{u}(s)\rangle\, ds \\ &\le \|u(t_0)\|^2_{L^2} + \|u\|^2_{L^2((0,T),H^1(\mathbb{R}^d))} + \|\dot{u}\|^2_{L^2((0,T),H^{-1}(\mathbb{R}^d))}. \end{aligned} \tag{6.10}$$

We can choose $\|u(t_0)\|^2_{L^2} = T^{-1}\int_0^T \|u(s)\|^2_{L^2} ds$ obtaining (6.8) for $C = \sqrt{1 + T^{-1}}$
The general case is obtained by considering a sequence $(u_n)$ in $C^1([0, T], H^1(\mathbb{R}^d))$ converging to $u$ in $L^2((0, T), H^1(\mathbb{R}^d)) \cap H^1((0, T), H^{-1}(\mathbb{R}^d))$. To get such a sequence, we can extend appropriately $u$ into a function in $L^2(\mathbb{R}, H^1(\mathbb{R}^d)) \cap H^1(\mathbb{R}, H^{-1}(\mathbb{R}^d))$, and then we can consider $u_n = \rho_{\epsilon_n} * u$ with $\epsilon_n \xrightarrow{n\to\infty} 0$. Then this sequence satisfies the desired properties.
Then (6.8) implies that $(u_n)$ is a Cauchy sequence in $C^0([0, T], L^2(\mathbb{R}^d))$. The limit is necessarily $u$, which satisfies (6.8). Also by a limit, we conclude that $u$ satisfies the equality in (6.10), for any fixed $t_0 \in [0, T]$. This implies $\|u(t)\|^2_{L^2} \in AC([0, T])$ and formula (6.9).

□

*Proof of uniqueness and of $u(t) \in C^0([0,\infty), L^2)$ in 2d.* We first claim that for any $d = 2$ solution we have

$$\partial_t u \in L^2((0,T), H^{-1}(\mathbb{R}^2, \mathbb{R}^2)) \text{ for any } T > 0. \tag{6.11}$$

Let us assume this for the moment. Since from (6.7) we have $u \in L^2((0,T), H^1(\mathbb{R}^2, \mathbb{R}^2))$, then $u \in L^2((0,T), H^1(\mathbb{R}^2, \mathbb{R}^2)) \cap H^1((0,T), H^{-1}(\mathbb{R}^2, \mathbb{R}^2))$ for any $T > 0$. By Lemma 6.6 we have $u \in C^0([0,T], L^2)$ for any $T > 0$, and so $u(t) \in C^0([0,\infty), L^2)$.
We now assume that there are two solutions $u$ and $v$ with $u(0) = v(0)$ and we set $w := u - v$. Both $u$ and $v$ satisfy (6.4). We claim that we can take as test function $w$, obtaining

$$\begin{aligned}
\langle u(t), w(t)\rangle &= \int_0^t (-\langle \nabla u, \nabla w\rangle + \langle u, \partial_t w\rangle - \langle \operatorname{div}(u \otimes u), w\rangle_{L^2})\, dt' \text{ and} \\
\langle v(t), w(t)\rangle &= \int_0^t (-\langle \nabla v, \nabla w\rangle + \langle v, \partial_t w\rangle - \langle \operatorname{div}(v \otimes v), w\rangle_{L^2})\, dt'
\end{aligned} \tag{6.12}$$

To prove the claim, notice that there exists a sequence of test functions $\Psi_n$ which converges to $w$ in

$$L^2((0,T), H^1) \cap H^1((0,T), H^{-1}) \cap C([0,T], L^2).$$

This implies that (6.4) with the $\Psi_n$ converge to the above formulas, where we have taken in account $w(0) = 0$ and where we used also estimates like, see Lemma 6.7 below,

$$\begin{aligned}
\int_0^t \langle \operatorname{div}(u \otimes u)(t'), w(t')\rangle_{L^2} dt' &\le C \int_0^t \|\nabla u(t')\|_{L^2} \|u(t')\|_{L^2} \|\nabla w(t')\|_{L^2} dt' \\
&\le C \|\nabla u\|_{L^2((0,t),L^2)} \|\nabla w\|_{L^2((0,t),L^2)} \|u\|_{L^\infty((0,t),L^2)},
\end{aligned}$$

and an analogous one for the other nonlinear term.
Taking the difference of the two formulas in (6.12), we obtain

$$\|w(t)\|_{L^2}^2 = \int_0^t \left(-\|\nabla w(t')\|_{L^2}^2 + \langle w(t'), \partial_t w(t')\rangle - \langle \operatorname{div}(u \otimes u)(t') + \operatorname{div}(v \otimes v)(t'), w(t')\rangle_{L^2}\right) dt'.$$

Formula (6.11) for any solution and Lemma 6.6 imply $\|w(t)\|_{L^2}^2 \in AC([0,T])$ with $\frac{d}{dt}\|w(t)\|_{L^2}^2 = 2\langle w(t), \partial_t w(t)\rangle$. Hence

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 = \langle \operatorname{div}(v \otimes v) - \operatorname{div}(u \otimes u), w\rangle = \\
&= -\langle \operatorname{div}(w \otimes v) - \operatorname{div}(u \otimes w), w\rangle = -\langle \partial_k(w^k v^j) - \partial_k(u^k w^j), w^j\rangle \\
&= -\left\langle \partial_k v^j, w^k w^j\right\rangle \le \|\nabla v\|_{L^2} \|w\|_{L^4}^2 \le c\|\nabla v\|_{L^2} \|w\|_{L^2} \|\nabla w\|_{L^2} \\
&\le c^2 \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2,
\end{aligned}$$

where in the 3rd line we applied Gagliardo Nirenberg in dimension 2. From the last formula we obtain

$$\frac{d}{dt}\|w\|_{L^2}^2 \leq 2c^2\|\nabla v\|_{L^2}^2\|w\|_{L^2}^2 \text{ which by Gronwall yields}$$
$$\|w\|_{L^2}^2 \leq e^{2c^2\int_0^t \|\nabla v(t')\|_{L^2}^2 dt'}\|w(0)\|_{L^2}^2 = 0.$$

To complete the proof we need to prove claim (6.11). We apply (6.4) for $\Psi(t,x) = \phi(x) \in C^\infty_{c\sigma}(\mathbb{R}^2,\mathbb{R}^2)$ and obtain

$$\langle u(t),\phi\rangle - \langle u(0),\phi\rangle = \int_0^t \left(\langle \triangle u(t'),\phi\rangle\rangle - \langle \mathbb{P}\mathrm{div}(u\otimes u)(t'),\phi\rangle\right) dt.$$

The above formula extends to any $\phi \in H^1(\mathbb{R}^2,\mathbb{R}^2)$.
We want to use Lemma A.29, which states that if $u, g \in L^1(I,X)$ are such that

$$\langle u(t_2), f\rangle_{XX^*} - \langle u(t_1), f\rangle_{XX^*} = \int_{t_1}^{t_2} \langle g(s), f\rangle_{XX^*}\, ds \text{ for any } f \in X^*,$$

with $X$ a Banach space, then $\partial_t u = g$ in $\mathcal{D}'(I,X) := \mathcal{L}(\mathcal{D}(I,\mathbb{R}),X)$.
Here we apply Lemma A.29 taking $X = H^{-1}(\mathbb{R}^2,\mathbb{R}^2)$ and its dual $X^* = H^1(\mathbb{R}^2,\mathbb{R}^2)$. Obviously, we have

$$\|\triangle u\|_{L^1((0,T),H^{-1})} \leq \sqrt{T}\|u\|_{L^2((0,T),H^1)}.$$

Notice that the above inequality does not depend on the dimension. The treatment of the nonlinear terms, depends on the dimension and is based on $\dot H^{\frac12}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, which depends on the dimension, and is

$$\|\mathbb{P}\mathrm{div}(u\otimes u)\|_{L^1((0,T),H^{-1})} \leq \sqrt{T}\|u\otimes u\|_{L^2((0,T),L^2)} = \sqrt{T}\|\|u\|_{L^4}^2\|_{L^2(0,T)} \lesssim \sqrt{T}\|\|u\|_{\dot H^{\frac12}}^2\|_{L^2(0,T)}$$
$$\leq \sqrt{T}\|u\|_{L^\infty((0,T),L^2)}\|\nabla u\|_{L^2((0,T),L^2)},$$

where in the last inequality we used the interpolation $\|u\|_{\dot H^{\frac12}}^2 \leq \|u\|_{L^2}\|\nabla u\|_{L^2}$.

So we can apply Lemma A.29 obtaining that

$$\partial_t u = -\triangle u + \mathbb{P}\mathrm{div}(u\otimes u) \text{ in } \mathcal{D}'((0,T),H^{-1})$$

and furthermore that (6.11) is true.

□

Notice that if we apply formally the operator $\mathbb{P}$ to equation (6.3) we obtain formally

$$\begin{cases} u_t - \triangle u = \mathcal{Q}_{NS}(u,u) \\ u(0,x) = u_0(x) \end{cases} (t,x) \in [0,\infty)\times\mathbb{R}^d \tag{6.13}$$

where we set

$$\mathcal{Q}_{NS}(u,v) := -\frac12\mathbb{P}(\mathrm{div}(u\otimes v)) - \frac12\mathbb{P}(\mathrm{div}(v\otimes u)). \tag{6.14}$$

Here notice that

$$\mathbb{P}(\mathrm{div}(u\otimes v))^j = \sum_{l=1}^d \partial_l\left((u^l v^j) - \frac{1}{\triangle}\sum_{k=1}^d \partial_j\partial_k(u^l v^k)\right). \tag{6.15}$$

## 6.1 Proof of Theorem 6.3

We will need the following elementary lemma.

**Lemma 6.7.** *Let $d = 2, 3$. Then the trilinear form*

$$(u, v, \varphi) \in (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \to \langle div(u \otimes v), \varphi \rangle_{L^2} \in \mathbb{R} \tag{6.16}$$

*extends into a unique bounded trilinear form $(H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d$ which satisfies for a fixed $C$*

$$\langle div(u \otimes v), \varphi \rangle_{L^2} \le C \|\nabla u\|_{L^2}^{\frac{d}{4}} \|\nabla v\|_{L^2}^{\frac{d}{4}} \|u\|_{L^2}^{1-\frac{d}{4}} \|v\|_{L^2}^{1-\frac{d}{4}} \|\nabla \varphi\|_{L^2} \tag{6.17}$$

*If furthermore $div\ u = 0$ then*

$$\langle div(u \otimes v), v \rangle_{L^2} = 0. \tag{6.18}$$

*Proof.* Recall that from (6.2) we have $\text{div}(u \otimes v)^j := \sum_{k=1}^d \partial_k (u^k v^j)$. Then for fields like in (6.16) we have

$$\langle \text{div}(u \otimes v), \varphi \rangle_{L^2} = \sum_{j=1}^d \langle \text{div}(u \otimes v)^j, \varphi^j \rangle_{L^2} = \sum_{j=1}^d \langle \sum_{k=1}^d \partial_k (u^k v^j), \varphi^j \rangle_{L^2} = -\sum_{j=1}^d \sum_{k=1}^d \langle u^k v^j, \partial_k \varphi^j \rangle_{L^2}.$$

Now the r.h.s. can be bounded by

$$|\langle u^k v^j, \partial_k \varphi^j \rangle_{L^2}| \le \|u^k v^j\|_{L^2} \|\nabla \varphi\|_{L^2} \le \|u^k\|_{L^4} \|v^j\|_{L^4} \|\nabla \varphi\|_{L^2}.$$

Finally, we apply Gagliardo-Nirenberg inequality writing

$$\|u^k\|_{L^4} \le C \|\nabla u^k\|_{L^2}^{\frac{d}{4}} \|u^k\|_{L^2}^{1-\frac{d}{4}}.$$

The same holds for $v^j$. Then we obtain (6.17), obviously with a different $C$. This implies that the form in (6.16) is continuous and, by density of $C_c^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$, it extends in a unique way.
Next, we write for $\varphi = v$

$$\begin{aligned} \langle \text{div}(u \otimes v), v \rangle_{L^2} &= -\sum_{j=1}^d \sum_{k=1}^d \langle u^k v^j, \partial_k v^j \rangle_{L^2} \\ &= -2^{-1} \sum_{j=1}^d \sum_{k=1}^d \langle u^k, \partial_k (v^j)^2 \rangle_{L^2} = 2^{-1} \sum_{j=1}^d \langle (\text{div} u) v^j, v^j \rangle_{L^2} = 0. \end{aligned}$$

Notice that this formal computation (the Leibnitz rule used for the 2nd equality requires some explaining) is certainly rigorous for $v \in (C_c^\infty(\mathbb{R}^d))^d$. On the other hand inequality (6.17) yields (6.18) by a density argument also for $v \in (H^1(\mathbb{R}^d))^d$. $\square$

We consider now a sort of regularization of the NS equation. In the sequel we consider only case $\boxed{d=3}$.

Using a smooth mollificator $\rho \in C_c^\infty(\mathbb{R}^d,[0,1])$ s.t. $\int \rho(x)dx = 1$ and with $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$, we consider

$$\begin{cases} u_t - \triangle u = -\mathbb{P}\left(\rho_\varepsilon * u \cdot \nabla u\right) \\ \qquad u(0) = \rho_\varepsilon * u_0. \end{cases} \tag{6.19}$$

If we are in the framework of Theorem 4.2, then

$$u = e^{t\triangle}\rho_\varepsilon * u_0 - \Phi_\varepsilon(u)(t) \text{ where } \Phi_\varepsilon(u)(t) := \int_0^t e^{(t-t')\triangle}\mathbb{P}\left(\rho_\varepsilon * u \cdot \nabla u\right) dt' . \tag{6.20}$$

**Lemma 6.8.** *Equation* (6.20) *has exactly one maximal solution. This solution $u$ is global in time, with*

$$u \in C^0([0,+\infty), L^2(\mathbb{R}^d,\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d) \tag{6.21}$$

*Furthermore, $u$ solves the energy identity* (6.6).

Before we prove Lemma 6.8 we state a useful abstract lemma.

**Lemma 6.9.** *Let $X$ be a Banach space and $B: X^2 \to X$ a continuous bilinear map. Let $\alpha < \frac{1}{4\|B\|}$ where $\|B\| = \sup_{\|x\|=\|y\|=1}\|B(x,y)\|$. Then for any $x_0 \in X$ in $D_X(0,\alpha)$ (the open ball of center 0 and radius $\alpha$ in $X$) there exists a unique $x \in \overline{D}_X(0,2\alpha)$ s.t. $x = x_0 + B(x,x)$.*

*Proof.* We consider the map

$$x \to x_0 + B(x,x). \tag{6.22}$$

We will frame this as a fixed point problem in $\overline{D}_X(0,2\alpha)$.
First of all, we claim that the map (6.22) leaves $\overline{D}_X(0,2\alpha)$ invariant. Indeed

$$\|x_0 + B(x,x)\| \le \|x_0\| + \|B(x,x)\| \le \|x_0\| + \|B\|\|x\|^2 \le \alpha\overbrace{(1 + \underbrace{4\|B\|\alpha}_{<1})}^{\le 2} < 2\alpha.$$

Next, we check that the map (6.22) is a contraction. Indeed

$$\|B(x,x) - B(y,y)\| \le \|B(x-y,x)\| + \|B(y,x-y)\| \le 4\alpha\|B\|\|x-y\|$$

where $4\alpha\|B\| < 1$. So the map (6.22) has a unique fixed point in $\overline{D}_X(0,2\alpha)$.

□

*Proof of Lemma 6.8.* Let, for $T \in \mathbb{R}_+$,

$$X := L^\infty\left([0,T], H(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)\right)$$

and

$$B(u,v) := -\int_0^t e^{(t-t')\triangle}\mathbb{P}\left(\rho_\epsilon * v\cdot\nabla u\right)dt'$$

Then by Theorem 4.2

$$\|B(u,v)\|_{L^\infty([0,T],L^2)\cap L^2([0,T],\dot H^1)} \le C\|\rho_\epsilon * v\cdot\nabla u\|_{L^2([0,T],\dot H^{-1})} \lesssim C\|\rho_\epsilon * vu\|_{L^2([0,T],L^2)}$$
$$\le C\sqrt{T}\|\rho_\epsilon * v\|_{L^\infty([0,T],L^\infty(\mathbb{R}^d))}\|u\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \le C_\varepsilon\sqrt{T}\|v\|_{L^\infty([0,T],L^2(\mathbb{R}^d))}\|u\|_{L^\infty([0,T],L^2(\mathbb{R}^d))}.$$

Then using

$$\|e^{t\triangle}\rho_\varepsilon * u_0\|_{L^\infty([0,T],L^2)\cap L^2([0,T],\dot H^1)} \le \|e^{t\triangle}u_0\|_{L^\infty(\mathbb{R}_+,L^2)\cap L^2(\mathbb{R}_+,\dot H^1)} \le C_0\|u_0\|_{L^2},$$

and picking $T = T\left(\|u_0\|_{L^2}\right)$ such that $4C_\varepsilon C_0\sqrt{T\left(\|u_0\|_{L^2}\right)}\|u_0\|_{L^2} < 1$ we obtain from Lemma 6.9 the existence of a solution of (6.20) in $X$. Furthermore this solution is unique and is in $C^0([0,T],H(\mathbb{R}^d,\mathbb{R}^d))$ by Theorem 4.2. Let us consider the maximal solution

$$u \in C^0([0,T^*),L^2(\mathbb{R}^d,\mathbb{R}^d))$$

and let us suppose that $T^* < +\infty$. Then we claim that

$$\lim_{t\to T^*}\|u(t)\|_{L^2} = +\infty. \tag{6.23}$$

In fact, suppose that (6.23) false. Then there exists an $M$ and a sequence $t_n \to T^*$ with $\|u(t_n)\|_{L^2} \le M$. Then for $n$ such that $t_n + T(M) > T^*$, let

$$w(t) := \begin{cases} u(t) \text{ for } 0 \le t \le t_n \\ v(t-t_n) \text{ for } t_n \le t \le t_n + T(M) \end{cases}$$

with $v$ the solution of

$$v(t) = e^{t\triangle}u(t_n) - \int_0^t e^{(t-t')\triangle}\mathbb{P}\left(\rho_\epsilon * v\cdot\nabla v\right)dt'\,.$$

Then in fact $w$ solves (6.20), by uniqueness it coincides with $u$ in $[0,T^*)$, and hence we can extend $u$ beyond $T^*$ getting a contradiction. Hence, if $T^* < +\infty$ we have (6.23).
Now we discuss the fact that (6.23) is impossible. To see this we consider the identity (4.5)

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(t')\|_{L^2}^2 dt' = \|\rho_\varepsilon * u_0\|_{L^2}^2 - 2\int_0^t \langle \rho_\epsilon * u\cdot\nabla u, u\rangle dt'.$$

The last term cancels out, because by (6.18)

$$\langle \rho_\epsilon * u\cdot\nabla u, u\rangle = \langle \rho_\epsilon * u^k\partial_k u^j, u^j\rangle = \langle \partial_k(\rho_\epsilon * u^k u^j), u^j\rangle = \langle \operatorname{div}(\rho_\epsilon * uu), u\rangle = 0,$$

so that we get the energy identity (6.6). This prevents the blowup (6.23) and completes the proof of Lemma 6.8.

□

We consider now a sequence $\varepsilon_n \to 0^*$ and denote by $u_n$ the corresponding sequence of solutions provided by Lemma 6.8. In particular, we have

$$\langle u_n(t), \Psi(t)\rangle = \int_0^t \left(\langle u_n, \triangle\Psi\rangle + \langle u_n, \partial_t\Psi\rangle - \langle \mathbb{P}\left(\rho_{\epsilon_n} * u_n \cdot \nabla u_n\right), \Psi\rangle\right) dt' + \langle \rho_\varepsilon * u_0, \Psi(0)\rangle \tag{6.24}$$

for any $\Psi \in C^\infty_{c\sigma}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$.
Let us focus here on $d = 3$. Then the $u_n$ belong to the spaces in (6.21) with norms uniformly bounded by $\|u_0\|_{L^2}$. Then $u_n \in L^r(\mathbb{R}_+, L^q(\mathbb{R}^3))$ for $\frac{3}{q} + \frac{2}{d} = \frac{3}{2}$. In particular, for $q = r = \frac{10}{3}$ by (4.5) we obtain

$$\|u_n\|_{L^{\frac{10}{3}}(\mathbb{R}^3\times\mathbb{R}_+)} \le \|u_0\|_{L^2}.$$

By the weak pre–compactness of bounded subsets of $L^{\frac{10}{3}}(\mathbb{R}^3 \times \mathbb{R}_+)$ this implies that, up to a subsequence, there exists $u \in L^{\frac{10}{3}}(\mathbb{R}^3 \times \mathbb{R}_+)$ s.t. $u_n \rightharpoonup u$ in $L^{\frac{10}{3}}(\mathbb{R}^3 \times \mathbb{R}_+)$. Our aim is to show that $u$ satisfies (6.4) by taking the limit in (6.24). Clearly we have

$$\lim_{n\to+\infty} \int_0^t \left(\langle u_n, \triangle\Psi\rangle + \langle u_n, \partial_t\Psi\rangle\right) dt' = \int_0^t \left(\langle u, \triangle\Psi\rangle + \langle u, \partial_t\Psi\rangle\right) dt'.$$

We will prove the following result.

**Proposition 6.10.** *We have $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d, \mathbb{R}^d)) \cap L^2_{loc}(\mathbb{R}_+, H^1(\mathbb{R}^d, \mathbb{R}^d))$, div $u = 0$ and for any $T > 0$ and any compact subset $K \subset \mathbb{R}^d$ we have*

$$\lim_{n\to\infty} \int_{[0,T]\times K} |u_n(t,x) - u(t,x)|^2 dtdx = 0. \tag{6.25}$$

*Moreover, for any $\psi \in C^0([0,\infty), H^1(\mathbb{R}^d, \mathbb{R}^d))$ we have $\langle u_n, \psi\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \to \langle u, \psi\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d)}$ in $L^\infty_{loc}([0,\infty))$, that is*

$$\lim_{n\to\infty} \|\langle u_n(t) - u(t), \psi(t)\rangle\|_{C^0([0,T])} = 0 \text{ for any } T. \tag{6.26}$$

Notice that (6.26) implies the weak continuity $u \in C^0([0,+\infty), L^2_w)$ and

$$\lim_{n\to+\infty} \langle u_n(t), \Psi(t)\rangle = \langle u(t), \Psi(t)\rangle$$

so that, to complete the proof that $u$ satisfies (6.4) what will be left is

$$\lim_{n\to+\infty} \int_0^t \langle u_n, \rho_{\epsilon_n} * u_n \cdot \nabla\Psi\rangle dt' = \int_0^t \langle u, u \cdot \nabla\Psi\rangle dt', \tag{6.27}$$

which will also follow from Proposition 6.10.

*Proof of Proposition 6.10.* Fix an arbitrary $T > 0$ and an arbitrary compact subset $K$ of $\mathbb{R}^d$. It is enough to prove the following claim.

**Claim 6.11.** The set formed by the elements of the sequence $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T]\times K,\mathbb{R}^d)$.

*Proof of Claim 6.11.* We will show the following statement, which is equivalent to Claim 6.11.

**Claim 6.12.** For any $\varepsilon > 0$ there exists a finite family of balls of the space $L^2([0,T]\times K,\mathbb{R}^d)$ which have radius $\varepsilon$ and whose union covers the set $\{u_n\}_{n\in\mathbb{N}}$.

*Proof of Claim 6.12.* First of all, if we want to approximate $\{u_n\}_{n\in\mathbb{N}}$ with $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ for a fixed $n_0$, we can use the fact that for any $n_0$ and any $n$ we have

$$\|u_n - \mathbf{P}_{n_0}u_n\|^2_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)} = \int_0^T \|u_n - \mathbf{P}_{n_0}u_n\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^d)}dt$$
$$\leq n_0^{-2}\int_0^T \|\nabla u_n - \nabla\mathbf{P}_{n_0}u_n\|^2_{L^2(\mathbb{R}^d)}dt \leq n_0^{-2}\int_0^T \|\nabla u_n\|^2_{L^2(\mathbb{R}^d)}dt \leq n_0^{-2}\|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

Hence we can choose $n_0$ large enough s.t.

$$\|u_n - \mathbf{P}_{n_0}u_n\|_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)} < \frac{\varepsilon}{2} \text{ for all } n\in\mathbb{N}. \tag{6.28}$$

Now consider $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Then Claim 6.12 is a consequence of

**Claim 6.13.** $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T]\times K,\mathbb{R}^d)$.

Indeed Claim 6.13 implies that for any $\varepsilon > 0$ there is a finite number of balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\frac{\varepsilon}{2})$ which cover $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Hence by (6.28) we conclude that for any $\varepsilon > 0$ the balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\varepsilon)$ cover $\{u_n\}_{n\in\mathbb{N}}$ and so we get Claim 6.12.
*Proof of Claim 6.13.* It will be a consequence of the following stronger claim.

**Claim 6.14.** $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T],(L^2(K))^d) \subset L^\infty([0,T],(L^2(K))^d)$.

*Proof of Claim 6.14.* To get this result we want to apply the Ascoli–Arzela Theorem (for which a sufficient condition for a sequence of continuous functions $f_n : K \to X$, with $K$ compact metric space and $X$ a complete metric space, to admit a subsequence that converges uniformly to a continuous function $f : K \to X$ is that it is equicontinuous and $\{f_n(k)\}_n$ is relatively compact for any $k \in K$ [1]). So it is enough to show that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$ and that for any $t\in[0,T]$ the sequence $\{\mathbf{P}_{n_0}u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$.
First of all we want to show that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$. This will follow from Hölder inequality (since $\frac{4}{d} > 1$ if $d = 2,3$) and from the following claim.

[1]The proof goes as follows. One first considers a dense countable subset $\mathcal{N}$ of $K$. Then by a diagonal argument, one considers a subsequence $\{f_{n_m}\}$ s.t. $\{f_{n_m}(k)\}$ converges for any $k\in\mathcal{N}$ to a limit that we denote by $f(k)$. Using equicontinuity and the completeness of $X$ it is easy to see that $\{f_{n_m}(k)\}$ converges for any $k\in K$. We denote again by $f(k)$ the limit. Finally, using equicontinuity we conclude that $f : K\to X$ is continuous

**Claim 6.15.** There exists a fixed constant $C = C(n_0)$ s.t.

$$\|(\mathbf{P}_{n_0} u_n)_t\|_{L^{\frac{4}{d}}([0,T],L^2(\mathbb{R}^d))} \leq C \text{ for all } n.$$

*Proof of Claim 6.15.* We apply $\mathbf{P}_{n_0}$ to (6.19) and we obtain

$$(\mathbf{P}_{n_0} u_n)_t = -\mathbf{P}_{n_0}\mathbb{P}\left(\rho_{\epsilon_n} * u_n \cdot \nabla u_n\right) + \mathbf{P}_{n_0}\triangle u_n.$$

We have

$$\|\mathbf{P}_{n_0}\triangle u_n\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \leq n_0^2 \|u_n\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \leq n_0^2 \|u_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}$$

and, by Gagliardo-Nirenberg inequality,

$$\begin{aligned}
&\|\mathbf{P}_{n_0}\mathbb{P}\left(\rho_{\epsilon_n} * u_n \cdot \nabla u_n\right)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \leq \|\mathbf{P}_{n_0}\partial_j\left(u_{ni}\ \rho_{\epsilon_n} * u_{nj}\right)\overrightarrow{e}_i\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \\
&\leq n_0 \sum_{j,i=1}^{d} \|u_{ni}\rho_{\epsilon_n} * u_{nj}\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \leq C n_0 \|\rho_{\epsilon_n} * u_n\|_{L^4(\mathbb{R}^d,\mathbb{R}^d)} \|u_n\|_{L^4(\mathbb{R}^d,\mathbb{R}^d)} \\
&\leq C n_0 \|u_n\|^2_{L^4(\mathbb{R}^d,\mathbb{R}^d)} \leq C' n_0 \left(\|\nabla u_n\|^{\frac{d}{4}}_{L^2}\|u_n\|^{1-\frac{d}{4}}_{L^2}\right)^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|(\mathbf{P}_{n_0} u_n)_t\|_{L^{\frac{4}{d}}([0,T],L^2(\mathbb{R}^d,\mathbb{R}^d))} &\leq n_0^2 T^{\frac{d}{4}} \|u_0\|_{(\mathbb{R}^d,\mathbb{R}^d)} \\
&+ C' n_0 \|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],L^2(\mathbb{R}^d,\mathbb{R}^d))} \|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2(\mathbb{R}^d))} \leq C
\end{aligned}$$

for some constant $C$ independent of $n$ by the energy equality (6.6).
Hence we have concluded the proof that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(\mathbb{R}^d))^d)$.
To complete the proof of Claim 6.14 we need to show that for any $t \in [0,T]$ the sequence $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$. It is here that we will exploit the fact that $K$ is a compact subspace of $\mathbb{R}^d$.
We know that $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^d,\mathbb{R}^d)$ for any $t \in [0,T]$. This follows immediately from $\|\mathbf{P}_{n_0} u_n(t)\|_{H^1} \leq n_0\|u_n(t)\|_{L^2} \leq n_0\|u_0\|_{L^2}$, which follows from the energy equality (6.6) which guarantees $\|u_n(t)\|_{L^2} \leq \|u_0\|_{L^2}$. We recall now the following.

**Claim 6.16.** The restriction map $H^1(\mathbb{R}^d) \to L^2(K)$ is compact for any compact $K$ .

*Sketch of proof* Indeed this is equivalent at showing that

$$\mathcal{T} f := \chi_K \mathcal{F}^* \left(\frac{f}{\langle\xi\rangle}\right) \text{ is compact as } L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

We have $\mathcal{T} f = \int \mathcal{K}(x,\xi) f(\xi) d\xi$ with integral *kernel* $\mathcal{K}(x,\xi) := \chi_K(x)\langle\xi\rangle^{-1}e^{-\mathrm{i}x\cdot\xi}$. It is easy to see that $\mathcal{T}_n \overset{n\to\infty}{\to} \mathcal{T}$ in the operator norm where the $\mathcal{T}_n$ has kernel $\mathcal{K}_n(x,\xi) := \chi_K(x)\langle\xi\rangle^{-1}e^{-\mathrm{i}x\cdot\xi}\chi_{B(0,n)}(\xi)$. Since $\mathcal{K}_n \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$, it follows that $\mathcal{T}_n$ is a Hilbert–Schmidt operator, with $\|\mathcal{T}_n\|_{HS} := \|\mathcal{K}_n\|_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)}$. It is easy to show that $\|\mathcal{T}_n\|_{L^2\to L^2} \le \|\mathcal{T}_n\|_{HS}$. $\mathcal{K}_n$ is the limit in $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ of elements in $L^2(\mathbb{R}^d_x)\otimes L^2(\mathbb{R}^d_\xi)$. The latter ones are integral kernels of finite rank operators and their operators converge in the Hilbert–Schmidt norm, and so also in the $\|\cdot\|_{L^2\to L^2}$ norm, to $\mathcal{T}_n$. We conclude that there is a sequence of finite rank operators which converges in the operator norm to $\mathcal{T}$, which then is compact. □

It follows that $\{\mathbf{P}_{n_0}u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(K,\mathbb{R}^d)$ for any $t\in[0,T]$.

Hence the hypotheses of the Ascoli–Arzela Theorem have been checked and we can conclude that Claim 6.14, that is the claim that $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T],L^2(K,\mathbb{R}^d))$, is true. □

By the above series of Claims and by $u_n \rightharpoonup u$ in $L^{\frac{10}{3}}(\mathbb{R}^3\times\mathbb{R}_+)$, we conclude (6.25).

We turn now to the proof of (6.26).

Fix a function $\psi \in C^0([0,\infty),H^1(\mathbb{R}^d,\mathbb{R}^d))$. For a given $n_0$ consider

$$g_n(t) := \langle u_n(t),\psi(t)\rangle_{L^2(\mathbb{R}^d)} \text{ and } g_n^{(n_0)}(t) := \langle \mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{L^2(\mathbb{R}^d)}.$$

Then for any $\epsilon>0$ and any fixed $T>0$ there exists $n_0$ s.t.

$$\|(\mathbf{P}_{n_0}-1)\psi(t)\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} < \epsilon.$$

This and $\|u_n(t)\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \le \|u_0\|_{L^2(\mathbb{R}^d)}$ imply

$$\|g_n - g_n^{(n_0)}\|_{L^\infty([0,T]} \le \|u_0\|_{L^2(\mathbb{R}^d)}\epsilon.$$

Furthermore, for any fixed $T>0$ there exists a compact $K$ s.t.

$$\|\psi(t)\|_{L^\infty([0,T],L^2(\mathbb{R}^d\backslash K))} < \epsilon.$$

Then, if we set $g_n^{(n_0,K)}(t) := \langle \mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{L^2(K,\mathbb{R}^d)}$ we have

$$\|g_n^{(n_0,K)} - g_n^{(n_0)}\|_{L^\infty([0,T]} \le \|u_0\|_{L^2(\mathbb{R}^d)}\epsilon.$$

We claim that

$$\mathbf{P}_{n_0}u_n \to \mathbf{P}_{n_0}u \text{ in } C^0([0,T],L^2(K,\mathbb{R}^d)). \tag{6.29}$$

Indeed, by Claim 6.14, and by a diagonal argument, we know that there exists a $v$ s.t. $\mathbf{P}_{n_0}u_n \to v$ in $C^0([0,T],L^2(K,\mathbb{R}^d))$ for any $T$ and $K$. It is easy to conclude that $v\in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and that $\mathbf{P}_{n_0}u_n \rightharpoonup v$ therein. On the other hand, we know that $u_n\to u$ in $L^2([0,T]\times K,\mathbb{R}^d)$. This implies that $u_n \rightharpoonup u$ in $L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. In turn, this implies $\mathbf{P}_{n_0}u_n \rightharpoonup \mathbf{P}_{n_0}u$ in $L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. But then this implies $v=\mathbf{P}_{n_0}u$ in $L^2([0,T]\times K,\mathbb{R}^d)$, and so we get (6.29).

In turn, (6.29) implies

$$\{g_n^{(n_0,K)}\}_n = \langle \mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{L^2(K)} \xrightarrow{n\to+\infty} \langle \mathbf{P}_{n_0}u(t),\psi(t)\rangle_{L^2(K)} \text{ in } C^0([0,T]).$$

But then also

$$
\begin{aligned}
&\|\langle u_n(t),\psi(t)\rangle_{L^2(\mathbb{R}^d)}-\langle u(t),\psi(t)\rangle_{L^2(\mathbb{R}^d)}\|_{L^\infty([0,T]}\\
&\le \|\langle \mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{L^2(K)}-\langle \mathbf{P}_{n_0}u(t),\psi(t)\rangle_{L^2(K)}\|_{L^\infty([0,T]}+2\|u_0\|_{L^2(\mathbb{R}^d)}\epsilon\\
&+\|\langle u(t),(1-\mathbf{P}_{n_0})\psi(t)\rangle_{L^2(\mathbb{R}^d)}\|_{L^\infty([0,T]}+\|\langle u(t),(1-\chi_K)\psi(t)\rangle_{L^2(\mathbb{R}^d)}\|_{L^\infty([0,T]}\le\\
&\le \|\langle \mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{L^2(K)}-\langle \mathbf{P}_{n_0}u(t),\psi(t)\rangle_{L^2(K)}\|_{L^\infty([0,T]}+4\|u_0\|_{L^2(\mathbb{R}^d)}\epsilon.
\end{aligned}
$$

Since $\epsilon$ is arbitrarily small, it follows that we obtain that $g_n$ converges to $\langle u(t),\psi(t)\rangle_{L^2(\mathbb{R}^d,\mathbb{R}^d)}$ in $L^\infty([0,T])$, and hence in $C^0([0,T])$. In particular we have shown that $u\in C^0([0,\infty),L^2_w(\mathbb{R}^d,\mathbb{R}^d))$. The proof of Proposition 6.10 is completed. □

*We will now show that* $\text{div}_x u(t)=0$ *for all* $t$. Notice that we knew already that

$$\lim_{n\to\infty}\int_{[0,T]\times\mathbb{R}^d}(u_n(t,x)-u(t,x))\cdot\Phi(t,x)dtdx=0 \text{ for all } \Phi\in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d).$$

For $\Phi(t,x)=\chi(t)\nabla\psi(x)$ we have from the above limit

$$\int_{[0,T]}dt\chi(t)\int_{\mathbb{R}^d}\text{div}_x u(t,x)\psi(x)dx=0 \text{ for all } \psi\in C^\infty_c(\mathbb{R}^d,\mathbb{R}) \text{ and any } \chi\in C^\infty([0,T],\mathbb{R}),$$

This implies that

$$\int_{\mathbb{R}^d}\text{div}_x u(t,x)\psi(x)dx=0 \text{ for a.e. } t.$$

In fact, $u\in C^0([0,\infty),L^2_w(\mathbb{R}^d,\mathbb{R}^d))$ proves (6.26) and is independent of what we are discussing right here, the integral on the l.h.s. is continuous in $t$. This integral equals 0 for all t, and not just for a.a. t. Since this is true for all $t$ and for all $\psi\in C^\infty_c(\mathbb{R}^d,\mathbb{R})$, it follows that $\text{div}_x u(t,x)=0$ for all $t$.

*We now prove that u satisfies the energy inequality* (6.7).

Notice also that, up to a subsequence, $u_n(t,x)\xrightarrow{n\to+\infty}u(t,x)$ for almost any $(t,x)$, see p. 94 [2], and $\nabla u_n\rightharpoonup\nabla u$ as $n\to+\infty$ in $L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$. We claim that, since we assume we have extracted a subsequence, to that $u_n(t,x)\xrightarrow{n\to+\infty}u(t,x)$ for almost any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, this implies that for almost any $t$ we have $u_n(t,x)\xrightarrow{n\to+\infty}u(t,x)$ for a.e. $x$. Indeed, if this was not the case, setting $w(t,x):=\limsup_n|u_n(t,x)-u(t,x)|$, there would exist $J\subset\mathbb{R}_+$ with measure $|J|>0$ and with $\int_{\mathbb{R}^d}w(t,x)dx>0$ for $t\in J$, which would imply $\int_{\mathbb{R}_+\times\mathbb{R}^d}w(t,x)dtdx>0$, and so $w>0$ on a subset of $\mathbb{R}_+\times\mathbb{R}^d$ of positive measure. But we know that $w=0$ a.e. in $\mathbb{R}_+\times\mathbb{R}^d$ and this proves our claim.

Then the energy inequality (6.6) for all $u_n$ implies by Fatou

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)}+2\int_0^t\|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)}dt'\le\|u_0\|^2_{L^2(\mathbb{R}^d)}, \tag{6.7}$$

where here for the 1st term in the l.h.s. we apply the classical Fatou theorem for a sequence of integrable functions converging pointwise to a function, see [2, Lemma 4.1], while for the 2nd term in the l.h.s. we apply claim (iii) Proposition 3.5 [2].

### 6.1.1 End of the proof of Leray's Theorem 6.3

Proposition 6.10 has provided us with a divergence free function

$$u \in L^\infty([0,\infty), L^2(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2([0,\infty), \dot H^1(\mathbb{R}^d,\mathbb{R}^d)) \cap C^0([0,\infty), L^2_w(\mathbb{R}^d,\mathbb{R}^d))$$

which satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{6.7}$$

To finish with the proof, we need to prove

$$\lim_{n\to+\infty} \int_0^t \langle u_n, \rho_{\epsilon_n} * u_n \cdot \nabla\Psi\rangle dt' = \int_0^t \langle u, u\cdot\nabla\Psi\rangle dt'. \tag{6.27}$$

We observe that, since $\Psi \in C^1([0,\infty), H^1(\mathbb{R}^d,\mathbb{R}^d))$, for any $\varepsilon > 0$ there is a compact set $K \subset \mathbb{R}^d$ s.t.

$$\sup_{s\in[0,T]} \|\nabla\Psi(s,\cdot)\|_{L^2(\mathbb{R}^d\setminus K)} < \varepsilon. \tag{6.30}$$

(6.30) is elementary to prove and it is assumed in the sequel.
By Hölder, (6.30), Gagliardo–Nirenberg and the energy equality (6.6) we have

$$\begin{aligned}
&|\int_0^t ds \int_{\mathbb{R}^d\setminus K} \rho_{\epsilon_n} * u_n(s,x)\otimes u_n(s,x) : \nabla\Psi(s,x)dx| \le \int_0^T ds \|\rho_{\epsilon_n} * u_n\otimes u_n\|_{L^2(\mathbb{R}^d)} \|\nabla\mathbb{P}\Psi(s)\|_{L^2(\mathbb{R}^d\setminus K)} \\
&\le T^{\frac{4-d}{4}} \|\rho_{\epsilon_n} * u_n\otimes u_n\|_{L^{\frac{4}{d}}([0,T],L^2(\mathbb{R}^d))} \|\nabla\Psi\|_{L^\infty([0,T],L^2(\mathbb{R}^d\setminus K))} \\
&\le T^{\frac{4-d}{4}} \| \|u_n\|^2_{L^4(\mathbb{R}^d)} \|_{L^{\frac{4}{d}}(0,T)} \varepsilon \lesssim \varepsilon T^{\frac{4-d}{4}} \| \|u_n\|^{2(1-d/4)}_{L^2(\mathbb{R}^d)} \|\nabla u_n\|^{d/2}_{L^2(\mathbb{R}^d)} \|_{L^{\frac{4}{d}}(0,T)} \\
&\lesssim \varepsilon T^{\frac{4-d}{4}} \|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],L^2(\mathbb{R}^d))} \|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2(\mathbb{R}^d))} \le \varepsilon T^{\frac{4-d}{4}} \|u_0\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence, to prove (6.27) it is enough to show for any compact set $K \subset \mathbb{R}^d$

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t ds \int_K \rho_{\epsilon_n} * u_n(s,x)\otimes u_n(s,x) : \nabla\mathbb{P}\Psi(s,x)dx \\
&= \int_0^t ds \int_K u(s,x)\otimes u(s,x) : \nabla\mathbb{P}\Psi(s,x)dx.
\end{aligned} \tag{6.31}$$

The limit (6.31) is a consequence of

$$\lim_{n\to\infty} \rho_{\epsilon_n} * u_n \otimes u_n = u\otimes u \text{ in } L^1([0,T], L^2(K))$$

which in turn is a consequence of

$$\lim_{n\to\infty} u_n = u \text{ in } L^2([0,T], L^4(K)). \tag{6.32}$$

Let us consider $\chi \in C_c^\infty(\mathbb{R}^d, [0,1])$ s.t. $\chi = 1$ in $K$, $\Omega := \mathrm{supp}\chi$ and with $\|\nabla\chi\|_{L^\infty(\mathbb{R}^d)} \le 1$. Then by Gagliardo Nirenberg we have

$$\|f\|_{L^4(K)} \le C\|f\|_{L^2(\Omega)}^{1-d/4}(\|\chi\nabla f\|_{L^2(\mathbb{R}^d)} + \|f\nabla\chi\|_{L^2(\mathbb{R}^d)})^{d/4} \le C\|f\|_{L^2(\Omega)}^{1-d/4}\|f\|_{H^1(\mathbb{R}^d)}^{d/4}.$$

Using this inequality and Hölder inequality with $\dfrac{1}{2} = \dfrac{4-d}{8} + \dfrac{d}{8}$,

$$\begin{aligned}
&\|u-u_n\|_{L^2([0,T],L^4(K))} \lesssim \| \|u-u_n\|_{L^2(\Omega)}^{1-\frac{d}{4}} \|u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \|_{L^2(0,T)} \\
&\le \| \|u-u_n\|_{L^2(\Omega)}^{1-\frac{d}{4}} \|_{L^{\frac{8}{4-d}}(0,T)} \| \|u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \|_{L^{\frac{8}{d}}(0,T)} \\
&= \|u-u_n\|_{L^2([0,T],L^2(\Omega))}^{1-\frac{d}{4}} \|u-u_n\|_{L^2([0,T],H^1(\mathbb{R}^d))}^{\frac{d}{4}} \\
&\le (2(1+\sqrt{T})\|u_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)})^{\frac{d}{4}} \|u-u_n\|_{L^2([0,T],L^2(\Omega))}^{1-\frac{d}{4}} \xrightarrow{n\to+\infty} 0
\end{aligned}$$

where the limit holds because $u_n \xrightarrow{n\to+\infty} u$ in $L^2([0,T], L^2(\Omega, \mathbb{R}^d))$, by Proposition 6.10. This yields (6.32) and so also (6.31).
This completes the proof of Leray's Theorem 6.3.

□

*Remark* 6.17. The solutions we have found do not satisfy only the energy inequality (6.7), but in fact the more general inequality

$$\|u(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}^2 + 2\int_s^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' \le \|u(s)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}^2 \text{ for any } 0 \le s < t. \tag{6.33}$$

We will check later that

$$\text{for a.a. } t \text{ we have } \|u_n(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \xrightarrow{n\to+\infty} \|u(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)} \tag{6.34}$$

so that (6.33) is proved in analogy to the proof of (6.7), exploiting the fact that the sequence $u_n$ satisfies

$$\|u_n(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}^2 + 2\int_s^t \|\nabla u_n(t')\|_{(L^2(\mathbb{R}^d))^{d^2}}^2 dt' = \|u_n(s)\|_{L^2(\mathbb{R}^d,\mathbb{R}^d)}^2. \tag{6.35}$$

Notice this interesting continuity from the right.

**Lemma 6.18.** *If $u(t)$ is a Leray–Hopf solution for $d = 3$ then for any $s \ge 0$ we have $u(t) \xrightarrow{t\to s^+} u(s)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.*

*Proof.* From (6.35) we have $\limsup_{t\to s^+} \|u(t)\|_{L^2(\mathbb{R}^3)} \le \|u(s)\|_{L^2(\mathbb{R}^3)}$. On the other hand, since by weak continuity $u(t) \overset{t\to s^+}{\rightharpoonup} u(s)$, by Fathou's Lemma we have $\liminf_{t\to s^+} \|u(t)\|_{L^2(\mathbb{R}^3)} \ge$

$\|u(s)\|_{L^2(\mathbb{R}^3)}$. Hence we have the limit $\lim_{t\to s^+} \|u(t)\|_{L^2(\mathbb{R}^3)} = \|u(s)\|_{L^2(\mathbb{R}^3)}$. This and $u(t) \overset{t\to s^+}{\rightharpoonup}$ $u(s)$ yield $u(t) \xrightarrow{t\to s^+} u(s)$ in $L^2(\mathbb{R}^3,\mathbb{R}^3)$.

□

The proof of the claim in (6.34) follows from the following lemma.

**Lemma 6.19.** *For any $T>0$ and any $\epsilon>0$ there exists $R=R(u_0,T,\epsilon)$ such that*

$$\int_{|x|\geq R} |u_\varepsilon(x,t)|^2 dx < \epsilon^2 \text{ for all } \varepsilon \in (0,1) \tag{6.36}$$

*where $u_\varepsilon$ is the solution to* (6.20).

We will prove Lemma (6.19) later at the end of Sect. 11. We show now how it implies the claim in (6.34). Fix a $T>0$ and consider the sequence $u_n$ of the proof of Leray's Theorem. For any $\epsilon>0$ fix the $R$ of Lemma 6.19. Then $\|u_n\|_{L^\infty((0,T),L^2(|x|\geq R)} < \epsilon$ for all $n$. This implies also $\|u\|_{L^\infty((0,T),L^2(|x|\geq R)} \leq \liminf \|u_n\|_{L^\infty((0,T),L^2(|x|\geq R)} < \epsilon$ by Fathou's Lemma. On the other hand, we have $u_n \xrightarrow{n\to+\infty} u$ in $L^2((0,T)\times D_{\mathbb{R}^d}(0,R))$. The latter implies that (extracting a subsequence) $u_n(t) \xrightarrow{n\to+\infty} u(t)$ in $L^2(D_{\mathbb{R}^d}(0,R))$ for a.a. $t\in[0,T]$. In fact, it is easy to show that there is a set of full measure $J\subseteq \mathbb{R}_+$ such that (extracting a subsequence) $u_n(t) \xrightarrow{n\to+\infty} u(t)$ in $L^2(K)$ for any compact $K \subset\subset \mathbb{R}^d$. This coupled with $\|u_n\|_{L^\infty((0,T),L^2(|x|\geq R)} < \epsilon$ and $\|u\|_{L^\infty((0,T),L^2(|x|\geq R)} < \epsilon$ yields (6.34) for all the $t\in J$.

# 7 Initial datum in $V(\mathbb{R}^d)$

**Theorem 7.1** (Local existence of regular solutions 3d)**.** *There exists a constant $c_0>0$ such that for $u_0 \in V(\mathbb{R}^3) := H^1(\mathbb{R}^3,\mathbb{R}^3)\cap H(\mathbb{R}^3)$ there exists a $T > c_0\|\nabla u_0\|_{L^2}^{-4}$ s.t. one of the Leray's solutions satisfies $u\in L^\infty([0,T],V)$ and $\nabla^2 u \in L^2([0,T],L^2)$.*
*Furthermore, this solution $u$ satisfies the energy equality*

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\int_s^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' = \|u(s)\|^2_{L^2(\mathbb{R}^d)} \text{ for any } 0\leq s<t\leq T. \tag{7.1}$$

*Proof.* We consider the solution $u$ obtained from the limit of the sequence $u_n$ defined by (6.19), and which we can write as

$$\dot{u}_n + \mathbb{P}\left((\rho_{\epsilon_n} * u_n)\cdot\nabla u_n\right) - \triangle u_n = 0 \quad , \quad u_n(0) = \rho_{\epsilon_n} * u_0. \tag{7.2}$$

Applying $\langle \cdot, -\triangle u_n\rangle$ we obtain

$$\begin{aligned} &2^{-1}\frac{d}{dt}\|\nabla u_n\|^2_{L^2} + \|\triangle u_n\|^2_{L^2} = \langle (\rho_{\epsilon_n} * u_n)\cdot\nabla u_n, \triangle u_n\rangle \leq \|(\rho_{\epsilon_n} * u_n)\cdot\nabla u_n\|_{L^2}\|\triangle u_n\|_{L^2} \\ &\leq \|u_n\|_{L^\infty}\|\nabla u_n\|_{L^2}\|\triangle u_n\|_{L^2} \leq c\|\nabla u_n\|^{\frac32}_{L^2}\|\triangle u_n\|^{\frac32}_{L^2} \leq C\|\nabla u_n\|^6_{L^2} + \frac12\|\triangle u_n\|^2_{L^2}, \end{aligned} \tag{7.3}$$

where we used Agmon's inequality $\|u_n\|_{L^\infty(\mathbb{R}^3)} \le \|\nabla u_n\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla^2 u_n\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}$, see (2.30), and Young's inequality $ab \le \frac{a^4}{4\lambda^4} + \frac{3}{4}\lambda^{\frac{4}{3}} b^{\frac{4}{3}}$, where we choose $\lambda$ so that $\frac{3}{4}\lambda^{\frac{4}{3}} = 1/2$. We obtain

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \|\triangle u_n\|_{L^2}^2 \le C\|\nabla u_n\|_{L^2}^6.$$

From this we derive

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 \le C\|\nabla u_n\|_{L^2}^6 \text{ with } \|\nabla u_n(0)\|_{L^2}^2 = \|\rho_{\epsilon_n} * \nabla u_0\|_{L^2}^2 \le \|\nabla u_0\|_{L^2}^2. \tag{7.4}$$

Let us consider the ODE

$$\frac{d}{dt}X = CX^3 \text{ with } X_\varepsilon(0) = \|\nabla u_0\|_{L^2}^2.$$

The equation is separable, so the general solution is obtained writing $\frac{dX}{X^3} = Cdt$ and integrating separately, so that

$$-\frac{1}{2X^2} + \frac{1}{2X^2(0)} = Ct \Longrightarrow X(t) = \frac{X(0)}{\sqrt{1-2CtX^2(0)}} = \frac{\|\nabla u_0\|_{L^2}^2}{\sqrt{1-2Ct\|\nabla u_0\|_{L^2}^4}}.$$

We claim

$$\|\nabla u_n(t)\|_{L^2}^2 \le X(t) \text{ for any } n \in \mathbb{N} \text{ and for any } 0 \le t < (2C\|\nabla u_0\|_{L^2}^4)^{-1}. \tag{7.5}$$

Now, we have

$$\frac{d}{dt}\left(\|\nabla u_n\|_{L^2}^2 - X\right) \le \lambda\left(\|\nabla u_n\|_{L^2}^2 - X\right) \text{ with } \lambda := C\left(X^2 + X\|\nabla u_n\|_{L^2}^2 + \|\nabla u_n\|_{L^2}^4\right)$$

Integrating, and using $\|\nabla u_n(0)\|_{L^2}^2 - X(0) \le 0$, we obtain

$$\|\nabla u_n(t)\|_{L^2}^2 - X(t) \le \int_0^t \lambda\left(\|\nabla u_n\|_{L^2}^2 - X\right) dt'$$

for any $0 \le t < (2C\|\nabla u_0\|_{L^2}^4)^{-1}$. But then, we can apply Gronwall's Lemma 2.31 (here the function $\lambda$ satisfies the hypotheses in Lemma 2.31) and conclude that the claim in (7.5) is true.
So there exists a $T$ like in the statement s.t.

$$\|u_n(t)\|_{H^1}^2 + \int_0^t \|u_n(t')\|_{H^2}^2 dt' \le C_{T,\|u_0\|_{H^1}} \text{ for } t \in [0,T]. \tag{7.6}$$

Recall that we had $u_n$ convergent to $u$ in various ways. By Banach–Alaoglu there exists a subsequence which is $*$–weakly convergent in $L^\infty([0,T], H^1)$ and is weakly convergent in

$L^2([0,T],H^2)$. This and various forms of Fathou lemma, see in [2] Proposition 3.5 for the weak topology and Proposition 3.13 for the $*$–weak topology, implies that

$$\|u(t)\|^2_{H^1} + \int_0^t \|u(t')\|^2_{H^2} dt' \le C_{T,\|u_0\|_{H^1}} \text{ for } t \in [0,T]. \tag{7.7}$$

We turn to the proof of the energy identity (7.1). We first claim that

$$\text{div}(u \otimes u), \partial_t u \in L^2((0,T),L^2). \tag{7.8}$$

Let us assume this claim. Next, we claim that

$$\int_0^T \langle \partial_t u - \triangle u + \text{div}(u \otimes u), w \rangle \, dt = 0 \text{ for all } w \in L^2((0,T),H). \tag{7.9}$$

Let us assume also (7.9). Then apply (7.9) for $w = \chi_{[s,t]} u$. Then we get

$$\int_s^t \left( \langle \partial_t u, u \rangle + \|\nabla u\|^2_{L^2} \right) dt' = 0$$

where we use $\langle \text{div}(u \otimes u), u \rangle = 0$, from (6.18). Next, we can apply Lemma 6.6 and conclude that $\|u(t)\|^2_{L^2} \in AC([0,T])$ with $\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\langle u(t), \dot{u}(t) \rangle$. This yields (7.1).

Let us now prove (7.8). We have

$$\begin{aligned} &\|\text{div}(u \otimes u)\|_{L^2((0,T),L^2)} \lesssim \| \|\nabla u\|_{L^2} \|u\|_{L^\infty} \|_{L^2(0,T)} \le \|u\|_{L^\infty((0,T),H^1)} \|u\|_{L^2((0,T),L^\infty)} \\ &\lesssim \|u\|_{L^\infty((0,T),H^1)} \|u\|_{L^2((0,T),H^2)} < \infty \end{aligned}$$

using Sobolev's embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and (7.7). Next, we apply (6.4) for $\Psi(t,x) = \phi(x) \in C^\infty_{c\sigma}(\mathbb{R}^3,\mathbb{R}^3)$ and obtain

$$\langle u(t), \phi \rangle - \langle u(0), \phi \rangle = \int_0^t \left( \nu \langle \triangle u(t'), \phi \rangle) - \langle \mathbb{P}\text{div}(u \otimes u)(t'), \phi \rangle \right) dt.$$

This extends to any $\phi \in L^2(\mathbb{R}^3,\mathbb{R}^3)$. Then we can apply Lemma A.29 [2] for $X = L^2(\mathbb{R}^3,\mathbb{R}^3)$, concluding the following, which completes the proof of (7.8):

$$\partial_t u = \triangle u - \mathbb{P}\text{div}(u \otimes u) \text{ in } \mathcal{D}'((0,T),L^2). \tag{7.10}$$

We turn to the proof of (7.9). There exists a sequence of test functions $\Psi_n \to w$ in $L^2((0,T),H)$, which satisfy

$$\begin{aligned} \langle u(T), \Psi_n(T) \rangle - \langle u_0, \Psi_n(0) \rangle = \int_0^T &\left( \langle \triangle u(t'), \Psi_n(t') \rangle + \langle u(t'), \partial_t \Psi_n(t') \rangle \right. \\ &\left. - \langle \text{div}(u \otimes u)(t'), \Psi_n(t') \rangle \right) dt'. \end{aligned}$$

---

[2] Recall that Lemma A.29 states that if $u, g \in L^1(I,X)$ are such that

$$\langle u(t_2), f \rangle_{XX^*} - \langle u(t_1), f \rangle_{XX^*} = \int_{t_1}^{t_2} \langle g(s), f \rangle_{XX^*} \, ds \text{ for any } f \in X^*,$$

with $X$ a Banach space, then $\partial_t u = g$ in $\mathcal{D}'(I,X)$.

Integration by parts, which can be proved like in [2, Corollary 8.10], yields

$$\langle u(T), \Psi_n(T)\rangle - \langle u_0, \Psi_n(0)\rangle - \int_0^T \langle u(t'), \partial_t \Psi_n(t')\rangle dt' = - \int_0^T \langle \partial_t u, \Psi_n\rangle \, dt',$$

so that we obtain

$$\int_0^T \langle \partial_t u - \triangle u + \mathrm{div}(u \otimes u), \Psi_n\rangle \, dt' = 0$$

and for $n \to \infty$ we obtain (7.9).

□

**Theorem 7.2** (Uniqueness of weak solutions)**.** *Let $u_0 \in V(\mathbb{R}^3)$ and let $u \in L^\infty([0,T], V)$ and $\nabla^2 u \in L^2([0,T], L^2)$ be a solution discussed in the proof of Theorem 7.1. Consider also a weak solution $v$ with initial datum $u_0$ and satisfying the energy inequality* (6.7)*. Then $u = v$ in $[0,T]$.*

*Furthermore, we have $\|\nabla u(t)\|_{L^2} \xrightarrow{t \nearrow T^*} \infty$ if the lifespan $T^* = \sup\{T \text{ s.t. } u \in L^\infty([0,T], V)\}$ is $T^* < \infty$.*
*Finally, there exists a constant $\varepsilon_0 > 0$ s.t. if*

$$\|\nabla u_0\|_{L^2} \|u_0\|_{L^2} < \varepsilon_0 \tag{7.11}$$

*then the statements in Theorem 7.1 and here are valid for any $T > 0$.*

*Proof.* From (7.9) we have

$$\int_0^T (\langle \partial_t u, v\rangle + \langle \nabla u, \nabla v\rangle + \langle \mathrm{div}(u \otimes u), v\rangle) \, dt = 0.$$

We claim now that we can treat $u$ as a test function for $v$, so that

$$\int_0^t (\langle \nabla v, \nabla u\rangle - \langle v, \partial_t u\rangle + \langle \mathrm{div}(v \otimes v), u\rangle) \, dt' = \|u_0\|_{L^2}^2 - \langle v(t), u(t)\rangle, \tag{7.12}$$

so that adding the two equations we have

$$\int_0^t (2\langle \nabla v, \nabla u\rangle + \langle \mathrm{div}(u \otimes u), v\rangle + \langle \mathrm{div}(v \otimes v), u\rangle) \, dt' = \|u_0\|_{L^2}^2 - \langle v(t), u(t)\rangle. \tag{7.13}$$

Let us assume (7.12) and let us continue the proof.
Set $w = v - u$ and substitute in the identities

$$2\langle \nabla v, \nabla u\rangle = \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 - \|\nabla w\|_{L^2}^2 \ ,$$
$$\langle v(t), u(t)\rangle = 2^{-1}\|u(t)\|_{L^2}^2 + 2^{-1}\|v(t)\|_{L^2}^2 - 2^{-1}\|w(t)\|_{L^2}^2,$$

which are the same as the expansion $(a-b)^2 = a^2 + b^2 - 2ab$, and

$$\langle \mathrm{div}(u \otimes u), v\rangle + \langle \mathrm{div}(v \otimes v), u\rangle = \langle \mathrm{div}(w \otimes w), u\rangle,$$

which follows from

$$\begin{aligned}&\langle v^j \partial_j v^k, u^k\rangle + \langle u^j \partial_j u^k, v^k\rangle = \langle v^j \partial_j v^k, u^k\rangle - \langle u^j \partial_j v^k, u^k\rangle = \langle w^j \partial_j v^k, u^k\rangle \\ &= \langle w^j \partial_j w^k, u^k\rangle + \langle w^j \partial_j u^k, u^k\rangle = \langle w^j \partial_j w^k, u^k\rangle = \langle \mathrm{div}(w \otimes w), u\rangle.\end{aligned}$$

Then rearranging, we obtain the equality

$$\begin{aligned}&2^{-1}\|w(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla w\|^2 - \langle \mathrm{div}(w \otimes w), u\rangle\right) dt' \\ &= 2^{-1}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|^2 - 2^{-1}\|u_0\|_{L^2}^2 \qquad (7.14)\\ &+ 2^{-1}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|^2 - 2^{-1}\|v_0\|_{L^2}^2 \le 0, \qquad (7.15)\end{aligned}$$

where the inequality follows from the Energy identity (7.1) and the Energy inequality (6.7). Then

$$\begin{aligned}&\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w\|^2 dt' \le 2\int_0^t \langle \mathrm{div}(w \otimes w), u\rangle dt' \\ &\le 2\int_0^t \|u\|_{L^\infty} \|\nabla w\|_{L^2} \|w\|_{L^2} dt' \le \int_0^t \|u\|_{H^2}^2 \|w\|_{L^2}^2 dt' + \int_0^t \|\nabla w\|_{L^2}^2 dt'.\end{aligned}$$

Absorbing, as usual, the very last term in the 2nd term of the l.h.s., we obtain

$$\|w(t)\|_{L^2}^2 \le \int_0^t \|u\|_{H^2}^2 \|w\|_{L^2}^2 dt'$$

which, by Gronwall inequality, yields $\|w(t)\|_{L^2}^2 \equiv 0$.
Next, suppose the $T^*$ in the statement of the lemma is $T^* < \infty$. If there is no blow up, there exists $C > 0$ and a $t' < T^*$ with $\|\nabla u(t')\|_{L^2}^4 < C$ and $T^* - t' < c_0/C$ (that because there are a $C$, a sequence $t_n \xrightarrow{n \nearrow \infty} T^*$ with $\|\nabla u(t_n)\|_{L^2}^4 < C$). In particular, for $\mathbf{v}$ a solution as of Theorem 7.1 with initial value $\mathbf{v}(t') = u(t')$, we have $\mathbf{v} \in L^\infty((t', t' + c_0/C), V)$. But by the uniqueness $\mathbf{v} = u$ in $[t', T^*)$, so $u$ extends into a solution in $u \in L^\infty([0, t' + c_0/C), V)$, $u \in L^2([0, t' + c_0/C), H^2)$, yielding a contradiction. Therefore, we must have $\|\nabla u(t)\|_{L^2} \xrightarrow{t \nearrow T^*} \infty$ if $T^* < \infty$.

We now need to address formula (7.12). We have $u \in H^1((0,t), L^2)$ for $t \in (0,T)$, see (7.8), $u \in L^\infty((0,t), H^1)$ and $u \in L^2((0,t), H^2)$, see (7.7), and we can consider a sequence of test functions $\Psi_n \xrightarrow{n \to \infty} u$ in all these spaces. Starting from

$$\int_0^t \left(\langle \nabla v, \nabla \Psi_n\rangle - \langle v, \partial_t \Psi_n\rangle + \langle \mathrm{div}(v \otimes v), \Psi_n\rangle\right) dt' = \langle u_0, \Psi_n(0)\rangle - \langle v(t), \Psi_n(t)\rangle,$$

for $n \nearrow \infty$ it is easy to see that all the terms linear in $v$ converge to the corresponding ones in (7.12). Also the nonlinear term converges, as a consequence of

$$\begin{aligned}\int_0^t \langle \mathrm{div}(v\otimes v), \Psi_n - u\rangle_{L^2} dt' &\le C\int_0^t \|\nabla v\|_{L^2}\|v\|_{L^2}\|\Psi_n - u\|_{L^\infty} dt' \\ &\le C\|\nabla v\|_{L^2((0,t),L^2)}\|v\|_{L^\infty((0,t),L^2)}\|\Psi_n - u\|_{L^2((0,t),H^2)}\end{aligned}$$

by Sobolev's embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.

We finally turn to the proof of the last statement of the theorem, that is the global regularity for small initial data, that is $T^* = \infty$. From (7.9) we obtain for any $T \in (0, T^*)$

$$\int_0^T \langle \partial_t u - \triangle u + \mathrm{div}(u\otimes u), -\triangle u\rangle\, dt = 0$$

that is

$$\int_0^T \left(\langle \partial_t \nabla u, \nabla u\rangle + \|\triangle u\|_{L^2}^2 - \langle \mathrm{div}(u\otimes u), \triangle u\rangle\right) dt = 0$$

Now, notice that $\nabla u \in L^2([0,T], H^1)$ and $\partial_t \nabla u \in L^2([0,T], H^{-1})$. Then we can apply Lemma 6.6 obtaining that $\|\nabla u\|_{L^2}^2 \in AC([0,T])$ with $\frac{d}{dt}\|\nabla u\|_{L^2}^2 = 2\langle \partial_t \nabla u, \nabla u\rangle$. Proceeding as in (7.3), adjusting Young's inequality $ab \le \frac{a^8}{8} + \frac{7b^{\frac{8}{7}}}{8}$ and using interpolation to get the last line, we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla u\|_{L^2}^2 + 2\|\triangle u\|_{L^2}^2 = 2\langle \mathrm{div}(u\otimes u), \triangle u\rangle \le c\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\triangle u\|_{L^2}^{\frac{3}{2}} = c\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\triangle u\|_{L^2}^{\frac{3}{2}} \\ &\le c\|u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}\|\triangle u\|_{L^2}^{\frac{7}{4}} \le \mathbf{C}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^8 + \|\triangle u\|_{L^2}^2 \\ &\le \mathbf{C}_1\|u\|_{L^2}^4\|\nabla u\|_{L^2}^4\|\triangle u\|_{L^2}^2 + \|\triangle u\|_{L^2}^2,\end{aligned}$$

so that

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 \le \|\triangle u\|_{L^2}^2\left(\mathbf{C}_1\|u\|_{L^2}^4\|\nabla u\|_{L^2}^4 - 1\right).$$

Since from (7.1) we have $\frac{d}{dt}\|u\|_{L^2}^2 = -2\|\nabla u\|_{L^2}^2$, $\|u\|_{L^2}$ is decreasing. We have, using Leinbitz rule for products of $AC$ functions, see Corollary 8.10 [2],

$$\frac{d}{dt}\left(\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\right) \le \|u\|_{L^2}^2\|\triangle u\|_{L^2}^2\left(\mathbf{C}_1\|u\|_{L^2}^4\|\nabla u\|_{L^2}^4 - 1\right) - 2\|\nabla u\|_{L^2}^4.$$

If $\|u\|_{L^2}^4\|\nabla u\|_{L^2}^4 \le \mathbf{C}_1^{-1}$ then $\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2$ is strictly decreasing in any interval $[0,T]$ with $T \in (0,T^*)$, and so also in $[0,T^*)$. Then

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\triangle u\|_{L^2}^2 \le \mathbf{C}_1\|u_0\|_{L^2}^4\|\nabla u_0\|_{L^2}^4\|\triangle u\|_{L^2}^2 \le \varepsilon_0^4\|\triangle u\|_{L^2}^2$$

so that, if $\varepsilon_0^4 \leq 1/2$, we get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\triangle u\|_{L^2}^2 \leq 0 \text{ in } [0,T^*)$$

and so also

$$\|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\triangle u\|_{L^2}^2 dt' \leq \|\nabla u_0\|_{L^2}^2 \text{ in } [0,T^*) \ .$$

This obviously contradicts the blow up $\|\nabla u(t)\|_{L^2} \xrightarrow{t\nearrow T^*} \infty$ if $T^* < \infty$, and hence $T* = \infty$. This completes the proof of the global existence of small solutions.

□

**Theorem 7.3** (Global existence of regular solutions 2d)**.** *For any $u_0 \in V(\mathbb{R}^2)(= H^1(\mathbb{R}^2,\mathbb{R}^2)\cap H(\mathbb{R}^2)$ we have $u \in L^\infty([0,T],V)$ and $\nabla^2 u \in L^2([0,T],L^2)$ for all $T > 0$.*

*Proof.* The fact that locally for some $T > 0$ we have $u \in L^\infty([0,T],V)$ and $\nabla^2 u \in L^2([0,T],L^2)$ and that we have $\|\nabla u(t)\|_{L^2} \xrightarrow{t\nearrow T^*} \infty$ whenever the lifespan $T^* = \sup\{T$ s.t. $u \in L^\infty([0,T],V)\}$ is $T^* < \infty$, can be proved as above and is skipped here. So we need to prove $T^* = \infty$ by showing there cannot be finite time blow up. Now we consider

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|_{L^2}^2 + 2\|\triangle u\|_{L^2}^2 = 2\left\langle \operatorname{div}(u\otimes u), \triangle u\right\rangle \lesssim \|u\|_{L^4}\|\nabla u\|_{L^4}\|\triangle u\|_{L^2} \\
&\lesssim \|u\|_{\dot H^{\frac{1}{2}}}\|\nabla u\|_{\dot H^{\frac{1}{2}}}\|\triangle u\|_{L^2} \leq \left(\|u\|_{L^2}\|\nabla u\|_{L^2}\right)^{\frac{1}{2}}\left(\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\right)^{\frac{1}{2}}\|\triangle u\|_{L^2} \\
&\lesssim \|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\triangle u\|_{L^2}^{\frac{3}{2}} \leq C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \|\triangle u\|_{L^2}^2,
\end{aligned}$$

where we used Young's inequality $ab \leq \frac{a^4}{4\lambda^4} + \frac{3\lambda^{\frac{4}{3}}b^{\frac{4}{3}}}{4}$ adjusting $\lambda$. By absorbing the last term in the 2nd term of the l.h.s. we obtain

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\triangle u\|_{L^2}^2 \leq c_\nu\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \leq C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\left(\|\nabla u\|_{L^2}^2 + \int_0^t \|\triangle u(s)\|_{L^2}^2 ds\right).$$

From Gronwall's inequality we obtain

$$\|\nabla u\|_{L^2}^2 + \int_0^t \|\triangle u(s)\|_{L^2}^2 ds \leq e^{C\|u\|^2_{L^\infty(0,\infty),L^2)}\int_0^\infty \|\nabla u\|_{L^2}^2 ds}\|\nabla u_0\|_{L^2}^2$$

which yields the desired result.

□

**Theorem 7.4** (Higher spacial regularity)**.** *Let $u \in L^\infty([0,T],V)$ with $\nabla^2 u \in L^2([0,T],L^2)$ be a solution like in Theorem 7.1 or Theorem 7.3. Suppose that $u_0 \in V \cap H^m(\mathbb{R}^d)$ with $m \geq 2$. Then $u \in L^\infty([0,T],H^m(\mathbb{R}^d))$ and $u \in L^2([0,T],H^{m+1}(\mathbb{R}^d))$.*

*Proof.* We can go back to the framework of Theorem 7.1 with the sequence of regularizations. We claim that we can generalize (7.6) into

$$\|u_n(t)\|^2_{H^k} + \int_0^t \|u_n(t')\|_{H^{k+1}} dt' \le C_{k,T,\|u_0\|_{H^k}} \text{in } [0,T] \text{ for all } 1 \le k \le m. \tag{7.16}$$

We have already case $k = 1$. Suppose $2 \le k \le m$ and we have case $k-1$. We apply $\langle \cdot, u_n \rangle_{H^k}$ to

$$\dot{u}_n + \mathbb{P}\left((\rho_{\epsilon_n} * u_n) \cdot \nabla u_n\right) - \triangle u_n = 0 \quad , \quad u_n(0) = \rho_{\varepsilon_n} * u_0. \tag{7.2}$$

and obtain

$$\begin{aligned}
2^{-1}\frac{d}{dt}\|u_n\|_{H^k} + \|\nabla u_n\|^2_{H^k} &= \langle (\rho_{\epsilon_n} * u_n) \cdot \nabla u_n, u_n \rangle_{H^k} \le \|(\rho_{\epsilon_n} * u_n) \cdot \nabla u_n\|_{H^k} \|u_n\|_{H^k} \\
&\le \|u_n\|_{H^k} \|\nabla u_n\|_{H^k} \|u_n\|_{H^k} \le \frac{1}{2}\|u_n\|^4_{H^k} + \frac{1}{2}\|\nabla u_n\|^2_{H^k},
\end{aligned}$$

where we used the fact that, since $k > d/2$ for $d = 2, 3$, $H^k$ is an algebra. So

$$\frac{d}{dt}\|u_n\|_{H^k} + \|\nabla u_n\|^2_{H^k} \le \|u_n\|^2_{H^k}\|u_n\|^2_{H^k}.$$

From this and Gronwall we obtain

$$\|u_n(t)\|_{H^k} + \int_0^t \|\nabla u_n\|^2_{H^k} ds \le e^{\int_0^t \|u_n\|^2_{H^k} ds} \|u_0\|^2_{H^k} \le C_{k,T\|u_0\|_{H^k}} \text{in } [0,T],$$

where, in the exponent, is uniformly bounded in $n$ for $t \in [0,T]$ because of (7.16) with $k$ replaced by $k-1$.
Recall, now, that we had $u_n$ convergent to $u$ in various ways. We can take a subsequence, which by Banach–Alaoglu is $*$–weakly convergent in $L^\infty([0,T], H^k)$ and is weakly convergent in $L^2([0,T], H^{k+1})$. This implies that

$$\|u(t)\|^2_{H^k} + \int_0^t \|u(t')\|_{H^{k+1}} dt' \le C_{k,T,\|u_0\|_{H^k}} \text{ in } [0,T] \text{ for all } 1 \le k \le m. \tag{7.17}$$

□

**Corollary 7.5.** *Let $u \in L^\infty([0,T], V)$ with $\nabla^2 u \in L^2([0,T], L^2)$ be a solution like in Theorem 7.1 or Theorem 7.3. Then, for any $m$ we have $u \in C^\infty((0,T], H^m(\mathbb{R}^d))$.*

*Proof.* We have seen in Theorem 7.1 that $u$ solves distributionally the NS equation, see (7.10) and that $\partial_t u \in L^2((0,T), L^2)$, see (7.8), and we know $u \in L^2([0,T], H^2)$. Obviously $u \in H^1((0,T), L^2) \cap L^2([0,T], H^2)$ is equivalent to $\left\langle \sqrt{-\triangle} \right\rangle u \in L^2([0,T], H^1) \cap H^1([0,T], H^{-1})$. The latter, by Lemma 6.6, implies $\left\langle \sqrt{-\triangle} \right\rangle u \in C^0([0,T], L^2)$. Equivalent conclusion is $u \in C^0([0,T], H^1)$. Then,

$$\begin{aligned}
&\text{for any } t_n \in (0,T),\ u \text{ is the unique solution in } L^\infty(([t_n,T], V) \cap L^2([t_n,T], H^2) \\
&\text{of NS with initial values } u(t_n).
\end{aligned} \tag{7.18}$$

Now, for any $\epsilon > 0$ there exists a $t_2 \in (0, \epsilon)$ s.t. $u(t_2) \in H^2$ and applying (7.18) and Theorem 7.4, we conclude $u \in L^\infty([t_2, T], H^2(\mathbb{R}^d))$ and $u \in L^2([t_2, T], H^3(\mathbb{R}^d))$. So there exists $t_3 \in (t_2, \epsilon)$ s.t. $u(t_3) \in H^3$, and proceeding by induction we get that for any $n$ there exists $t_n \in (0, \epsilon)$ s.t. $u(t_n) \in H^n$, so that $u \in L^\infty([t_n, T], H^n(\mathbb{R}^d)) \cap L^2([t_n, T], H^{n+1}(\mathbb{R}^d))$. Recalling $\partial_t u = \triangle u - \mathbb{P}\text{div}(u \otimes u)$ in $\mathcal{D}'((0, T), L^2)$, from $u \in L^\infty([t_{m+2}, T], H^{m+2}(\mathbb{R}^d))$ we derive $\partial_t u \in L^\infty([t_{m+2}, T], H^m)$ and so $u \in C^0([t_{m+2}, T], H^m)$. So we conclude $u \in C^0([\epsilon, T], H^m)$ for any $m$ and, by the arbitrariness of $\epsilon > 0$, $u \in C^0((0, T], H^m)$ for any $m$. Notice that this implies $\partial_t u = \triangle u - \mathbb{P}\text{div}(u \otimes u)$ in $C^0((0, T], H^m)$ for any $m$. In other words $u \in C^1((0, T], H^m)$ for any $m$. It is easy to conclude, proceeding by induction, that we have $u \in C^\infty((0, T], H^m)$ and that for all $j$

$$\partial_t^j u = \triangle \partial_t^{j-1} u - \mathbb{P} \sum_{k=0}^{j-1} \binom{j-1}{k} \text{div}(\partial_t^k u \otimes \partial_t^{j-1-k} u).$$

□

Notice that the proof of Lemma 7.4 [14] is incomplete, because it is based on the last displayed formula of p. 152 [14], where the uniformity in $n$ is left untreated both in the text and in the exercises, and seems non trivial.

## 7.1 Structure of the singular set

We consider a digression on the singular set of Leray–Hopf solutions in $d = 3$.

**Lemma 7.6** (Compactness of Singular Set)**.** *Given a Leray–Hopf solution $u$ there exists a $T_* \geq 0$ such that $u \in C^\infty((T_*, +\infty) \times \mathbb{R}^3, \mathbb{R}^3)$.*

*Proof.* Since $u \in L^\infty(\mathbb{R}_+, L^2)$ and $\nabla u \in L^2(\mathbb{R}_+, L^2)$, we known that there is a $T^* > 0$ s.t. $\|\nabla u(T^*)\|_{L^2} \|u(T^*)\|_{L^2} < \varepsilon$, with $\varepsilon > 0$ the constant in (7.11). From Remark 6.2 we know that $u$ is a weak solution of the NS in $[T^*, \infty)$ with initial value $u(T^*) \in V$. By the smalleness condition (7.11) in Theorem 7.1 we know that there exists a solution $v \in L^\infty([T^*, \infty), V)$ and $\nabla^2 v \in L^2([T^*, \infty), L^2)$ of the NS with initial value $u(T^*) \in V$. Notice that, as a Leray–Hopf solution, see Remark 6.17, in particular $u$ satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2 \int_s^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \leq \|u(T^*)\|^2_{L^2(\mathbb{R}^d)} \text{ for any } T^* < t.$$

By the Uniqueness theorem of weak solutions 7.2, we know that $u = v$ in $[T^*, \infty)$. Finally, from Corollary 7.5 we know $u \in C^\infty((T_*, +\infty) \times \mathbb{R}^3, \mathbb{R}^3)$.

□

**Definition 7.7.** Consider a Leray–Hopf solution $u$. We say that a time $t_0 \geq 0$ is regular if there exists a neighborhood $I$ of $t_0$ in $[0, \infty)$ with $\nabla u \in L^\infty(I, L^2)$. If $t_0$ is not regular, it is called singular. We denote by $\mathcal{R}$ the set of regular times, and by $\mathcal{T}$ the set of singular times.

It is quite obvious that $\mathcal{R}$ is open in $[0,\infty)$, and hence that $\mathcal{T}$ is closed. From Lemma 7.6 we know that $\mathcal{T}$ is compact.

In this section we consider two simple results about $\mathcal{T}$, one about box–counting dimension and the other about Hausdorff measure.

Let us start with the box–counting dimension.

**Definition 7.8.** Consider a compact subspace $X$ of $\mathbb{R}^d$ and for any $\epsilon > 0$ denote by $N(X,\epsilon)$ the smallest number of open balls of radius $\epsilon$ needed to cover $X$. Then the (upper) box–counting dimension of $X$ is

$$\dim_B(X) := \limsup_{\epsilon\to 0^+} \left(-\log_\epsilon N(X,\epsilon)\right) = \limsup_{\epsilon\to 0^+} \frac{\log N(X,\epsilon)}{-\log\epsilon}. \tag{7.19}$$

**Lemma 7.9.** *For a compact subspace $X$ of $\mathbb{R}^d$ we have $\dim_B(X) = \dim'_B(X)$, where*

$$\dim'_B(X) := \limsup_{\epsilon\to 0^+} \frac{\log M(X,\epsilon)}{-\log\epsilon}$$

*with $M(X,\epsilon)$ the largest number of disjoint open balls of radius $\epsilon$ with centers at points of $X$.*

*Proof.* First of all, we have $M(X,\epsilon) \le N(X,\epsilon)$ (so that $\dim_B(X) \ge \dim'_B(X)$). Indeed, let us consider a family of disjoint balls $\{D(x_j,\epsilon)\}_{j=1}^{M(X,\epsilon)}$, with $x_j \in X$. If $\{D(y_k,\epsilon)\}_{j=1}^{N(X,\epsilon)}$ is a cover of $X$, it is also a cover of $\{x_1,...,x_{M(X,\epsilon)}\}$. It is not possible to have a $D(y_k,\epsilon)$ which contains two distinct $x_i \neq x_j$, because this would imply $|x_i - x_j| < 2\epsilon$ , while we know that $|x_i - x_j| \ge 2\epsilon$. So $M(X,\epsilon) \le N(X,\epsilon)$.

Next, we have $M(X,\epsilon/3) \ge N(X,\epsilon)$. This follows by the proof of Vitali's lemma, Theorem 3.1. In fact, given a cover $\{D(x_j,\epsilon/3)\}$ of $X$, we know that we can extract a family of of disjoint balls, which we will label as $\{D(x_j,\epsilon/3)\}_{j=1}^L$, such that $\{D(x_j,\epsilon)\}_{j=1}^L$ is a cover of $X$. Then $M(X,\epsilon/3) \ge L \ge N(X,\epsilon)$. So

$$\begin{aligned}\dim'_B(X) &\le \dim_B(X) \le \limsup_{\epsilon\to 0^+} \frac{\log M(X,\epsilon/3)}{-\log\epsilon} = \limsup_{\epsilon\to 0^+} \frac{\log M(X,\epsilon/3)}{-\log\epsilon/3}\,\frac{-\log\epsilon/3}{-\log\epsilon/3 - \log 3}\\ &= \dim'_B(X).\end{aligned}$$

□

*Example* 7.10.

1. $\dim_B([0,1]^d) = d$, $\dim_B([0,1]^j \times \{0\}^{d-j}) = j$.
2. We have $\dim_B S_k = \frac{1}{k+1}$ for $S_k = \{n^{-k} : n \in \mathbb{N}\}$, $k > 0$, see [14].
3. For $C$ the usual Cantor ternary set, we have $\dim_B(C) = \frac{\log(2)}{\log(3)}$.

**Lemma 7.11.** *Let $K(\epsilon)$ be either $M(X,\epsilon)$ or $N(X,\epsilon)$. Then*

1. *if $d' \in (0, \dim_B(X))$ there is a sequence $\epsilon_j \to 0$ s.t. $K(\epsilon_j) \ge \epsilon_j^{-d'}$ while*

*2. if $d'' > \dim_B(X)$ there is $\epsilon_0 > 0$ s.t. $K(\epsilon) \leq \epsilon^{-d''}$ for all $\epsilon \in (0, \epsilon_0)$.*

*Proof.* By the properties of lim sup there exists a sequence $\epsilon_j \to 0$ s.t. $-\log_{\epsilon_j} K(\epsilon_j) \to \dim_B(X)$. So, if $d' < \dim_B(X)$,we have $\log_{\epsilon_j} K(\epsilon_j) < -d'$, that is $K(\epsilon_j) > \epsilon_j^{-d'}$ for $j \gg 1$.

Let now $d'' > \dim_B(X)$. Then we claim $K(\epsilon) \leq \epsilon^{-d''}$ for all $\epsilon \in (0, \epsilon_0)$ for an appropriate $\epsilon_0 > 0$. If this is false, there exists a sequence $\epsilon_j \to 0$ s.t.$K(\epsilon_j) > \epsilon_j^{-d''}$. Then $\frac{\log K(\epsilon_j)}{-\log \epsilon_j} > d''$. But then $\dim_B(X) = \limsup_{\epsilon \to 0^+} \frac{\log K(\epsilon)}{-\log \epsilon} \geq \liminf_j \frac{\log K(\epsilon_j)}{-\log \epsilon_j} \geq d'' > \dim_B(X)$.

□

Now we have the following result.

**Proposition 7.12.** *Given a Leray–Hopf solution $u$ of NS in $d = 3$, then $\dim_B(\mathcal{T}) \leq 1/2$.*

*Proof.* Fix $\epsilon > 0$ and let us consider a family of disjoint 1− dimensional balls $\{D(t_j, \epsilon)\}_{j=1}^{M(\mathcal{T},\epsilon)}$, with $t_j \in \mathcal{T}$, with $t_1 < ... < t_{M(\mathcal{T},\epsilon)}$. For $c_0$ the constant in Theorem 7.1, we claim that

$$t_j - t \geq c_0 \|\nabla u(t)\|_{L^2}^{-4} \text{ for } t \in (t_j - \epsilon, t_j), \tag{7.20}$$

where we set $\|\nabla u(t)\|_{L^2} = \infty$ in the 0 measure set of points $t$ where $u(t) \notin H^1$. Notice that for $u(t) \notin H^1$, (7.20) is obviously true. To prove (7.20) observe that if there exists a $t$ for which (7.20) is false, then we would have $t_j - t < c_0 \|\nabla u(t)\|_{L^2}^{-4}$, which automatically implies that $\|\nabla u(t)\|_{L^2} < \infty$ and $u(t) \in H^1$. But then, by Theorem 7.1, there exists a solution $v \in L^\infty([t, t+T], H^1)$ to the NS with $v(t) = u(t)$ and with $T > c_0 \|\nabla u(t)\|_{L^2}^{-4} > t_j - t$. This means that $t_j \in (t, t+T)$. But since $u$ is a Leray–Hopf solution, by Theorem 7.2 we have $u = v$ in $[t, t+T)$. But then we get a contradiction to $t_j \in \mathcal{T}$.
From (7.20) we obtain $\|\nabla u(t)\|_{L^2}^2 \geq \frac{\sqrt{c_0}}{\sqrt{t_j - t}}$ in $(t_j - \epsilon, t_j)$. For $T$ sufficiently large, such that $u$ is smooth in $(T, \infty)$, using the energy inequality

$$\begin{aligned}
\|u_0\|_{L^2(\mathbb{R}^d)}^2 &\geq 20 \int_0^T \|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 dt \geq 20 \sum_{j=1}^{M(\mathcal{T},\epsilon)} \int_{t_j - \epsilon}^{t_j} \|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 dt \\
&\geq 20\sqrt{c_\nu} \sum_{j=1}^{M(\mathcal{T},\epsilon)} \int_{t_j - \epsilon}^{t_j} \frac{1}{\sqrt{t_j - t}} dt = 40\sqrt{c_0} M(\mathcal{T}, \epsilon)\sqrt{\epsilon}.
\end{aligned}$$

So $M(\mathcal{T}, \epsilon) < \frac{\|u_0\|_{L^2(\mathbb{R}^d)}^2}{4\nu\sqrt{c_0}} \epsilon^{-\frac{1}{2}}$ which, implies

$$\dim_B(\mathcal{T}) = \limsup_{\epsilon \to 0^+} \frac{\log M(X, \epsilon)}{-\log \epsilon} \leq \lim_{\epsilon \to 0^+} \frac{\log \left( \frac{\|u_0\|_{L^2(\mathbb{R}^d)}^2}{4\nu\sqrt{c_0}} \epsilon^{-\frac{1}{2}} \right)}{-\log \epsilon} = \frac{1}{2}.$$

□

**Definition 7.13.** Given a subset $X \subset \mathbb{R}^d$ set for $s > 0$ and $\delta > 0$

$$\mu_{s,\delta}(X) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } (U_j))^s : \ \{U_j\}_{j=1}^{\infty} \text{ is an open cover of } X \text{ with diam } (U_j) \leq \delta \text{ for all } j \right\}$$

Notice that $\mu_{s,\delta}(X)$ is decreasing in $\delta$. The we call $s$–dimensional Hausdorff measure of $X$ the number

$$\mathcal{H}^s(X) = \lim_{\delta \to 0^+} \mu_{s,\delta}(X).$$

The Hausdorff measure of $X$ is

$$\dim_H(X) = \inf\{s > 0 : \mathcal{H}^s(X) = 0\}.$$

*Remark* 7.14. Notice that $\mu_{s,\delta}(X) = \mu'_{s,\delta}(X)$ if we set

$$\mu'_{s,\delta}(X) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } (U_j))^s : \ \{U_j\}_{j=1}^{\infty} \text{ is an open cover of } X \right.$$
$$\left. \text{with diam } (U_j) \leq \delta \text{ for all } j \text{ and all the } U_j \text{ are convex} \right\}.$$

Indeed, any open set is contained in an open convex set with the same diameter.

**Lemma 7.15.** *We have* $\dim_H(X) \leq \dim_B(X)$.

*Proof.* Let $d = \dim_B(X)$ and $s > d$. Let $s > z > d$. Then, by Lemma 7.11 there is $\epsilon_0 > 0$ s.t. $N(X, \epsilon) \leq \epsilon^{-z}$ for all $\epsilon \in (0, \epsilon_0)$. Since we can cover $X$ with $N(X, \epsilon)$ balls of radius $\epsilon$, we have

$$\sum_{j=1}^{N(X,\epsilon)} (2\epsilon)^s = 2^s N(X, \epsilon)\epsilon^s \leq 2^s \epsilon^{s-z} \xrightarrow{\epsilon \to 0^+} 0$$

$\Longrightarrow \mathcal{H}^s(X) = 0$. This implies $\dim_H(X) := \inf\{s > 0 : \mathcal{H}^s(X) = 0\} \leq \dim_B(X)$.

**Lemma 7.16** (Isodiametric Inequality)**.** *The Lebesgue measure of an open convex set in* $\mathbb{R}^d$ *of diameter* $D$ *is at most the volume* $c_d D^d$ *of the ball of radius* $D/2$ *.*

See [6, Sect. 2.2]. □

**Theorem 7.17.** *In* $\mathbb{R}^d$, *for* $\mathcal{L}^d$ *the Lebesgue measure,* $\mathcal{L}^d = c_d \mathcal{H}^d$.

*Proof.* Here we will only prove $\mathcal{L}^d(K) \leq c_d \mathcal{H}^d(K)$ for any $K \subset\subset \mathbb{R}^d$. Given $\epsilon > 0$, we can cover $K \subseteq \cup_{j=1}^{\infty} U_j$ with $U_j$ open convex sets and with $\sum_{j=1}^{\infty} (\text{diam } (U_j))^d \leq \mathcal{H}^d(K) + \epsilon$. Then $|K| \leq \sum_{j=1}^{\infty} |U_j| \leq c_d \sum_{j=1}^{\infty} (\text{diam } (U_j))^d \leq c_d \left(\mathcal{H}^d(K) + \epsilon\right)$. This implies $\mathcal{L}^d(K) \leq c_d \mathcal{H}^d(K)$.

**Proposition 7.18.** *Given a Leray–Hopf solution $u$ of NS in $d = 3$, then $\mathcal{H}^{1/2}(\mathcal{T}) = 0$.*

*Proof.* Since $\mathcal{H}^1(\mathcal{T}) = 0$, we can cover $\mathcal{T}$ by a finite or a numerable family of disjoint intervals $\{[t_k, t_k+\epsilon_k]\}_{k\in\mathbb{N}}$ with $\sum \epsilon_k < \delta$ for any preassigned $\delta > 0$. Possibly picking $\epsilon_k = 0$, we can assume that $t_k + \epsilon_k \in \mathcal{T}$. Indeed, if $[t_k, t_k + \epsilon_k] \cap \mathcal{T} = \emptyset$ we can discard the interval, while if $\exists\, t_k' \in [t_k, t_k+\epsilon_k] \cap \mathcal{T}$, then we can replace $[t_k, t_k+\epsilon_k]$ with $[t_k, t_k']$. By the discussion in Proposition 7.12 we have $\|\nabla u(t)\|^2_{L^2} \geq \frac{\sqrt{c_0}}{\sqrt{t_k+\epsilon_k-t}}$ in $[t_k, t_k + \epsilon_k]$. Then

$$\int_{\cup_k [t_k, t_k+\epsilon_k]} \|\nabla u(t)\|^2_{L^2} dt \geq \sqrt{c_0} \sum_{k=1}^{\infty} \int_{t_k}^{t_k+\epsilon_k} \frac{dt}{\sqrt{t_k + \epsilon_k - t}} = 2\sqrt{c_\nu} \sum_{k=1}^{\infty} \sqrt{\epsilon_k}.$$

But, by absolute integrability, the l.h.s. can be made smaller than any given $\epsilon > 0$. Then $\mathcal{H}^{1/2}(\mathcal{T}) = 0$.

□

## 7.2 Serrin's condition

We have the following theorem.

**Theorem 7.19.** *Let $u$ be a solution in $d = 3$ of the type in Leray's Theorem 6.3 and suppose that*

$$u \in L^r((0,T), L^s(\mathbb{R}^3)) \text{ where } \frac{2}{r} + \frac{3}{s} = 1, \text{ with } r \geq 2 \text{ and } s > 3. \tag{7.21}$$

*Then $u \in C^\infty((0,T] \times \mathbb{R}^3, \mathbb{R}^3)$ and $u$ is in $[\epsilon, T]$ for any $\epsilon \in (0,T)$ also a solution in the sense of Theorem 7.1. Furthermore, if $v$ is another solution of the type in Leray's Theorem 6.3 satisfying Serrin's condition, for possibly different exponents (still with $s > 3$) in $(0,T)$ and with the same initial value, we have $u = v$ in $[0,T]$.*

*Remark* 7.20. The case $L^\infty((0,T), L^3(\mathbb{R}^3))$ is relatively recent, [5], is more complicated to prove and will not be considered here.

*Remark* 7.21. Notice that any $u$ like in Leray's Theorem 6.3, we have

$$u \in L^r((0,T), L^s(\mathbb{R}^d)) \text{ where } \frac{2}{r} + \frac{d}{s} = \frac{d}{2}, \text{ with } r \geq 2. \tag{7.22}$$

Notice that of the endpoint cases $(r,s) = (\infty, 2)$ follows from the Energy Inequality and for $d = 3$, similarly $(r,s) = (2,6)$ follows from the Energy Inequality and, additionally, from Sobolev's Embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. Notice that in dimension $d = 2$ the case $(r,s) = (2,\infty)$ is true, again from Sobolev's Embedding $\dot{H}^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$.

However, for $d = 3$ there is a difference between (7.22) and (7.21).

*Proof.* Let us start by assuming $u_0 \in V$. Then we know that there exists $T^* > 0$ s.t. $u \in L^\infty([0,T_1], V)$ if $T_1 \in (0,T^*)$ and that $u \in C^\infty((0,T^*) \times \mathbb{R}^3, \mathbb{R}^3)$. So for the regularity part of the lemma, it is enough to show that $T < T^*$. Suppose the opposite, that is $\infty > T \geq T^*$. Then, recall that there is the blow up $\|\nabla u(t)\|_{L^2} \xrightarrow{t \nearrow T^*} \infty$.

We have in $[0,T^*)$, by (7.9),

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+2\|\triangle u\|_{L^2}^2=2\left\langle \operatorname{div}(u\otimes u),\triangle u\right\rangle\le c\|u\|_{L^s}\|\nabla u\|_{L^{\frac{2s}{s-2}}}\|\triangle u\|_{L^2},$$

where $\frac{1}{s}+\frac{s-2}{2s}+\frac{1}{2}=1$ and where $s>3$ implies $\frac{2s}{s-2}\in[2,6)$. Then, by Hölder's (we use $\dfrac{s-2}{2s}=\dfrac{\frac{s-3}{s}}{2}+\dfrac{\frac{3}{s}}{6}$ and $\|f\|_{\frac{2s}{s-2}}\le\|f\|_{L^2}^{\frac{s-3}{s}}\|f\|_{L^6}^{\frac{3}{s}}$) and Young's (using $1=\dfrac{s-3}{2s}+\dfrac{s+3}{2s}$) inequalities and by Sobolev's immersion $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}&c\|u\|_{L^s}\|\nabla u\|_{L^{\frac{2s}{s-2}}}\|\triangle u\|_{L^2}\le c\|u\|_{L^s}\|\nabla u\|_{L^2}^{\frac{s-3}{s}}\|\nabla u\|_{L^6}^{\frac{3}{s}}\|\triangle u\|_{L^2}\\&\le c'\|u\|_{L^s}\|\nabla u\|_{L^2}^{\frac{s-3}{s}}\|\triangle u\|_{L^2}^{\frac{s+3}{s}}\le c''\left(\|u\|_{L^s}\|\nabla u\|_{L^2}^{\frac{s-3}{s}}\right)^{\frac{2s}{s-3}}+\left(\|\triangle u\|_{L^2}^{\frac{s+3}{s}}\right)^{\frac{2s}{s+3}}.\end{aligned}$$

Then we conclude

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2\le\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\triangle u\|_{L^2}^2\le c''\|u\|_{L^s}^{\frac{2s}{s-3}}\|\nabla u\|_{L^2}^2$$

which, by Gronwall's inequality, in $[0,T^*)$ yields

$$\|\nabla u(t)\|_{L^2}^2\le\|\nabla u(0)\|_{L^2}^2e^{c''\int_0^t\|u(t')\|_{L^s}^{\frac{2s}{s-3}}dt'}\le\|\nabla u(0)\|_{L^2}^2e^{c''\|u\|_{L^r([0,T],L^s)}^{\frac{2s}{s-3}}}.$$

But this contradicts the blow up and shows that $T^*>T$. Hence the regularity $u\in C^\infty((0,T]\times\mathbb{R}^3,\mathbb{R}^3)$ is proved when $u_0\in V$. More generally, if $u_0\notin V$, we can consider a sequence $t_n\searrow 0$ with $u(t_n)\in V$, from this and the uniqueness Theorem 7.2 conclude $u\in C^\infty((t_n,T]\times\mathbb{R}^3,\mathbb{R}^3)$ for all $n$, and hence also $u\in C^\infty((0,T]\times\mathbb{R}^3,\mathbb{R}^3)$. The statement that $u$ in $[\epsilon,T]$, for any $\epsilon\in(0,T)$, is also a solution in the sense of Theorem 7.1, has been assumed implicitly in this proof, but is easily proved using Theorem 7.2.

Next, we turn to the discussion of the uniqueness in the present theorem. So let us consider a solution $v$ like in the statement. We claim that $u$ can be used as test function for $v$ and $v$ can be used as a test function of $u$ in the formula (6.4).
Assuming the claim, we have

$$\begin{aligned}&\int_0^t\left(\langle\nabla v,\nabla u\rangle-\langle v,\partial_t u\rangle+\langle\operatorname{div}(v\otimes v),u\rangle\right)dt'=\|u_0\|_{L^2}^2-\langle v(t),u(t)\rangle\text{ and}\\&\int_0^t\left(\langle\nabla v,\nabla u\rangle-\langle\partial_t v,u\rangle+\langle\operatorname{div}(u\otimes u),v\rangle\right)dt'=\|u_0\|_{L^2}^2-\langle v(t),u(t)\rangle.\end{aligned}$$

We can write the above as

$$\begin{aligned}&\int_0^t\left(\langle\nabla v,\nabla u\rangle-\langle v,\partial_t u\rangle+\langle\operatorname{div}(v\otimes v),u\rangle\right)dt'=\|u_0\|_{L^2}^2-\langle v(t),u(t)\rangle\text{ and}\\&\int_0^t\left(\langle\nabla v,\nabla u\rangle+\langle v,\partial_t u\rangle+\langle\operatorname{div}(u\otimes u),v\rangle\right)dt'=0\end{aligned}$$

where in the 2nd equality we have used the fact that $u, v \in C^\infty((0,T], L^2)$, which follows from the 1st part of the proof, integration by parts with the information that $u(t) \xrightarrow{t\to 0^+} u_0$ and $v(t) \xrightarrow{t\to 0^+} u_0$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.Adding the two equations, we obtain the same equation

$$\int_0^t (2\langle \nabla v, \nabla u\rangle + \langle \text{div}(u\otimes u), v\rangle + \langle \text{div}(v\otimes v), u\rangle)\, dt' = \|u_0\|_{L^2}^2 - \langle v(t), u(t)\rangle. \tag{7.13}$$

of the uniqueness Theorem 7.2. Proceeding with the same algebraic manipulations, we arrive to

$$\begin{aligned} &2^{-1}\|w(t)\|_{L^2}^2 + \int_0^t \big(\|\nabla w\|^2 - \langle \text{div}(w\otimes w), u\rangle\big)\, dt' \\ &= 2^{-1}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|^2 - 2^{-1}\|u_0\|_{L^2}^2 \qquad (7.23)\\ &+ 2^{-1}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|^2 - 2^{-1}\|v_0\|_{L^2}^2 \le 0, \qquad (7.24) \end{aligned}$$

where the inequality follows from the fact that both $u$ and $v$ satisfy the Energy Inequality (6.7), and so the last two lines are both $\le 0$.
Therefore we get for $w = v - u$

$$\|w(t)\|_{L^2}^2 + 2\int_0^t \|\nabla w\|^2 dt' \le 2\int_0^t \langle \text{div}(w\otimes w), u\rangle dt'$$

Like above in the proof of this theorem, we bound

$$\begin{aligned} &2\text{div}(w\otimes w), u\rangle \le c\|u\|_{L^s}\|w\|_{L^{\frac{2s}{s-2}}}\|\nabla w\|_{L^2} \\ &\le c\|u\|_{L^s}\|w\|_{L^2}^{\frac{s-3}{s}}\|\nabla w\|_{L^2}^{\frac{s+3}{s}} \le c''\|u\|_{L^s}^r\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \text{ where } r = \frac{2s}{s-3}. \end{aligned}$$

Then

$$\|w(t)\|_{L^2}^2 \le c''\int_0^t \|u(t')\|_{L^s}^r \|w(t')\|_{L^2}^2 dt'$$

implies by Gronwall $w(t) \equiv 0$ in $[0,T]$, proving uniqueness.

Now we have to prove the claim that $v$ (and $u$) can be used as test functions in (6.4). Suppose that $v$ is a weak solution like in Leray's Theorem satisfying the Serrin condition and let $u$ be a weak solution like in Leray's Theorem. We know that $v \in C^\infty([\epsilon, T])$ for any $\epsilon \in (0,T)$. So, for $t \in (\epsilon, T)$ we get

$$\int_\epsilon^t \langle u, \partial_t v\rangle dt' = \langle u(\epsilon), v(\epsilon)\rangle - \langle u(t), v(t)\rangle + \int_\epsilon^t \langle \nabla u(t'), \nabla v(t')\rangle dt' + \int_\epsilon^t \langle \text{div}(u\otimes u), v\rangle dt'.$$

Let us consider now the limit $\epsilon \to 0^+$. We know by the right continuity Lemma 6.18 that $u(\epsilon) \xrightarrow{\epsilon\to 0^+} u_0$ and $v(\epsilon) \xrightarrow{\epsilon\to 0^+} v_0$ in $L^2(\mathbb{R}^3,\mathbb{R}^3)$ so that

$$(\langle u(\epsilon), v(\epsilon)\rangle - \langle u(t), v(t)\rangle) \xrightarrow{\epsilon\to 0^+} (\langle u_0, v_0\rangle - \langle u(t), v(t)\rangle) .$$

Next, $u,v \in L^2((0,t),H^1)$ implies

$$\int_\epsilon^t \langle \nabla u(t'), \nabla v(t')\rangle dt' \xrightarrow{\epsilon\to 0^+} \int_0^t \langle \nabla u(t'), \nabla v(t')\rangle dt'.$$

Finally, we show that

$$\lim_{\epsilon\to 0^+} \int_\epsilon^t \langle \operatorname{div}(u\otimes u), v\rangle dt' \text{ exists and is finite.} \tag{7.25}$$

The above limits are sufficient to prove

$$\int_0^t \langle u, \partial_t v\rangle dt' = \langle u_0, v_0\rangle - \langle u(t), v(t)\rangle + \int_0^t \langle \nabla u(t'), \nabla v(t')\rangle dt' + \int_0^t \langle \operatorname{div}(u\otimes u), v\rangle dt',$$

and hence the claim. To prove (7.25) it is sufficient to show that

$$I := \int_0^t |\langle \operatorname{div}(u\otimes u), v\rangle| dt' < \infty.$$

We bound, using $r' = \frac{2s}{s+3}$,

$$\begin{aligned} I &\lesssim \int_0^t \|v\|_{L^s} \|u\|_{L^2}^{\frac{s-3}{s}} \|\nabla u\|_{L^2}^{\frac{s+3}{s}} dt' \lesssim \int_0^t \|v\|_{L^s}^r dt' + \int_0^t \|u\|_{L^2}^{\frac{(s-3)r'}{s}} \|\nabla u\|_{L^2}^{\frac{(s+3)r'}{s}} dt' \\ &\le \|v\|_{L^r((0,t),L^s)}^r + \|u\|_{L^\infty(\mathbb{R}_+,L^2)}^{\frac{2s-6}{s+3}} \int_0^t \|\nabla u\|_{L^2}^2 dt < \infty. \end{aligned}$$

□

# 8 Well posedness in Sobolev spaces

In Sections 8–10 we follow [1]. The theory is mostly due to T.Kato. The approach will be different and the results will partially overlap with the ones in previous sections. To explain the approach we go back to equation (6.13) and observe that if $\mathcal{Q}_{NS}(u,u)$ is a force like the $f$ in (4.1), we can interpret the solutions of (6.13) as solutions of a linear heat equation (4.1). More specifically, if we denote by $B(u,v)$ the weak solution of

$$\begin{cases} \partial_t B(u,v) - \triangle B(u,v) = \mathcal{Q}_{NS}(u,v) \\ B(u,v)|_{t=0} = 0. \end{cases} \tag{8.1}$$

then, when we are within the scope of the theory of Sect. 4, the solutions of (6.13) can be rewritten as

$$u = e^{t\triangle}u_0 + B(u,u). \tag{8.2}$$

In fact in Sections 8–10, for us the Navier Stokes equation will be equation (8.2). In Sect. 10 we will give an explicit formula to the operator $B(u,v)$. It is an integral operator whose integral kernel is the so called Oseen kernel. We will try to solve the problem by means of a fixed point argument. Specifically, we will look for an appropriate Banach space $X_T$ of functions defined in $[0,T]\times\mathbb{R}^d$, for a subspace $\mathcal{E}\subset\mathcal{S}'(\mathbb{R}^d,\mathbb{R}^d)$ such that $u_0\in\mathcal{E}$ implies $e^{\nu t\triangle}u_0\in X_T$ and furthermore $\|e^{\nu t\triangle}u_0\|_{X_T}\xrightarrow{T\to 0^+}0$, and we will use Lemma 6.9.

In this section we will discuss the case $X = X_T = L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$ and space of initial data $\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)$ and use the abstract Lemma 6.9 to prove the following well posedness result.

**Theorem 8.1.** *For any $u_0\in\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)$ there exists a $T$ and a solution of* (8.2) *with $u\in L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$. This solution is unique. Furthermore we have*

$$u\in C([0,T],\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)),\ \nabla u\in L^2([0,T],\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)). \tag{8.3}$$

*Let $T_{u_0}$ be the lifespan of the solution. Then:*

*(1) there exists a $c$ s.t.*

$$\|u_0\|_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)}\le c\Rightarrow T_{u_0}=\infty;$$

*(2) if $T_{u_0}<\infty$ then*

$$\int_0^{T_{u_0}}\|u(t)\|^4_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}dt=\infty. \tag{8.4}$$

*(3) if $T_{u_0}<\infty$ then*

$$\int_0^{T_{u_0}}\|\nabla u(t)\|^2_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)}dt=\infty. \tag{8.5}$$

*Moreover, if $u$ and $v$ are solutions, then*

$$\begin{aligned}&\|u(t)-v(t)\|^2_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)}+\int_0^t\|\nabla(u-v)(s)\|^2_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)}ds\\ &\le\|u_0-v_0\|^2_{\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)}e^{C\int_0^t\left(\|u(t')\|^4_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}+\|v(t')\|^4_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}\right)dt'}\end{aligned} \tag{8.6}$$

*where $C$ is a fixed constant.*

*Remark* 8.2. Notice that the following transformation preserves the solutions of the Navier Stokes equation:

$$u\left(t,x\right)\mapsto u_\lambda\left(t,x\right):=\lambda u\left(\lambda^2t,\lambda x\right), \tag{8.7}$$

Furthermore, notice that the norms of $u$ in the spaces in (8.3) coincide with the analogous norms of $u_\lambda$ in the interval $[0, T/\lambda^2]$. Notice also that the norm of $u_0(x)$ in $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$ coincides with the norm of $u_0(x/\lambda)$ in the same space. So the space $\dot{H}^{\frac{d}{2}-1}$ is an example of space *critical* for the Navier Stokes equation. One obvious consequence of this is the following: there exists no function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ s.t. $T_{u_0} \geq T(\|u_0\|_{\dot{H}^{\frac{d}{2}-1}})$ for all $u_0 \in \dot{H}^{\frac{d}{2}-1}$.

*Remark* 8.3. While for $d = 2$ the solutions provided by Theorem 8.1 are exactly Leray's solutions, for $d = 3$ we could have $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3)$ with $u_0 \notin L^2(\mathbb{R}^3, \mathbb{R}^3)$. The corresponding solutions of the Navier Stokes equations provided by Theorem 8.1 are not Leray's solutions.

*Remark* 8.4. We will prove in Sect. 10 that the solutions provided by Theorem 8.1 are in $C^\infty((0, T) \times \mathbb{R}^d, \mathbb{R}^d)$.

*Remark* 8.5. Notice that the finite lifespan (8.4) is relevant only for $d = 3$. Furthermore, if $T_{u_0} < \infty$, it has been shown that

$$\|u\|_{L^\infty([0,T_{u_0}], \dot{H}^1(\mathbb{R}^3,\mathbb{R}^3))} = \infty,$$

but the proof is a much harder.

We will assume for the moment Theorem 8.1 and prove the following.

# 9 Proof of Theorem 8.1

This section is devoted to the proof of this theorem. First we have the following lemma.

**Lemma 9.1.** *Let $d = 2, 3$. There exists a constant $C > 0$ s.t.*

$$\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d,\mathbb{R}^d)} \leq C\|u\|_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}\|v\|_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}. \tag{9.1}$$

*Proof.* If $d = 2$ we have

$$\begin{aligned}
\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{-1}} &\leq \sum_{j,k=1}^{2} \left( \|\partial_k(u^k v^j)\|_{\dot{H}^{-1}} + \|\partial_k(v^k u^j)\|_{\dot{H}^{-1}} \right) \\
&\leq 2\sum_{j,k}^{2} \|u^k v^j\|_{L^2} \leq C\|u\|_{L^4}\|v\|_{L^4} \leq C\|u\|_{\dot{H}^{\frac{1}{2}}}\|v\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}$$

by the Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ , since $\frac{1}{4} = \frac{1}{2} - \frac{\frac{1}{2}}{2}$. This yields (9.1) for $d = 2$. For $d = 3$

$$\begin{aligned}
\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} &\leq \sum_{j,k}^{2} \left( \|\partial_k(u^k v^j)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} + \|\partial_k(v^k u^j)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \right) \\
&\lesssim \|(\nabla u)v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} + \|u\nabla v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|(\nabla u)v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|u\nabla v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}
\end{aligned}$$

where we are using the Sobolev embedding $\dot H^{\frac12}(\mathbb{R}^3)\subset L^3(\mathbb{R}^3)$ (since $\frac13=\frac12-\frac{\frac12}{3}$) which in turn by duality implies $L^{\frac32}(\mathbb{R}^3)\subset \dot H^{-\frac12}(\mathbb{R}^3)$.
Hence, by $\frac23=\frac12+\frac16$ and Hölder,

$$\|\mathcal{Q}_{NS}(u,v)\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)}+\|u\|_{L^6(\mathbb{R}^3)}\|\nabla v\|_{L^2(\mathbb{R}^3)}\le 2\|u\|_{\dot H^1(\mathbb{R}^3)}\|v\|_{\dot H^1(\mathbb{R}^3)}.$$

This yields (9.1) for $d=3$. □

A straightforward consequence of Lemma 9.1 is the following for $C$ the constant in Lemma 9.1.

**Lemma 9.2.** *Let $d=2,3$. Then for $u,v\in L^4([0,T],(\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$ we have*

$$\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2}(\mathbb{R}^d,\mathbb{R}^d))}\le C\|u\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))}\|v\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))} \tag{9.2}$$

□

*Proof of Theorem 8.1.* By Theorem 4.4 we have for $s=\frac d2-1$ and $p=4$

$$\begin{aligned}&\|B(u,v)\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}=\|\|B(u,v)\|_{\dot H^{s+\frac2p}}\|_{L^p(0,T)}\lesssim\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{s-1})}\\&=\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2})}\le C\|u\|_{L^4([0,T],\dot H^{\frac{d-1}{2}}}\|v\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}.\end{aligned} \tag{9.3}$$

So in the Banach space $X=L^4([0,T],\dot H^{\frac{d-1}{2}})$ we have $\|B\|\le C$. Obviously this is the same as $\frac{1}{4C}\le\frac{1}{4\|B\|}$. Our strategy is to prove

$$\|e^{t\triangle}u_0\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}<\frac{1}{4C}\le\frac{1}{4\|B\|} \tag{9.4}$$

where $e^{t\triangle}u_0$ plays the role of $x_0$ in the abstract Lemma 6.9.
If (9.4) happens, that is if the l.h.s.of (9.4) is less than an $\alpha<\frac{1}{4\|B\|}$, then by Lemma 6.9 we can conclude that problem (8.2) admits a unique solution in $L^4([0,T],\dot H^{\frac{d-1}{2}})$ with norm less than $2\alpha<\frac{\nu^{\frac34}}{2C}$.
We consider two distinct proofs of (9.4). The 1st, simpler, is valid only if $\|u_0\|_{\dot H^{\frac d2-1}}$ is sufficiently small and shows that (9.4) holds for all $T$. In the second proof, which is general, we drop the assumption that $\|u_0\|_{\dot H^{\frac d2-1}}$ is small, and we prove (9.4) for $T$ sufficiently small.
**Step 1: small initial data.** By Theorem 4.4 we have for $s=\frac d2-1$ and $p=4$

$$\|e^{mt\triangle}u_0\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}=\|\|e^{mt\triangle}u_0\|_{\dot H^{s+\frac2p}}\|_{L^p(0,T)}\le m|u_0\|_{\dot H^s}=m\|u_0\|_{\dot H^{\frac d2-1}}. \tag{9.5}$$

So, if $\|u_0\|_{\dot H^{\frac d2-1}}<\frac{1}{4C}$ then (9.4) is true for any $T>0$. In particular $T_{u_0}=\infty$ and we have just proved (1) in Theorem 8.1.

**Step 2: possibly large initial data**. Now we consider the case when $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ is possibly large. We consider a low–high energy decomposition: $u_0 = \mathbf{P}_\rho u_0 + \chi_{\sqrt{-\triangle}\geq\rho} u_0$ where we pick $\rho = \rho_{u_0}$ large enough so that

$$\|\chi_{\sqrt{-\triangle}\geq\rho} u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \frac{1}{8C}.$$

Then by (9.5) we get

$$\begin{aligned}
&\|e^{t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \leq \|e^{t\triangle}\chi_{\sqrt{-\triangle}\geq\rho} u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} + \|e^{t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \\
&< \frac{1}{8C} + \|e^{t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}
\end{aligned} \tag{9.6}$$

where we made the high energy contribution small by the choice of $\rho$ large.
We now exploit the fact that we have the freedom to choose $T$ small, in order to make the contribution to (9.6) small too. Indeed we have

$$\begin{aligned}
&\|e^{t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} = \|e^{t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \\
&= \|e^{t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})\sqrt{\rho}\frac{(-\triangle)^{\frac{1}{4}}}{\sqrt{\rho}}u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} \\
&\leq \sqrt{\rho}\|e^{t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} = \sqrt{\rho}\|e^{t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} \\
&\leq (\rho^2 T)^{\frac{1}{4}}\|e^{t\triangle}\mathbf{P}_\rho u_0\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} \leq (\rho^2 T)^{\frac{1}{4}}\|\mathbf{P}_\rho u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq (\rho^2 T)^{\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq \frac{1}{8C}
\end{aligned}$$

if we choose $T$ small enough so that the last inequality holds, that is if we choose $T$ such that

$$T \leq \left(\frac{1}{8\rho^{\frac{1}{2}}C\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}}\right)^4, \tag{9.7}$$

then all terms in the r.h.s. of (9.6) have been made small enough s.t.

$$\|e^{t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} < \frac{1}{4C} \leq \frac{1}{4\|B\|},$$

that is we obtained (9.4).

We have proved the 1st sentence in the statement of Theorem 8.1.
Now we turn to the proof that a solution $u \in L^4([0,T],\dot{H}^{\frac{d-1}{2}})$ satisfies (8.3).
By (9.1) we have $\mathcal{Q}_{NS}(u,u) \in L^2([0,T],\dot{H}^{\frac{d}{2}-2})$. Then it must be remarked that by its definition $B(u,u)$ is a solution in the sense of Definition 4.1 of the Heat Equation written above (8.2). Similarly, by Theorem 4.2 also $e^{\nu t\triangle}u_0$ is a solution of the homogeneous Heat Equation with initial value $u_0$. Hence, since $u$ satisfies (8.2), then $u$ is the solution of the Heat Equation (6.13), where the latter can be framed in terms of the theory in Sect. 4 for

$s = \frac{d}{2} - 1$. Then by Theorem 4.2 we have $u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$ and $\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}})$. This yields (8.3).
We turn now to the proof of (8.6). We consider two solutions $u$ and $v$, and set $w = u - v$. Then

$$\begin{cases} w_t - \triangle w = \mathcal{Q}_{NS}(w, u+v) \\ \qquad w(0) = u_0 - v_0 \end{cases}$$

where we used the symmetry $\mathcal{Q}_{NS}(u,v) = \mathcal{Q}_{NS}(v,u)$ and

$$\mathcal{Q}_{NS}(u-v, u+v) = \mathcal{Q}_{NS}(u,u) - \mathcal{Q}_{NS}(v,v) + \underbrace{\mathcal{Q}_{NS}(u,v) - \mathcal{Q}_{NS}(v,u)}_{0}.$$

By the energy estimate (4.5) for $s = \frac{d}{2} - 1$ we have

$$\Delta_w := \|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' = \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \langle \mathcal{Q}_{NS}(w, u+v), w\rangle_{\dot{H}^{\frac{d}{2}-1}}(t')dt'.$$

**Claim 9.3.** We have

$$\langle \mathcal{Q}_{NS}(a,b), c\rangle_{\dot{H}^{\frac{d}{2}-1}} \le C\|a\|_{\dot{H}^{\frac{d-1}{2}}} \|b\|_{\dot{H}^{\frac{d-1}{2}}} \|c\|_{\dot{H}^{\frac{d}{2}}}. \tag{9.8}$$

*Proof.* Indeed, trading derivatives we have

$$\langle \mathcal{Q}_{NS}(a,b), c\rangle_{\dot{H}^{\frac{d}{2}-1}} \le \|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}} \|c\|_{\dot{H}^{\frac{d}{2}}}$$

and by (9.1) we have

$$\|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}} \le C\|a\|_{\dot{H}^{\frac{d-1}{2}}} \|b\|_{\dot{H}^{\frac{d-1}{2}}}.$$

This proves Claim 9.3.

Now for $N(t) := \|u(t)\|_{\dot{H}^{\frac{d-1}{2}}} + \|v(t)\|_{\dot{H}^{\frac{d-1}{2}}}$ by Claim 9.3 we have

$$\Delta_w \le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} N(t') \|\nabla w(t')\|_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

By the interpolation estimate in Lemma 2.25 we have

$$\|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} \le \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} \|\nabla w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}}.$$

This implies

$$\Delta_w \le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} N(t') \|\nabla w(t')\|^{\frac{3}{2}}_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

Using the inequality $ab \le \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, which follows by

$$\log(ab) = \frac{1}{4}\log(a^4) + \frac{3}{4}\log(b^{\frac{4}{3}}) \le \log\left(\frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}\right),$$

we get

$$\begin{aligned}\text{the integrand} &= \left(\|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} N(t')\nu^{-\frac{3}{4}}\left(\frac{3}{4}\right)^{\frac{3}{4}}\right)\left(\frac{4}{3}\|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}}\right)^{\frac{3}{4}} \\ &\le \frac{3^3}{4^4}\|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t') + \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}}.\end{aligned}$$

Then

$$\Delta_w \le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{3^3}{4^4}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt' + \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

In other words, by the definition of $\Delta_w$

$$\begin{aligned}&\|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \cancel{2}\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \\ &\le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{3^3}{4^4}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt' + \cancel{\int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'}\end{aligned}$$

so that, if we set

$$X(t) := \|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'$$

we have

$$\begin{aligned}X(t) &\le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{3^3}{4^4}\int_0^t \|w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} N^4(t')dt' \\ &\le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + \frac{3^3}{4^4}\int_0^t X(t')N^4(t')dt'.\end{aligned}$$

So by Gronwall's inequality

$$\|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' \le \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} \exp\left(\frac{3^3}{4^4}\int_0^t N^4(t')dt'\right).$$

This proves the stability inequality (8.6)

We now consider the blow up criterion (8.4). Suppose that $u(t)$ is a solution in $[0,T)$ with

$$\int_0^T \|u(t)\|^4_{\dot{H}^{\frac{d-1}{2}}} dt < \infty.$$

Notice that then $u \in L^4([0,T], \dot{H}^{\frac{d-1}{2}})$ and

$$\|\mathcal{Q}_{NS}(u,u)\|_{L^2([0,T],\dot{H}^{\frac{d}{2}-2})} \le C\|u\|^2_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}. \tag{9.9}$$

We claim that we can extend $u(t)$ beyond $T$.

**Claim 9.4.** There exists a $\tau > 0$ s.t. $u$ extends in a solution in $L^4([0,T+\tau), \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$.

First of all we set

$$g(\xi) := \sup_{0\le t'\le T} |\widehat{u}(t',\xi)|.$$

**Claim 9.5.** We have $|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$.

*Proof of Claim 9.5.* By (4.15) for $s = \frac{d}{2}-1$ and by (9.1) we have

$$\begin{aligned}
\||\xi|^{\frac{d}{2}-1} g\|_{L^2} &= \left( \int_{\mathbb{R}^d} |\xi|^{d-2} \left( \sup_{0\le t'\le t} |\widehat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}} \\
&\le \|u_0\|_{\dot{H}^{\frac{d}{2}-1}} + \frac{1}{2^{\frac{1}{2}}} \|\mathcal{Q}_{NS}\|_{L^2([0,T],\dot{H}^{\frac{d}{2}-2})} \\
&\le \|u_0\|_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{2^{\frac{1}{2}}} \|u\|^2_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} < \infty.
\end{aligned}$$

This proves Claim 9.5.

*Proof of Claim 9.4.* Claim 9.5 implies

$$\int_{|\xi|\ge\rho} |\xi|^{d-2} |g(\xi)|^2 d\xi \xrightarrow{\rho\to+\infty} 0.$$

Thus there exists $\rho > 0$ s.t for any preassigned $c > 0$

$$\int_{|\xi|\ge\rho} |\xi|^{d-2} |\widehat{u}(t,\xi)|^2 d\xi < c^2 \text{ for all } t \in [0,T).$$

Now, recalling the splitting in high and low energies in the proof of the 1st sentence in the statement of Theorem 8.1, there exists a fixed $\tau > 0$ s.t. the lifespan of the solution with initial datum $u(t)$ is bounded below by $\tau$ independently of $t \in [0,T)$. Indeed there exists a $c_1 > 0$ independent from $t \in [0,T)$ s.t.

$$\left( \frac{1}{8\rho^{\frac{1}{2}} C \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}} \right)^4 > c_1 > 0.$$

This follows from the fact that

$$\|u(t)\|_{\dot{H}^{\frac{d}{2}-1}} \le \||\xi|^{\frac{d}{2}-1} g\|_{L^2} < \infty$$

So we can take $\tau = c_1$. Then $T_{u_0} \ge T + \tau$ and this yields Claim 9.4.

Let us now discuss the blow up criterion (8.5). Suppose that $T_{u_0} < \infty$ and that

$$C_{L2} := \int_0^{T_{u_0}} \|\nabla u(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} dt < \infty. \tag{9.10}$$

Since we have (8.4) and

$$L^4([0,T],\dot H^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)) \subseteq L^\infty([0,T],\dot H^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2([0,T],\dot H^{\frac{d}{2}}(\mathbb{R}^d,\mathbb{R}^d))$$

it follows that since we must have (8.4), then (9.10) implies that

$$\lim_{T\to T_{u_0}} \|u(t)\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} = \infty \tag{9.11}$$

For $0 \le t \le T < T_{u_0}$ we have, by (9.8) and interpolation,

$$\begin{aligned}
&\|u(t)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' = \|u(t_1)\|^2_{\dot H^{\sigma_s}} + 2\int_0^t \langle Q(u(t'),u(t')),u(t')\rangle_{\dot H^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C'_d \int_0^t \|u(t')\|^2_{\dot H^{\frac{d-1}{2}}} \|\nabla u(t')\|_{\dot H^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C_d \int_0^t \|u(t')\|_{\dot H^{\frac{d}{2}-1}} \|\nabla u(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'
\end{aligned} \tag{9.12}$$

and so

$$\|u\|^2_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} \leqslant \|u(0)\|^2_{\dot H^{\frac{d}{2}-1}} + C_d C_{L2} \|u\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})}.$$

But this means that

$$\|u\|_{L^\infty([0,T],\dot H^{\frac{d}{2}-1})} \le \frac{1}{2} C_d C_{L2} + \frac{1}{2}\sqrt{C_d^2 C_{L2}^2 + 4\|u(0)\|^2_{\dot H^{\frac{d}{2}-1}}} < \infty,$$

contradicting (9.11). This contradiction proves the blow up criterion (8.5).
The proof of Theorem 8.1 is completed. □

Theorem 8.1 yields also an alternative proof of Leray's Theorem 6.5 for $d = 2$.

**Corollary 9.6.** *In the case $d = 2$, Theorem 8.1 implies Leray's Theorem 6.5 for $d = 2$*

*Proof.* By the Leray's Theorem 6.3 we know that given a divergence free $u_0 \in L^2(\mathbb{R}^2,\mathbb{R}^2)$ there are weak solutions in the sense of Leray with $u \in L^\infty([0,\infty),L^2(\mathbb{R}^2,\mathbb{R}^2))$ and $\nabla u \in L^2([0,\infty),L^2(\mathbb{R}^2,\mathbb{R}^4))$. Interpolating, for each such a solution we have

$$\| \|u\|_{\dot H^{\frac{1}{2}}} \|_{L^4_t} \le \| \|u\|^{\frac{1}{2}}_{L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2} \|_{L^4_t} \le \|u\|^{\frac{1}{2}}_{L^\infty_t L^2} \|\nabla u\|^{\frac{1}{2}}_{L^2_t L^2}$$

and so we obtain also $u \in L^4([0,\infty),\dot H^{\frac{1}{2}}(\mathbb{R}^2,\mathbb{R}^2))$.
By Lemma 9.2 we know that this implies

$$\mathcal{Q}_{NS}(u,u) \in L^2([0,\infty),\dot H^{-1}(\mathbb{R}^2,\mathbb{R}^2)).$$

Notice that the right hand side of (6.13) satisfies the hypothesis of the force term in the linear heat equation (4.1). As a weak solution of the Navier Stokes equation in the sense of Definition 6.1, $u$ is then also a solution of the linear heat equation (4.1) in the sense of

Definition 4.1. This means that it is also a solution of (8.2). Since by Theorem 8.1 such solution is a unique, we conclude that the solution of Leray's Theorem 6.3 in the case $d = 2$ is unique. Furthermore by Theorem 8.1 we know also that $u \in C^0([0, \infty), L^2(\mathbb{R}^2, \mathbb{R}^2))$.
We now turn to the energy identity. By Leray's Theorem 6.3 we know that

$$\|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^2)} dt' \leq \|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

We want now to prove that $\leq$ can be replaced by $=$ in this formula. As we have mentioned above, $u$ solves in the sense of Definition 4.1 the problem

$$\partial_t u - \nu \triangle u = \mathcal{Q}_{NS}(u, u) \text{ with } \mathcal{Q}_{NS}(u, u) \in L^2(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^2, \mathbb{R}^2)),$$

Then, by Theorem 4.2 for $s = 0$ the identity (4.5) yields

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(t')\|^2_{L^2} dt' = \|u_0\|^2_{L^2} + 2\int_0^t \langle \mathcal{Q}_{NS}(u(t'), u(t')), u(t')\rangle_{L^2} dt'.$$

By Lemma 6.7 we have the cancelation

$$\langle \mathcal{Q}_{NS}(u, u), u\rangle = \langle \mathbb{P}(\text{div}(u \otimes u), u\rangle = \langle \text{div}(u \otimes u), u\rangle = 0.$$

This completes the proof, by giving the energy identity. □

# 10 The case of initial data in $L^3(\mathbb{R}^3)$

It is possible to prove the following theorem.

**Theorem 10.1.** *For any divergence free $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $T > 0$ and a unique solution $u \in C^0([0, T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ of*

$$u = e^{t\triangle} u_0 + B(u, u). \tag{8.2}$$

*Furthermore there exists a $\varepsilon_3 > 0$ s.t. for $\|u_0\|_{L^3} < \varepsilon_3$ we have $T = \infty$. Furthermore, if $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$, the life span is the same of Theorem 8.1.*

**Exercise 10.2.** Prove that the mapping $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3) \to L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

**Exercise 10.3.** Prove that the subspace of divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ is closed in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Prove the same for with $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ replaced by $L^3(\mathbb{R}^3, \mathbb{R}^3)$.

**Exercise 10.4.** Prove that the Sobolev embedding from the subspace of divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ to the subspace of divergence free vector fields in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

**Exercise 10.5.** Pick a divergence free $u_0$ belonging to $L^3(\mathbb{R}^3, \mathbb{R}^3)$ but not to $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Show that there exists a sequence of divergence free vector fields $\{u_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $u_0^{(n)} \to u_0$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$. Show also that $\|u_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$.

**Exercise 10.6.** Show that it is possible to define divergence free sequences $\{v_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $\|v_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$ and $\|v_0^{(n)}\|_{L^3} \to 0$.

*Remark* 10.7. For a sequence such as in Exercise 10.6, for $n \gg 1$ the corresponding solutions of the NS equation are globally defined in time by Theorem 10.13, while Theorem 8.1 is able to guarantee only on short intervals of time.

To prove Theorem 10.13 we will apply the abstract Lemma 6.9 in an appropriate Banach space $X$. The striking fact though, is that the space $X$ will not be of the form $C^0([0,T], L^3(\mathbb{R}^3, \mathbb{R}^3))$. This because if $X$ where this space, then the bilinear form $B$ defined by (8.1) is known not to be continuous. It turns out that to get the right Banach space $X$, has required a certain degree of imagination and insight.

**Definition 10.8** (Kato's Spaces)**.** For $p \in [d, \infty]$ and $T \in (0, \infty)$ we set

$$K_p(T) := \{u \in C^0((0,T], L^p(\mathbb{R}^d, \mathbb{R}^d)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} t^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\} \tag{10.1}$$

and for $p \in [1, d)$

$$K_p(T) := \{u \in C^0([0,T], L^p(\mathbb{R}^d, \mathbb{R}^d)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} t^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\}. \tag{10.2}$$

We denote by $K_p(\infty)$ the spaces defined as above, with $(0,T]$ replaced by $(0,\infty)$.

We recall that the solution of the heat equation $u_t - \nu\triangle u = 0$ is $e^{t\triangle} f = K_t * f$ where $K_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. Notice that $K_t(x) = t^{-\frac{d}{2}} K(t^{-\frac{1}{2}} x)$, where $K(x) := (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$ and where $\widehat{K}(\xi) = e^{-|\xi|^2}$.

Notice that for $u_0 \in L^d(\mathbb{R}^d)$ and $p \geq d$ we have from (1.13),

$$\|e^{t\Delta} u_0\|_{L^p(\mathbb{R}^d)} \leqslant (4\pi t)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{d}\right)} \|u_0\|_{L^d(\mathbb{R}^d)} \text{ for all } p \geq d, \tag{10.3}$$

it can be proved that $e^{t\Delta} u_0 \in C(\mathbb{R}_+, L^p)$, and so $e^{t\Delta} u_0 \in K_p(\infty)$.

**Lemma 10.9.** *Let $u_0 \in L^d(\mathbb{R}^d, \mathbb{R}^d)$ and $p > d$. Then*

$$\lim_{T \to 0} \|e^{t\Delta} u_0\|_{K_p(T)} = 0. \tag{10.4}$$

*Proof.* For any $\epsilon > 0$ there exists $\phi \in L^d(\mathbb{R}^d, \mathbb{R}^d) \cap L^p(\mathbb{R}^d, \mathbb{R}^d)$ s.t. $\|u - \phi\|_{L^d} < \epsilon$. Then by (10.3) we have

$$\|u - \phi\|_{K_p(T)} \leq (4\pi T)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{d}\right)} \epsilon.$$

Since $\|e^{t\Delta}\phi\|_{L^p} \leq \|\phi\|_{L^p}$, it follows

$$\|e^{t\nu\triangle}\phi\|_{K_p(T)} = \sup_{t \in (0,T]} t^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} \|e^{t\Delta}\phi\|_{L^p} \leq T^{\frac{d}{2}\left(\frac{1}{d} - \frac{1}{p}\right)} \|\phi\|_{L^p} \xrightarrow{T \to 0} 0.$$

□

**Lemma 10.10.** *Let $p$, $q$ and $r$ satisfy*

$$\begin{aligned} & 0 < \frac{1}{p} + \frac{1}{q} \leqslant 1 \\ & \frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{q} < \frac{1}{d} + \frac{1}{r} \end{aligned} \tag{10.5}$$

*Then the bilinear map $B$ defined in (8.1) maps $K_p(T) \times K_q(T) \to K_r(T)$ and there is a constant $C$ independent from $T$ s.t.*

$$\|B(u,v)\|_{K_r(T)} \leqslant C\|u\|_{K_p(T)}\|v\|_{K_q(T)}. \tag{10.6}$$

To prove Lemma 10.10 we consider for any $m = 1, ..., d$ the problem

$$\begin{cases} (L_m f)_t - \triangle L_m f = \mathbb{P}\partial_m f \\ L_m f(0,x) = 0 \end{cases} \tag{10.7}$$

($L_m f$ is by definition the solution of the above heat equation). Then by (4.7) and (6.15) for $c_{ijk}$ the constants s.t. $\widehat{\mathbb{P}u}^i = \sum_{j,k=1}^d c_{ijk}\xi_j\xi_k|\xi|^{-2}\widehat{u}^k$, we have

$$\widehat{L_m f^i}(t,\xi) = \sum_{j,k=1}^{d} c_{ijk} \int_0^t e^{-(t-t')|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}\widehat{f}^k(t',\xi)dt'. \tag{10.8}$$

This means, for $\Gamma_{jkm}(t,x)$ the inverse Fourier transform of $e^{-t|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$,

$$L_m f^i(t) = \sum_{j,k=1}^{d} c_{ijk} \int_0^t \Gamma_{jkm}(t-t') * \widehat{f}^k(t')dt'. \tag{10.9}$$

We claim the following.

**Claim 10.11.** We have for a fixed $C > 0$

$$|\Gamma_{jkm}(t,x)| \leqslant C(\sqrt{t} + |x|)^{-d-1}. \tag{10.10}$$

*Proof.* It is elementary that $\Gamma_{jkm}(t,x) = t^{-\frac{d+1}{2}}\Gamma_{jkm}(t^{-1/2}x)$ with $\widehat{\Gamma}_{jkm}(x) = e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Then (10.10) is a consequence of

$$|\Gamma_{jkm}(x)| \leqslant C(1 + |x|)^{-d-1}. \tag{10.11}$$

It is straightforward that $\Gamma_{jkm} \in C^\infty(\mathbb{R}^{d+1}) \cap L^\infty(\mathbb{R}^{d+1})$, because of the rapid decay to 0 at infinity of $e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Hence, to prove (10.11) it suffices to consider $|x| \gg 1$. For $\chi_0$ a smooth cutoff of compact support equal to 1 near 0 and with $\chi_1 := 1 - \chi_0$, we set

$$\begin{aligned} \Gamma_{jkm}(x) = & (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x}\chi_0\left(|x|\xi\right)e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi \\ & + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x}\chi_1\left(|x|\xi\right)e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi \end{aligned}$$

The 1st term in the r.h.s. is

$$\lesssim \int_{|\xi|\le|x|^{-1}} |\xi|\ d\xi \sim |x|^{-d-1}.$$

We next consider the other term, which we split as

$$(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} \chi_1\left(|x|\xi\right) \chi_0\left(\xi\right) e^{-|\xi|^2} \xi_j\xi_k\xi_m |\xi|^{-2} d\xi \tag{10.12}$$

$$+ (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} \chi_1\left(\xi\right) e^{-|\xi|^2} \xi_j\xi_k\xi_m |\xi|^{-2} d\xi. \tag{10.13}$$

Notice that the last line is $O(|x|^{-N})$ for any $N$. Indeed, $\chi_1\left(\xi\right) e^{-|\xi|^2} \xi_j\xi_k\xi_m |\xi|^{-2} \in \mathcal{S}(\mathbb{R}^d)$, and so also its Fourier transform (10.13) is rapidly decreasing.
Let us consider the term in (10.12). Set $L := \mathrm{i}\frac{x}{|x|^2}\cdot\nabla_\xi$ and notice that $Le^{-\mathrm{i}\xi\cdot x} = e^{-\mathrm{i}\xi\cdot x}$. Then, the term in (10.12) is

$$(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} L^{d+2} \left(\chi_1\left(|x|\xi\right) e^{-|\xi|^2} \xi_j\xi_k\xi_m |\xi|^{-2}\right) d\xi.$$

The absolute value of the integrand is for fixed $C$

$$\left|L^{d+2}\left(\cdots\right)\right| \le C|x|^{-d-2}|\xi|^{-d-1}.$$

Here we used that in the support of $\nabla_\xi\left(\chi_1\left(|x|\xi\right)\right)$ we have $|x|\sim|\xi|^{-1}$. So the last integral is bounded

$$\lesssim |x|^{-d-2} \int_{1\ge|\xi|\ge|x|^{-1}} |\xi|^{-d-1} d\xi \sim |x|^{-d-2}|x| = |x|^{-d-1}.$$

This completes the proof of Claim 10.11.

□

*Completion of proof of Lemma 10.10.* By (10.10) we have by Young's inequality for convolutions and Hölder's inequality for the tensor product of $u$ and $v$ the bound (here $\frac{1}{a} = 1 + \frac{1}{r} - \frac{1}{\beta}$ and $\frac{1}{\beta} = \frac{1}{p} + \frac{1}{q}$)

$$\begin{aligned}\|B(u,v)\|_{L^r} &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\otimes v(t')\right\|_{L^\beta} dt' \\ &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\right\|_{L^p} \left\|v(t')\right\|_{L^q} dt' \\ &\lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} (t')^{-\frac{d}{2}\left(\frac{2}{d}-\frac{1}{p}-\frac{1}{q}\right)} dt' \, \|u\|_{K_p(t)} \|v\|_{K_q(t)} \qquad (10.14)\end{aligned}$$

where in the 3rd line we used

$$\left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a(\mathbb{R}^d)} \lesssim \left\|(\sqrt{t-t'}+|x|)^{-d-1}\right\|_{L^a(\mathbb{R}^d)} = (t-t')^{-\frac{d+1}{2}} \left\|\left(1+\frac{|x|}{\sqrt{t-t'}}\right)^{-d-1}\right\|_{L^a(\mathbb{R}^d)}$$

$$= (t-t')^{-\frac{d+1}{2}}(t-t')^{\frac{d}{2a}} \left\|(1+|x|)^{-d-1}\right\|_{L^a(\mathbb{R}^d)} \sim (t-t')^{-\frac{d+1}{2}+\frac{d}{2}\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)}$$

$$= (t-t')^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}.$$

We then conclude

$$\|B(u,v)\|_{L^r} \leqslant Ct^{-\frac{d}{2}\left(\frac{1}{d}-\frac{1}{r}\right)} \|u\|_{K_p(t)} \|v\|_{K_q(t)} \tag{10.15}$$

where we used the fact that $\forall\ \alpha,\beta \in (-\infty,1)$ we have

$$\int_0^t (t-t')^{-\alpha}(t')^{-\beta}dt' = C(\alpha,\beta)t^{1-\alpha-\beta} \text{ for all } t>0 \text{ and for } C(\alpha,\beta) := \int_0^1 (1-t')^{-\alpha}(t')^{-\beta}dt'. \tag{10.16}$$

and

$$\frac{1}{2}+\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)+\frac{d}{2}\left(\frac{2}{d}-\frac{1}{p}-\frac{1}{q}\right) = \frac{1}{2}+\frac{d}{2}\left(\frac{2}{d}-\frac{1}{r}\right) = \frac{1}{2}+1-\frac{d}{2r}$$

$$= 2-\frac{1}{2}-\frac{d}{2r} = 1+1-\frac{d}{2r} = 1+\frac{d}{2}\left(\frac{1}{d}-\frac{1}{r}\right).$$

Notice that in the inequalities in (10.5) we need:

- $\frac{1}{\beta} := \frac{1}{p}+\frac{1}{q} \leqslant 1$ in order for $u\otimes v$ to belong to the Lebesgue space $L^\beta(\mathbb{R}^d)$;
- $0 < \frac{1}{p}+\frac{1}{q}$ is needed because otherwise in (10.14) we get $(t')^{-1}$ and the integral is undefined;
- $\frac{1}{r} \leqslant \frac{1}{p}+\frac{1}{q}$ is needed for $a \geq 1$;
- $\frac{1}{p}+\frac{1}{q} < \frac{1}{d}+\frac{1}{r}$ is needed to get $-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right) > -1$ in the exponent of $(t-t')$ in (10.14).

□

We have the following fact.

**Proposition 10.12.** *For any $p \in (d,\infty]$ there exists a constant $\varepsilon_{p\nu} > 0$ s.t. if*

$$\|e^{t\triangle}u_0\|_{K_p(T)} < \varepsilon_{p\nu} \tag{10.17}$$

*then there exists and is unique $u$ in the ball of center 0 and radius $2\varepsilon_{p\nu}$ in $K_p(T)$ which satisfies* (8.2).

*Proof.* Setting $r = q = p$, we see that for $p > d$ we have $B : K_p(T) \times K_p(T) \to K_p(T)$ is bounded and with norm that admits a finite upper bound independent from $T$. The proof follows then from the abstract Lemma 6.9. $\square$

**Theorem 10.13.** *For any $u_0 \in L^d(\mathbb{R}^d, \mathbb{R}^d)$ there is a $T > 0$ and solution $u \in C^0([0,T), L^d(\mathbb{R}^d, \mathbb{R}^d))$ of (8.2) which is unique. Furthermore there exists a $\varepsilon_d > 0$ s.t. for $\|u_0\|_{L^d} < \varepsilon_d$ we have $T = \infty$. In the case $d = 2$, in particular, all solutions are defined for all $T > 0$.*

*Proof.* We have $e^{t\triangle}u_0 \in K_p(T)$ for any $p > d$, see (10.3). Furthermore, $\|e^{t\triangle}u_0\|_{K_p(T)} \xrightarrow{T\to 0} 0$ for $p > d$ by Lemma 10.9. Then we can apply Proposition 10.12 concluding that there exists a solution $u$ of (8.2) in $K_{2d}(T)$ for $T > 0$ small enough. Applying Lemma 10.10 for $p = q = 2d$ and $r = d$ we get $B(u,u) \in C^0([0,T], L^d)$, and so $u \in C^0([0,T], L^d)$.
We assume now that there are two solutions $u_1$ and $u_2$ in $C^0([0,T], L^d)$. We already know the uniqueness for $d = 2$, so we will focus uniquely on the case $d = 3$.

Setting $u_{21} = u_2 - u_1$ and $w_j = B(u_j, u_j)$ we have

$$\begin{cases} \partial_t u_{21} - \triangle u_{21} = f_{21} \\ u_{21}(0) = 0 \end{cases} \text{with}$$
$$f_{21} = 2Q(e^{t\triangle}u_0, u_{21}) + Q(w_2, u_{21}) + Q(w_1, u_{21}).$$

By $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$, which is the dual of Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we have

$$\|Q(u,v)\|_{\dot{H}^{-\frac{3}{2}}(\mathbb{R}^3)} \le \|u \otimes v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|u \otimes v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le \|u\|_{L^3}\|v\|_{L^3}.$$

Then, by (4.5) and entering the definition of $f_{21}$

$$\begin{aligned} &\|u_{21}(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + 2\int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt' \le 2\int_0^t \langle f_{21}(t'), u_{21}(t')\rangle_{\dot{H}^{-\frac{1}{2}}} dt' \\ &\le 4\int_0^t \|Q(e^{t'\triangle}u_0, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\ &+ 2\int_0^t \|Q(w_2, u_{21}) + Q(w_1, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt'. \end{aligned} \tag{10.18}$$

We bound the last line with, for $j = 1, 2$,

$$\begin{aligned} &2\int_0^t \|Q(w_j, u_{21})\|_{\dot{H}^{-\frac{3}{2}}} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \lesssim \|w_j\|_{K_3(t)} \int_0^t \|u_{21}(t')\|_{L^3} \|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}} dt' \\ &\lesssim \|w_j\|_{K_3(t)} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt', \end{aligned} \tag{10.19}$$

where in the last line we used Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$.
So, the last line of (10.18) is

$$\lesssim \left(\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)}\right) \int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}} dt'. \tag{10.20}$$

We split now

$$u_0 = u_0^{(1)} + u_0^{(2)} \text{ with } \|u_0^{(1)}\|_{L^3} < \epsilon \text{ and } u_0^{(2)} \in L^6 \cap L^3$$

and we bound similarly to (10.19)

$$\int_0^t \|Q(e^{t'\triangle}u_0^{(1)}, u_{21})\|_{\dot H^{-\frac32}} \|\nabla u_{21}(t')\|_{\dot H^{-\frac12}} dt' \lesssim \|u_0^{(1)}\|_{L^3} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt'.$$

Finally, we bound

$$\begin{aligned}
&\int_0^t \|Q(e^{t'\triangle}u_0^{(2)}, u_{21})\|_{\dot H^{-\frac32}} \|\nabla u_{21}(t')\|_{\dot H^{-\frac12}} dt' \\
&\le \int_0^t \|e^{t'\triangle}u_0^{(2)} \otimes u_{21}\|_{\dot H^{-\frac12}} \|\nabla u_{21}(t')\|_{\dot H^{-\frac12}} dt' \lesssim \int_0^t \|e^{t'\triangle}u_0^{(2)} \otimes u_{21}\|_{L^{\frac32}} \|\nabla u_{21}(t')\|_{\dot H^{-\frac12}} dt' \\
&\le \int_0^t \|e^{t'\triangle}u_0^{(2)}\|_{L^6} \, \|u_{21}\|_{L^2} \|\nabla u_{21}\|_{H^{-\frac12}} dt' \le \|u_0^{(2)}\|_{L^6} \int_0^t \|u_{21}\|^{\frac12}_{H^{-\frac12}} \|\nabla u_{21}\|^{\frac32}_{H^{-\frac12}} dt'.
\end{aligned}$$

So we get

$$\begin{aligned}
&\|u_{21}(t)\|^2_{\dot H^{-\frac12}} + 2\int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' \lesssim \left( \|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3} \right) \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt \\
&+ \frac{3}{4\mathbf{C}^{\frac43}} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' + \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac12}} dt'.
\end{aligned}$$

Taking $\mathbf{C}$ large, and $t$ small, so that $\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3} < 3\epsilon$ with $\epsilon$ sufficiently small, we obtain

$$\|u_{21}(t)\|^2_{\dot H^{-\frac12}} + \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' \lesssim \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac12}} dt'.$$

Gronwall's Inequality implies that $u_{21}(t') = 0$ for all $t' \in [0,t]$ with $t > 0$ sufficiently small. The above argument shows that the set

$$\{t \in [0,T) : u_{21} \equiv 0 \text{ in } [0,t]\} \tag{10.21}$$

is open (and, obviously, non empty) in $[0,T)$. On the other hand, since $u_{21} \in C^0([0,T), L^3(\mathbb{R}^3, \mathbb{R}^3))$, the set in (10.21) is also closed in $[0,T)$. Hence, since it is non empty because if contains 0, it coincides with $[0,T)$.

Next we turn to the global existence for small data. This follows $\|e^{t\Delta}u_0\|_{K_{2d}(\infty)} \le C_d\|u_0\|_{L^d(\mathbb{R}^d)}$ and Proposition 10.12 when $C_d\|u_0\|_{L^d(\mathbb{R}^d)} < \varepsilon_{2d}$.

□

*Remark* 10.14. Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3, \mathbb{R}^3)$. Then it can be proved that if $T_3 > 0$ is the lifespan of the corresponding solution $u \in C^0([0,T_3), L^3(\mathbb{R}^3, \mathbb{R}^3))$ provided by Theorem 10.13 and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 8.1, we have $T_3 = T_{u_0}$. We will prove the simpler result in Proposition 10.15.

**Proposition 10.15.** *Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3,\mathbb{R}^3)$. There there exists $\epsilon_{3\nu} > 0$ s.t. for $\|u_0\|_{L^3(\mathbb{R}^3)} < \epsilon_{3\nu}$ and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 8.1, we have $T_{u_0} = \infty$.*

*Proof.* Taking $\epsilon_{3\nu} > 0$ sufficiently small we can assume by Theorem 10.13 that $u \in C^0([0,\infty), L^3)$. In fact, if it is sufficiently small we can prove $\|u\|_{L^\infty([0,\infty),L^3)} < C_0\|u_0\|_{L^3}$ for a fixed $C_0 > 0$. Suppose that $T_{u_0} < \infty$. Then by Theorem 8.1 we have the blow up

$$\lim_{T \nearrow T_{u_0}} \int_0^T \|\nabla u(t)\|^2_{\dot H^{\frac12}} dt = \infty. \tag{10.22}$$

By Theorem 8.1 and by (4.5), for $0 < t \le T < T_{u_0}$ we have

$$\|u(t)\|^2_{\dot H^{\frac12}} + 2\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' = \|u_0\|^2_{\dot H^{\frac12}} + 2\int_0^t \langle u(t')\cdot\nabla u(t'), u(t')\rangle_{\dot H^{\frac12}} dt'. \tag{10.23}$$

By Sobolev's Embedding $\dot H^{\frac12}(\mathbb{R}^3,\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3,\mathbb{R}^3)$ we obtain

$$|\langle u\cdot\nabla u, u\rangle_{\dot H^{\frac12}}| = |\langle u\cdot\nabla u, \nabla u\rangle_{L^2}| \le \|u\|_{L^3}\|\nabla u\|^2_{L^3} \le C\|u\|_{L^3}\|\nabla u\|^2_{\dot H^{\frac12}}.$$

Then

$$\begin{aligned}&\|u(t)\|^2_{\dot H^{\frac12}} + 2\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}} + C\|u\|_{L^\infty(\mathbb{R}_+,L^3)}\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \\ &\le \|u_0\|^2_{\dot H^{\frac12}} + C_0C\|u_0\|_{L^3}\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt'.\end{aligned}$$

So, for $C_0C\|u_0\|_{L^3} < 1$, we get

$$\|u(t)\|^2_{\dot H^{\frac12}} + \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}},$$

which contradicts (10.22).

□

We will prove now the following.

**Lemma 10.16.** *The solutions $u \in C^0([0,T), L^d(\mathbb{R}^d,\mathbb{R}^d))$ in Theorem 10.13 are in $C^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$.*

*Proof.* A proof of this lemma is in [12, Proposition 15.1], but it uses Besov spaces so here we modify the argument. We know the result already for $d = 2$, so we consider only $d = 3$. We notice that $e^{t\triangle}u_0 \in K_r(\infty)$ for all $r \ge d$ and $e^{t\triangle}u_0 \in C^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$. We already know that, for $S > 0$ sufficiently small, we have $u \in K_{2d}(S)$, see the proof of Theorem 10.13. Then, using Lemma 10.10 we conclude that $B(u,u) \in K_r(S)$ for any $r \in [d,\infty)$ (notice $\frac1p + \frac1q < \frac1d + \frac1r$ in (10.5), where $p = q = 2d$ in our case). So $u \in K_r(S)$ for all $r \in [d,\infty)$. But then, applying again Lemma 10.10, we conclude that $u \in K_r(S)$ for all $r \in [d,\infty]$, and in particular $u \in L^\infty([t_0,S], L^r(\mathbb{R}^d))$ for any $t_0 \in (0,S)$ and any $r \in [d,\infty)$.

Let us fix an $r \in (2d, \infty)$ and let us prove by induction that $u \in L^\infty([t_0, S], W^{\frac{k}{2},r}(\mathbb{R}^d))$ for all $k \in \mathbb{N} \cup \{0\}$. We have shown this for $k = 0$, and let us suppose by induction that it is true for some $k$. Then we will show $u \in L^\infty([t_1, S], W^{\frac{k+1}{2},r}(\mathbb{R}^d))$ for any $t_0 < t_1 < S$. We can write

$$u(t) = e^{(t-t_0)\triangle}u(t_0) - \int_{t_0}^{t} e^{(t-s)\triangle}\mathbb{P}\nabla \cdot (u \otimes u)ds. \tag{10.24}$$

We know that $e^{(t-t_0)\triangle}u(t_0) \in C^\infty([t_1, S], W^{\frac{k+1}{2},r}(\mathbb{R}^d))$ for all $k$, so we focus on the integral. We write for $k \geq 1$

$$\begin{aligned}
&\|(-\triangle)^{\frac{1}{4}} \int_{t_0}^{t} e^{(t-s)\triangle}\mathbb{P}\nabla \cdot (u \otimes u)ds\|_{W^{\frac{k}{2},r}} \lesssim \int_{t_0}^{t} \|(-\triangle)^{-\frac{1}{2}}\nabla \cdot e^{(t-s)\triangle}(-\triangle)^{\frac{3}{4}}(u \otimes u)\|_{W^{\frac{k}{2},r}} ds \\
&\leq C_{d,r} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|e^{(t-s)\triangle}(-(t-s)\triangle)^{\frac{3}{4}}(u \otimes u)\|_{W^{\frac{k}{2},r}} ds \\
&\leq C'_{d,r,S,k} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u \otimes u\|_{W^{\frac{k}{2},r}} ds \leq C''_{d,r,S,k} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u\|^2_{W^{\frac{k}{2},r}} ds \\
&= 4C''_{d,r,S,k}(t-t_0)^{\frac{1}{4}} \|u\|^2_{L^\infty([t_0,S],W^{\frac{k}{2},r}(\mathbb{R}^d))},
\end{aligned}$$

where we exploited the Calderon Zygmund theory (for example, Theorem 3 at p. 96 in [18], and the relation between the constants $B$ and $A_p$ in that statement where, from the proof, $A_p = A_p(B)$). Next, for $k = 0$ we use Hölder's inequality to bound

$$C'_{d,r,S,0} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u \otimes u\|_{L^r} ds \leq C'_{d,r,S,0} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u\|^2_{L^{2r}} ds \leq C''_{d,r,S,0}(t-t_0)^{\frac{1}{4}} \|u\|^2_{L^\infty([t_0,S],L^{2r}(\mathbb{R}^d))}$$

while for $k \geq 1$ we use the fact that $W^{\frac{k}{2},r}(\mathbb{R}^d)$ is an algebra to bound

$$\begin{aligned}
&C'_{d,r,S,k} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u \otimes u\|_{W^{\frac{k}{2},r}} ds \leq C''_{d,r,S,k} \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|u\|^2_{W^{\frac{k}{2},r}} ds \\
&= 4C'''_{d,r,S,k}(t-t_0)^{\frac{1}{4}} \|u\|^2_{L^\infty([t_0,S],W^{\frac{k}{2},r}(\mathbb{R}^d))}.
\end{aligned}$$

Now we use a general result of the theory of semigroups which guarantees that for $f \in L^1((0,T), X)$, where $X$ is a Banach space where $e^{t\triangle}$ is a contraction semigroup, then

$$v(t) := -\int_{t_0}^{t} e^{(t-s)\triangle} f(s)ds$$

satisfies $\partial_t v = \triangle v + f(t)$ in $\mathcal{D}'((t_0,T), X)$, see [3, Proposition 4.1.6 (ii)]. In our case, since $u \in L^\infty([t_0, S], W^{k,r}(\mathbb{R}^d))$ for all $k$ and appropriate $r < \infty$, and $f = -\mathbb{P}\nabla \cdot (u \otimes u)$, we have $v' = \triangle v + f(t)$ in $\mathcal{D}'((t_0,T), W^{k,r})$, since $e^{t\triangle}$ is a contraction semigroup in any space $W^{k,r}$ for $r < \infty$.

Furthermore, the Hille–Yosida–Phillips Theorem, see [3, Theorem 3.1.1], guarantees that $\partial_t e^{t\triangle}u(t_0) = \triangle e^{t\triangle}u(t_0)$ in $\mathcal{D}'((t_0,T),W^{k,r})$ for all $k$ and our $r<\infty$.
Summing up, we obtain that

$$\partial_t u = \triangle u - \mathbb{P}\nabla\cdot(u\otimes u) \text{ in } \mathcal{D}'((0,S),W^{k,r}(\mathbb{R}^d)) \text{ for all } k. \tag{10.25}$$

Since the r.h.s. is in $L^\infty((t_0,S),W^{k,r}(\mathbb{R}^d))$ for all $k$, it follows that $u\in W^{1,\infty}((t_0,S),W^{k,r}(\mathbb{R}^d))$ for all $k$, which fed again in (10.25) yields $u\in W^{2,\infty}((t_0,S),W^{k,r}(\mathbb{R}^d))$, by applying Leibnitz rule like in Brezis [2, Corollary 8.10]. By induction, proceeding iteratively we get $u\in W^{l,\infty}((t_0,S),W^{k,r}(\mathbb{R}^d))$ for all $l$ and $k$, and so the statement.

□

# 11 Vorticity

We recall the following.

**Lemma 11.1.** *Suppose that $f\in\mathcal{S}(\mathbb{R}^3)$. Then $u\in\mathcal{S}'(\mathbb{R}^3)$ satisfies $-\triangle u = f$ if and only if*

$$u = K*f+h \text{ with } K(x) := \frac{1}{4\pi|x|} \tag{11.1}$$

*and $h(x)$ a harmonic polynomial.*

*Proof.* Notice that $-\triangle h = 0$ requires $|\xi|^2\widehat{h}=0$, that is, $\operatorname{supp}\widehat{h}=\{0\}$. But $\operatorname{supp}\widehat{h}=\{0\}$ implies $\widehat{h}=\sum_{|\alpha|\le k} a_\alpha\partial_\xi\delta$, with $k$ the order of $\widehat{u}$ and $a_\alpha$ arbitrary constants, see [8, Theorem 2.3.4]. Then $h$ is a degree $k$ harmonic polynomial.

Next, let us consider the tempered distribution given by $\widehat{v}=\frac{1}{|\xi|^2}\widehat{f}$. Now, recall from Lemma 2.23, that

$$\mathcal{F}(|\cdot|^{-\gamma})(\xi) = \frac{2^{\frac{d-\gamma}{2}}\Gamma\left(\frac{d-\gamma}{2}\right)}{2^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}+1\right)}|\xi|^{\gamma-d}.$$

So, for $\gamma=2$ and $d=3$, using $\Gamma(2)=1$ and $\Gamma(1/2)=\sqrt{\pi}$, we get $\mathcal{F}^{-1}(|\xi|^{-2})=\sqrt{\frac{\pi}{2}}\,\frac{1}{|x|}$. Recalling also the formula $\widehat{f*g}=(2\pi)^{\frac{3}{2}}\widehat{f}\widehat{g}$, we get

$$v = \frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{\pi}{2}}\,\frac{1}{|x|}*f = \frac{1}{4\pi}\,\frac{1}{|x|}*f.$$

By linearity, $u\in\mathcal{S}'(\mathbb{R}^3)$ satisfies $-\triangle u=f$ exactly if it is like in (11.1).

□

If we consider a field $u\in\mathcal{S}'(\mathbb{R}^3,\mathbb{R}^3)$, then its vorticity is $\omega:=\nabla\times u$.

**Lemma 11.2** (Biot–Savart Law)**.** *Let $u\in W^{1,p}(\mathbb{R}^3,\mathbb{R}^3)$ with $p\in(1,3)$ and with* $\operatorname{div}u=0$. *Then*

$$u = T\omega \text{ for } \omega=\nabla\times u, \tag{11.2}$$

*where*

$$T\omega := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)dy. \tag{11.3}$$

*Proof.* First of all, for divergent free vector–fields we have the identity $-\triangle u = \nabla \times \omega$.

Let us now assume $\omega \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then we claim

$$\frac{1}{|x|} * \partial_j \omega_k = \left(\partial_j \frac{1}{|x|}\right) * \omega_k. \tag{11.4}$$

Indeed, by applying the divergence theorem, we have

$$\begin{aligned}\frac{1}{|x|} * \partial_j \omega_k &= -\lim_{\epsilon\to 0^+}\int_{|x-y|\geq\epsilon} \omega_k(y)\partial_{y_j}\frac{1}{|x-y|}dy + \lim_{\epsilon\to 0^+}\int_{|x-y|=\epsilon}\frac{x_j-y_j}{|x-y|^2}\omega_k(y)dS\\ &= -\int_{\mathbb{R}^3}\omega_k(y)\partial_{y_j}\frac{1}{|x-y|}dy = \left(\partial_j\frac{1}{|x|}\right) * \omega_k.\end{aligned}$$

Still for $\omega \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$, from $-\triangle u = \nabla\times\omega$, from Lemma 11.1 and (11.4) we have

$$\begin{aligned}u &= \overrightarrow{e}_i \epsilon_{ijk}\frac{1}{4\pi}\;\frac{1}{|x|} * \partial_j\omega_k + h = \overrightarrow{e}_i\epsilon_{ijk}\frac{1}{4\pi}\;\left(\partial_j\frac{1}{|x|}\right) * \omega_k + h\\ &= -\overrightarrow{e}_i\epsilon_{ijk}\frac{1}{4\pi}\;\frac{x_j}{|x|^3} * \omega_k + h = T\omega + h,\end{aligned}$$

where the components of $h$ are harmonic polynomials. From the Hardy-Littlewood-Sobolev inequality, we have $\|T\omega\|_{L^q(\mathbb{R}^3)} \leq c\|\omega\|_{L^p(\mathbb{R}^3)}$ for $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$, if $1 < p < 3$. Since also $u \in L^q(\mathbb{R}^3, \mathbb{R}^3)$ it follows that also $h \in L^q(\mathbb{R}^3, \mathbb{R}^3)$ which, given that the coordinates of $h$ are polynomials, implies $u = K * (\nabla \times \omega) = T\omega$ in the case $\omega \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Let us consider a general $u$ like in the statement, with $\omega$ its vorticity.

Let $C_c^\infty(\mathbb{R}^3, \mathbb{R}^3) \ni \widetilde{\omega}_n \xrightarrow{n\to\infty} \omega$ in $L^p(\mathbb{R}^3, \mathbb{R}^3)$. Then $u_n = T\widetilde{\omega}_n \longrightarrow \widetilde{u} \in L^q(\mathbb{R}^3, \mathbb{R}^3)$, with $\widetilde{u} = T\omega$. We need to show that $u = \widetilde{u}$.

Notice that

$$\nabla\cdot u_n = \nabla\cdot T\omega_n = \nabla\cdot\left[(-\triangle)^{-1}(\nabla\times\widetilde{\omega}_n)\right] = (-\triangle)^{-1}\left[\nabla\cdot(\nabla\times\widetilde{\omega}_n)\right] = (-\triangle)^{-1}0 = 0.$$

This implies, in particular, since $u_n \xrightarrow{n\to\infty} \widetilde{u}$, we have $\nabla\cdot\widetilde{u} = 0$.

Next notice that $\mathbb{P}$, which is a Calderon–Zygmund operator, is a bounded operator inside $L^p(\mathbb{R}^3, \mathbb{R}^3)$. Thus, for

$$\widetilde{\omega}_n = \widetilde{\omega}_n^{(1)} + \widetilde{\omega}_n^{(2)}, \text{ with } \widetilde{\omega}_n^{(1)} := \mathbb{P}\widetilde{\omega}_n \text{ and } \widetilde{\omega}_n^{(2)} := (1-\mathbb{P})\widetilde{\omega}_n,$$

we have $\widetilde{\omega}_n^{(1)} \xrightarrow{n\to\infty} \omega$ and $\widetilde{\omega}_n^{(2)} \xrightarrow{n\to\infty} 0$ in $L^p(\mathbb{R}^3, \mathbb{R}^3)$.

From $u_n = K * (\nabla\times\widetilde{\omega}_n) = K * (\nabla\times\widetilde{\omega}_n^{(1)})$, we have

$$-\triangle u_n = \nabla\times\widetilde{\omega}_n^{(1)},$$

and for $\omega_n := \nabla \times u_n$ and $u_n = \mathbb{P} u_n = -\triangle^{-1} \nabla \times \omega_n$, we also have

$$-\triangle u_n = \nabla \times \omega_n.$$

Then $\widetilde{\omega}_n^{(1)} = \omega_n$ by

$$\widetilde{\omega}_n^{(1)} = \mathbb{P}\widetilde{\omega}_n^{(1)} = -\triangle^{-1}\nabla \times (\nabla \times \widetilde{\omega}_n^{(1)}) = -\triangle^{-1}\nabla \times (\nabla \times \omega_n) = \mathbb{P}\omega_n = \omega_n. \tag{11.5}$$

Hence we have proved that $\nabla \times u_n = \widetilde{\omega}_n^{(1)}$.
Now we show $\nabla \times \widetilde{u} = \omega$. Indeed this follows from $u_n \xrightarrow{n\to\infty} \widetilde{u}$ in $L^q(\mathbb{R}^3, \mathbb{R}^3)$ and from $\nabla \times u_n = \widetilde{\omega}_n^{(1)} \xrightarrow{n\to\infty} \omega$. Hence we conclude that $\nabla \times u = \nabla \times \widetilde{u}$. Using again formula (2.9) and proceeding like in (11.5), we conclude $u = \mathbb{P}u = \mathbb{P}\widetilde{u} = \widetilde{u}$.

□

Notice that for any $u \in \mathcal{S}'(\mathbb{R}^3, \mathbb{R}^3)$ we have $\nabla \cdot \omega = 0$. Indeed

$$\nabla \cdot (\nabla \times u) = (\epsilon_{ijk}\partial_i\partial_j)u_k = 0u_k = 0$$

by $\partial_i\partial_j = \partial_j\partial_i$ and by $\epsilon_{ijk} = -\epsilon_{jik}$.

As we know, a solution of NS formally satisfies

$$u_t - \triangle u + u \cdot \nabla u = -\nabla p\,.$$

Notice that if $u$ is regular,

$$(u \cdot \nabla)u = 2^{-1}\nabla|u|^2 - u \times \omega, \tag{11.6}$$

since indeed $(u \cdot \nabla)u = \overrightarrow{e}_i u_j \partial_j u_i$ , $2^{-1}\nabla|u|^2 = \overrightarrow{e}_i u_j \partial_i u_j$ and

$$\begin{aligned} u \times (\nabla \times u) &= \overrightarrow{e}_i \varepsilon_{ijk} u_j (\nabla \times u)_k = \overrightarrow{e}_i \varepsilon_{ijk}\varepsilon_{i'j'k} u_j \partial_{i'} u_{j'} = \overrightarrow{e}_i (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}) u_j \partial_{i'} u_{j'} \\ &= \overrightarrow{e}_i u_j \partial_i u_j - \overrightarrow{e}_i u_j \partial_j u_i. \end{aligned}$$

Summing up, we obtain (11.6).
From (11.6) we obtain

$$\nabla \times ((u \cdot \nabla)u) = -\nabla \times (u \times \omega) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u, \tag{11.7}$$

from div $u$ = div $\omega = 0$ and

$$\begin{aligned} \nabla \times (u \times \omega) &= \overrightarrow{e}_i \varepsilon_{ijk} \partial_j (u \times \omega)_k = \overrightarrow{e}_i \varepsilon_{ijk}\varepsilon_{i'j'k} (\omega_{j'} \partial_j u_{i'} + u_{i'} \partial_j \omega_{j'}) = \overrightarrow{e}_i (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})(\omega_{j'} \partial_j u_{i'} + u_{i'} \partial_j \omega_{j'}) \\ &= \overrightarrow{e}_i (\omega_j \partial_j u_i + u_i \partial_j \omega_j) - \overrightarrow{e}_i (\omega_i \partial_j u_j + u_j \partial_j \omega_i) = \overrightarrow{e}_i \omega_j \partial_j u_i - \overrightarrow{e}_i u_j \partial_j \omega_i. \end{aligned}$$

Then, applying $\nabla\times$ to the NS, we formally obtain

$$\omega_t - \triangle\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{11.8}$$

If we apply $\langle \cdot, \phi \rangle_{L^2_{tx}}$ to (11.8) with $\phi(t,x)$ a function in $C^\infty_c((0,\infty)\times\mathbb{R}^3, \mathbb{R}^3)$, then, exploiting $\nabla_x \cdot u = \nabla_x \cdot \omega = 0$, (11.8) implies

$$\int_0^\infty (\langle \omega, \partial_t \phi\rangle + \langle \omega, \triangle \phi\rangle + \langle \omega, u\cdot\nabla\phi\rangle - \langle u, \omega\cdot\nabla\phi\rangle)\, dt' = 0 \text{ for all } \phi \in C^\infty_c((0,\infty)\times\mathbb{R}^3, \mathbb{R}^3), \tag{11.9}$$

which is the weak form of the vorticity equation. The above discussion is purely heuristic, but we have the following.

**Lemma 11.3.** *Let $u$ be Leray Hopf solution of the NS, in the sense of Definition 6.1, with $u \in L^\infty(\mathbb{R}_+, L^2)$ and $\nabla u \in L^2(\mathbb{R}_+, L^2)$ and consider the vorticity $\omega$. Then, the pair $(u,\omega)$ satisfies* (11.9)*.*

*Proof.* For $\phi \in C^\infty_c((0,\infty)\times\mathbb{R}^3, \mathbb{R}^3)$ we have $\nabla\times\phi \in C^\infty_{c\sigma}((0,\infty)\times\mathbb{R}^3, \mathbb{R}^3)$. So, by (6.4), we have

$$\int_{\mathbb{R}_+} (-\langle u, \triangle\nabla\times\phi\rangle - \langle u, \nabla\times\partial_t\phi\rangle + \langle (u\cdot\nabla)u, \nabla\times\phi)\rangle)\, dt' = 0.$$

Integrating by parts, we have $\langle u, \triangle\nabla\times\phi\rangle = -\langle\omega, \triangle\phi\rangle$ and $\langle u, \nabla\times\partial_t\phi\rangle = -\langle\omega, \partial_t\phi\rangle$. Notice that the fact $u(t) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ for a.a. $t$ implies that formulas (11.6)–(11.6) are for a.a. $t$. This yields (11.9).

□

**Lemma 11.4** (Local Biot–Savart Law)**.** *Let $B$ be a bounded open subset of $\mathbb{R}^3$, consider a divergence free vector–field $u \in L^r(B, \mathbb{R}^3)$ with $\nabla u \in L^p(B)$, where $r \in [1,\infty]$ and $1 < p < \infty$. Let $\Omega$ be an open subset of $B$ with $\overline{\Omega} \subset\subset B$ and with boundary . Then*

$$u(x) = T(\chi_\Omega \omega) + h(x) \text{ for all } x \in \Omega, \tag{11.10}$$

*where $h$ is a harmonic vector–field in $\Omega$.*

*Proof.* We can start by defining $h$ by formula (11.10). We return to $h$ later. Let us consider an open ball $\mathbb{B}_1$ in $\Omega$, and another ball $\overline{\mathbb{B}_1} \subset\subset \mathbb{B}_2 \subset \overline{\mathbb{B}}_2 \subset\subset \Omega$. Then let $\varphi \in C^\infty_c(\Omega, [0,1])$, with $\varphi|_{\mathbb{B}_2} = \chi_{\mathbb{B}_2}$. Then we write

$$\begin{aligned}\widetilde{u} &:= T\nabla\times(\varphi u) = T(\varphi\omega) + T(\nabla\varphi\times u)\\ &= T(\chi_\Omega\omega) + \widetilde{h} \text{ where } \widetilde{h} = T((\varphi - \chi_\Omega)\omega) + T(\nabla\varphi\times u).\end{aligned}$$

Notice that $\widetilde{h}$ is harmonic inside $\mathbb{B}_1$. Indeed, inside $\mathbb{B}_2$

$$\triangle\widetilde{h} = -\frac{1}{4\pi}\triangle\int_{B\backslash\mathbb{B}_2} \frac{x-y}{|x-y|^3}\times\omega(y)(\varphi(y) - \chi_\Omega(y))dy - \frac{1}{4\pi}\triangle\int_{B\backslash\mathbb{B}_2} \frac{x-y}{|x-y|^3}\times(\nabla\varphi(y)\times u(y))dy = 0$$

by $\triangle_x \frac{x-y}{|x-y|^3} = 0$ (this follows from $\Delta r^{-1} = (\partial_r^2 + \frac{2}{r})r^{-1} = 0$ for $r \neq 0$, and then applying $\nabla$ to this equation), for $x \neq y$ and by differentiation with respect to a parameter in an integral. In $\mathbb{B}_2$ we have

$$\nabla \times (u - \widetilde{u}) = \nabla \times (\varphi u - T\nabla \times (\varphi u)) = 0, \tag{11.11}$$

where the 1st equality follows from $u = \varphi u$ in $\mathbb{B}_2$ and from the definition of $\widetilde{u}$, and where the 2nd equality follows from

$$\mathbb{P}(\varphi u) = T\nabla \times \mathbb{P}(\varphi u) = T\nabla \times (\varphi u) \tag{11.12}$$

where for the 1st equality we apply the Biot Savart Law, Lemma 11.2, to $\mathbb{P}(\varphi u)$, as we show now. Indeed we claim $\varphi u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$. Then, for $p \in (1, 3)$ the Biot Savart Law Lemma 11.2 applies to $\mathbb{P}(\varphi u) \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$. If $p \geq 3$, notice that $\varphi u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ is equivalent to $\varphi u \in W^{1,p}(B, \mathbb{R}^3)$. The latter implies $\varphi u \in W^{1,a}(B, \mathbb{R}^3)$ for any $a \leq p$, which again is equivalent to $\varphi u \in W^{1,a}(\mathbb{R}^3, \mathbb{R}^3)$ for any $a \leq p$, and in particular implies $\mathbb{P}(\varphi u) \in W^{1,a}(\mathbb{R}^3, \mathbb{R}^3)$ for any $1 < a \leq p$.

Now we need to prove the claim $\varphi u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$. By $u \in L^r(B, \mathbb{R}^3)$ with $\nabla u \in L^p(B)$, if the boundary $\partial B$ is smooth, by Poincaré–Wirtinger inequality we have

$$\|u - |B|^{-1} \int_B u\|_{L^p(B)} \leq C_B \|\nabla u\|_{L^p(B)}.$$

If the boundary $\partial B$ is not smooth, we can simply replace $B$ by another open domain $B'$ s.t. $\overline{\Omega} \subset\subset B' \subset \overline{B'} \subset\subset B$ with $\partial B'$ is smooth. With this we have completed the proof of (11.12) and of (11.11).

From (11.11) and from the usual identity

$$\triangle(u - \widetilde{u}) = \nabla(\nabla \cdot (u - \widetilde{u})) - \nabla \times (\nabla \times (u - \widetilde{u}))$$

we obtain $\triangle(u - \widetilde{u}) = 0$ in $\mathbb{B}_2$. So $u - \widetilde{u} = h_1$ with $h_1$ harmonic in $\mathbb{B}_2$. So $u = \widetilde{u} + h_1$ Thus we conclude that $u = \widetilde{u} + h_1 = T(\chi_\Omega \omega) + h$ with $h = \widetilde{h} + h_1$ harmonic vector–field in $\mathbb{B}_1$. This implies the statement of the lemma.

□

Recall that, for $\Omega$ an open subset of $\mathbb{R}^d$, the space $C^{k,\alpha}(\Omega)$ with $\alpha \in (0, 1)$ is the subspace of $C^k(\Omega) \cap W^{k,\infty}(\Omega)$ defined by the functions $f$ satisfying the additional conditions

$$\sup_{|\mu|=k} \sup_{x \neq y \text{ in } \Omega} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x - y|^\alpha} < \infty.$$

Let $B_R$ a ball of radius $R$ and a fixed center(which we can take to be 0) in $\mathbb{R}^3$.

**Lemma 11.5.** *Let $u \in L^\infty((0, T), L^2(B_R))$. Then, for any $R' < R$ we have:*

1. *for $\beta \in [2, \infty]$ and $k \in \{0, 1, ...\}$, $\omega \in L^\beta((0, T), W^{k,\infty}(B_R)) \Rightarrow u \in L^\beta((0, T), W^{k,\infty}(B_{R'}))$;*

2. *for* $\alpha \in (0,1)$ *and* $k \in \{0,1,2,...\}$, $\omega \in L^{\beta}((0,T), C^{k,\alpha}(B_R)) \Rightarrow u \in L^{\beta}((0,T), C^{k+1,\alpha'}(B_{R'}))$ *for any* $\alpha' \in (0,\alpha)$.

*Proof.* The proof of the first statement is elementary. We consider only case $k = 0$. We fix $R'' \in (R', R)$. Then, by Lemma 11.4 we have

$$u(x) = -\frac{1}{4\pi} \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y) dy + h(x) \text{ for all } x \in B_{R''},$$

Since $h$ is harmonic in $B_{R''}$, it follows that $h \in L^{\beta}((0,T), W^{n,\infty}(B_{R''}))$. Next, for $x \in B_R$ we have

$$\left| \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y) dy \right| \leq \int_{B_{R''}} \frac{1}{|x-y|^2} dy \|\omega\|_{L^\infty(B_R)} \leq \int_{B_{2R''}(x)} \frac{1}{|x-y|^2} dy \|\omega\|_{L^\infty(B_R)} = c_3 R'' \|\omega\|_{L^\infty(B_R)}.$$

The 2nd claim in the statement of Lemma 11.5 is more delicate. It is not restrictive to consider only $k = 0$. Using the above discussion, we do not need to worry about $h$. We consider $\varphi \in C_c^\infty(B_{R''}, [0,1])$, with $\varphi|_{B_{R'}} = 1$ in a slightly larger ball than $R'$. Then is

$$u(x) = \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y) dy = \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y)\varphi(y) dy + \int_{B_{R''}} \frac{x-y}{|x-y|^3} \times \omega(y)(1-\varphi(y)) dy$$

and it is elementary to see, that the 2nd integral on the r.h.s. is harmonic in $B_{R'}$. We look then at the 1st integral on the r.h.s. and we absorb $\varphi$ in $\omega$, simply assuming $\omega \in C_c^{0,\alpha}(B_R)$ and let us consider

$$v(x) := \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y) dy.$$

We have the following lemma.

**Lemma 11.6.** *Let* $K$ *be smooth in* $\mathbb{R}^d \backslash \{0\}$, *homogeneous of degree* $-(d-1)$. *Then*

$$\langle \partial_j K, \psi \rangle = P.V. \int_{\mathbb{R}^d} \partial_j K(y) \psi(y) dy - c_j \psi(0) \text{ for all } \psi \in C_c^\infty(\mathbb{R}^d), \tag{11.13}$$

*where* $c_j = \int_{|x|=1} K(x) x_j dS$.

*Proof.* We have

$$-\langle K, \partial_j \psi \rangle = -\lim_{\epsilon \to 0^+} \int_{|y| \geq \epsilon} K(y) \partial_j \psi(y) dy = \lim_{\epsilon \to 0^+} \int_{|y| \geq \epsilon} \partial_j K(y) \psi(y) dy + \lim_{\epsilon \to 0^+} \int_{|y| = \epsilon} K(y) \psi(y) \frac{y_j}{|y|} dS$$
$$= P.V. \int_{\mathbb{R}^d} \partial_j K(y) \psi(y) dy + \psi(0) \int_{|y|=1} K(y) y_j dS.$$

□

**Exercise 11.7.** For $K$ like in Lemma 11.6, that is smooth in $\mathbb{R}^d\backslash\{0\}$ and homogeneous of degree $-(d-1)$, we have $\int_{|x|=1} \partial_j K(x)dS = 0$ for any $j$. Show this in two ways. First way, by using the information that $P.V.\partial_j K \in \mathcal{S}'(\mathbb{R}^d)$. Second way, by a direct computation of the integral $\int_{|x|=1} \partial_j K(x)dS$.

ANSWER. Let us look only at the 2nd approach. It is enough to consider $j = d$. Let us consider cylindrical coordinates

$$x_d = x_d$$
$$(x_1, ..., x_{d-1}) = r\omega \text{ with } r > 0 \text{ and } \omega \in S^{d-2}$$

Then, for $x_d = \rho\cos\phi$ and $r = \rho\sin\phi$,

$$\int_{|x|=1} \partial_d K(x)dS = \int_{S^{d-2}} dS(\omega) \int_{\substack{x_d^2+r^2=1\\ r>0}} r^{d-2}\partial_d K d\ell = \int_{S^{d-2}} dS(\omega) \int_0^\pi \sin^{d-2}(\phi)(\cos\phi\partial_\rho K - \sin\phi\partial_\phi K)d\phi$$
$$= -\int_{S^{d-2}} dS(\omega) \int_0^\pi ((d-1)\sin^{d-2}(\phi)\cos\phi K + \sin^{d-1}(\phi)\partial_\phi K)d\phi$$
$$= -\int_{S^{d-2}} dS(\omega) \sin^{d-1}(\phi)K]_{\phi=0}^{\phi=\pi} = 0 \text{ by } \sin(0) = \sin(\pi) = 0.$$

Here we used $\partial_{x_d} K = \cos\phi\partial_\rho K - \frac{\sin\phi}{\rho}\partial_\phi K$. □

Returning to the proof of Lemma 11.5, we can assume initially that $\omega \in C_c^\infty(B_R, \mathbb{R}^3)$. Then, for a test field $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$,

$$\langle \partial_j v, \psi\rangle = -\langle v, \partial_j\psi\rangle = -\left\langle \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y)dy, \partial_j\psi \right\rangle = -\int_{\mathbb{R}^3\times\mathbb{R}^3} \varepsilon_{iab}\omega_b(y)\frac{x_a-y_a}{|x-y|^3}\partial_{x_j}\psi_i(x)dxdy$$
$$= -\lim_{\epsilon\to 0^+} \int_{|x-y|\geq\epsilon} \varepsilon_{iab}\omega_b(y)\frac{x_a-y_a}{|x-y|^3}\partial_{x_j}\psi_i(x)dxdy$$
$$= \lim_{\epsilon\to 0^+} \int_{|x-y|\geq\epsilon} \varepsilon_{iab}\omega_b(y)\partial_{x_j}\frac{x_a-y_a}{|x-y|^3}\psi_i(x)dxdy + \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^3} dy\varepsilon_{iab}\omega_b(y) \int_{|x-y|=\epsilon} \frac{x_a-y_a}{|x-y|^3}\frac{x_j-y_j}{|x-y|}\psi_i(x)dS(x)$$
$$= \lim_{\epsilon\to 0^+} \int_{|x-y|\geq\epsilon} \varepsilon_{iab}\omega_b(y)\partial_{y_j}\frac{y_a-x_a}{|x-y|^3}\psi_i(x)dxdy + \int_{|x-y|=1} \frac{(x_a-y_a)(x_j-y_j)}{|x-y|^4}dS(x) \int_{\mathbb{R}^3} dy\varepsilon_{iab}\omega_b(y)\psi_i(y)$$
$$= -\lim_{\epsilon\to 0^+} \int_{|x-y|\geq\epsilon} \varepsilon_{iab}\partial_{y_j}\omega_b(y)\frac{y_a-x_a}{|x-y|^3}\psi_i(x)dxdy$$
$$= \left\langle \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \partial_j\omega(y)dy, \psi \right\rangle.$$

So we conclude

$$\partial_j v(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \partial_j\omega(y)dy.$$

On the other hand, by Lemma 11.6

$$\nabla v(x) = P.V. \int_{\mathbb{R}^3} \nabla_x \frac{x-y}{|x-y|^3} \times \omega(y)dy + L\omega(x) = P.V. \int_{\mathbb{R}^3} \nabla_y \frac{y-x}{|y-x|^3} \times \omega(y)dy + L\omega(x)$$

with $L$ some fixed linear operator in $\mathbb{R}^3$. Obviously $\|L\omega\|_{C^{0,\alpha}(B_{R'})} \leq \|L\| \; \|\omega\|_{C^{0,\alpha}(B_R)}$, so the key term we need to bound is the P.V.

Let us define $H_{ik}(y-x)$ by

$$\overrightarrow{e}_i H_{ik}(y-x)\omega_k(y) = \overrightarrow{e}_i \varepsilon_{ijk} \frac{y_j - x_j}{|x-y|^3} \omega_k(y).$$

Then

$$P.V. \int_{\mathbb{R}^3} \partial_{y_a} \frac{y-x}{|y-x|^3} \times \omega(y)dy) = \overrightarrow{e}_i P.V. \int_{\mathbb{R}^3} \partial_{y_a} H_{ik}(y-x)\omega_k(y)dy.$$

An elementary computation shows that

$$K_{ik}^{(a)}(x) := \partial_{x_a} H_{ik}(x) = \varepsilon_{iak} \frac{1}{|x|^3} - 3\varepsilon_{ijk} \frac{x_j x_a}{|x|^5}.$$

These functions are homogeneous degree $-3$ and satisfy

$$\int_{|x|=1} K_{ik}^{(a)}(x)dS = \varepsilon_{iak} \int_{|x|=1} \frac{1}{|x|^3} dS - 3\varepsilon_{ijk} \int_{|x|=1} \frac{x_j x_a}{|x|^5} dS = 4\pi\varepsilon_{iak} - 3\varepsilon_{iak} \int_{|x|=1} x_3^2 dS$$
$$= 4\pi\varepsilon_{iak} - 6\pi\varepsilon_{iak} \int_0^\pi \cos^2(\phi)\sin(\phi)d\phi = 4\pi\varepsilon_{iak} - 6\pi\varepsilon_{iak} \frac{\cos^3(\phi)}{3}]_0^\pi = -4\pi\varepsilon_{iak} - 6\pi\varepsilon_{iak}\left(-\frac{2}{3}\right) = 0.$$

We claim now that for any $\alpha' \in (0,\alpha)$ there is a constant $C_{\alpha'}$ s.t. for all $x, x' \in B_{R'}$

$$\left| P.V. \int_{\mathbb{R}^3} K_{ik}^{(a)}(y)\omega(x-y)dy - P.V. \int_{\mathbb{R}^3} K_{ik}^{(a)}(y)\omega(x'-y)dy \right| \leq C_{\alpha'} \|\omega\|_{C_c^{0,\alpha}(B_R)} |x-x'|^{\alpha'}. \tag{11.14}$$

This will prove the second claim in the statement of Lemma 11.5 for $\omega \in C_c^\infty(B_R)$, but in fact by density this will extend to all $\omega \in C_c^{0,\alpha}(B_R)$.

The l.h.s. of (11.14) can be written as

$$\left| P.V. \int_{\mathbb{R}^3} K_{ik}^{(a)}(y) \left(\omega(x'-y) - \omega(x') - \omega(x-y) + \omega(x)\right) dy \right|$$

by the cancelation $\int_{|x|=1} K_{ik}^{(a)}(x)dS = 0$. It is elementary that

$$\left|\omega(x'-y) - \omega(x') - \omega(x-y) + \omega(x)\right| \leq 2\|\omega\|_{C_c^{0,\alpha}(B_R)} \min\{|y|^\alpha, |x'-x|^\alpha\}.$$

Then

$$\left|P.V.\int_{\mathbb{R}^3}K^{(a)}_{ik}(y)\omega(x'-y)dy - P.V.\int_{\mathbb{R}^3}K^{(a)}_{ik}(y)\omega(x-y)dy\right|$$
$$\lesssim \|\omega\|_{C^{0,\alpha}_c(B_R)}\int_{B_{2R}}\frac{1}{|y|^3}\min\{|y|^\alpha,|x'-x|^\alpha\}dy$$
$$\lesssim \|\omega\|_{C^{0,\alpha}_c(B_R)}\left(\int_0^{|x'-x|}|y|^{\alpha-1}d|y| + |x'-x|^\alpha\int_{|x'-x|}^{2R}|y|^{-1}d|y|\right)\approx \|\omega\|_{C^{0,\alpha}_c(B_R)}|x'-x|^\alpha\left|\log|x'-x|\right|.$$

□

The following result will be useful in the sequel.

**Lemma 11.8.** *Given a field $u\in L^r(\mathbb{R}^3,\mathbb{R}^3)$ for $r\in(2,\infty)$ there is a unique solution $p\in L^{\frac{r}{2}}(\mathbb{R}^3)$ of the equation*

$$-\triangle p = \partial_i\partial_j(u_iu_j) \tag{11.15}$$

*which is given by*

$$p = \frac{\partial_i}{\sqrt{-\triangle}}\frac{\partial_j}{\sqrt{-\triangle}}(u_iu_j) = R_iR_j(u_iu_j). \tag{11.16}$$

*It satisfies*

$$\|p\|_{L^{\frac{r}{2}}} \le C_r\sum_{i,j}\|u_iu_j\|_{L^{\frac{r}{2}}} \le C_r\|u\|^2_{L^r}. \tag{11.17}$$

*Proof.* The discussion is similar to that in Lemma 11.2. The estimates follow by the estimates on Rietz transformations. □

**Proposition 11.9.** *Consider a weak solution $u$ of NS in $d=3$ with $u\in L^\infty(\mathbb{R}_+,L^2)\cap L^2(\mathbb{R}_+,\dot H^1)$ and define the pressure $p\in L^1(\mathbb{R}_+,L^3)$ by the equation* (11.15)*. Then $u$ is a distributional solution in $\mathbb{R}_+\times\mathbb{R}^3$ of the equation*

$$u_t + u\cdot\nabla u - \triangle u = -\nabla p \tag{11.18}$$

*Proof.* Recall that $u$ satisfies equation (6.4), and thus, in particular,

$$\int_{\mathbb{R}_+}\left(\langle u,\triangle\Psi\rangle + \langle u,\partial_t\Psi\rangle - \langle \mathrm{div}(u\otimes u),\Psi\rangle\right)dt' = 0 \text{ for all } \Psi\in C^\infty_{c\sigma}(\mathbb{R}_+\times\mathbb{R}^3,\mathbb{R}^3).$$

Exploiting Remark 2.3, which states that $C^\infty_{c\sigma}(\mathbb{R}^3,\mathbb{R}^3)$ is dense in $V$, we claim that

$$\text{for any } T>0 \text{ the space } C^\infty_{c\sigma}((0,T)\times\mathbb{R}^3,\mathbb{R}^3) \text{ is dense in } C^1_c((0,T),V). \tag{11.19}$$

To prove (11.19), consider $\Phi\in C^1_c((0,T),V)$ and its derivative $\dot\Phi\in C^0_c((0,T),V)$. For any given $\epsilon>0$, let $0<t_0<t_1<...<t_M<T$, with $\dot\Phi=0$ outside $[t_0,t_M]$, and

$\|\dot\Phi(t) - \dot\Phi(s)\|_{H^1} < \epsilon$ for $t, s \in [t_{j-1}, t_j]$, for any $j = 1, ..., M$. For a $\delta > 0$ to be fixed later, let $\widetilde{\dot\Psi}(t_j) \in C^\infty_{c\sigma}(\mathbb{R}^3, \mathbb{R}^3)$ s.t. $\|\widetilde{\dot\Psi}(t_j) - \dot\Phi(t_j)\|_{H^1} < \delta$ for all $j = 1, ..., M$ and define $\widetilde{\dot\Psi}(t) = \frac{t_j - t}{t_j - t_{j-1}} \widetilde{\dot\Psi}(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} \widetilde{\dot\Psi}(t_j)$ for $t \in [t_{j-1}, t_j]$ and $\dot\Psi = 0$ outside $[t_0, t_M]$. Then $\|\widetilde{\dot\Psi}(t) - \dot\Phi(t)\|_{H^1} < \delta$ for all $t \in [0, T]$. We also have

$$\left\| \int_0^T \widetilde{\dot\Psi}(t) dt \right\|_{H^1} \leq T\delta$$

For $\theta \in C^0_c((0, T), [0, 1])$ a cutoff with $\int_0^1 \theta(t) dt = 1$, let

$$\widehat{\dot\Psi}(t) = \widetilde{\dot\Psi}(t) - \theta(t) \int_0^T \widetilde{\dot\Psi}(t) dt$$

Then

$$\|\widehat{\dot\Psi}(t) - \dot\Phi(t)\|_{H^1} < \delta(T + 1)$$

and for $\widehat\Psi(t) := \int_0^t \widehat{\dot\Psi}(t') dt' \in C^1_c((0, T), V)$ we have $\widehat\Psi(t) \in C^\infty_{c\sigma}(\mathbb{R}^3, \mathbb{R}^3)$ for any $t$ and

$$\|\widehat\Psi(t) - \Phi(t)\|_{H^1} < \delta(T + 1)T.$$

Next, taking a cutoff $\rho \in C^\infty_c(\mathbb{R}, [0, 1])$ with $\int_{\mathbb{R}} \rho(t) dt = 1$, we can assume that $\Psi(t) := \rho_h *_t \widehat\Psi(t)$ is in $C^\infty_c((0, T), V)$ and that

$$\|\widehat\Psi(t) - \Psi(t)\|_{C^1_c([0,T],V)} < \delta.$$

Then $\|\Phi(t) - \Psi(t)\|_{C^1_c([0,T],V)} \leq \delta(T + 1)^2 + \delta < \epsilon$, if we pick $\delta > 0$ small enough. This completes the proof of (11.19), since clearly $\Psi \in C^\infty_{c\sigma}((0, T) \times \mathbb{R}^3, \mathbb{R}^3)$.
By (11.19), we claim that

$$\int_{\mathbb{R}_+} \left(-\langle \nabla u, \nabla \Phi \rangle + \langle u, \partial_t \Phi \rangle - \langle \mathbb{P}\mathrm{div}(u \otimes u), \Phi \rangle\right) dt' = 0 \text{ for all } \Phi \in C^1_c(\mathbb{R}_+, V), \tag{11.20}$$

which, in particular, implies

$$\int_{\mathbb{R}_+} \left(\langle u, \triangle \Phi \rangle + \langle u, \partial_t \Phi \rangle - \langle \mathbb{P}\mathrm{div}(u \otimes u), \Phi \rangle\right) dt' = 0 \text{ for all } \Phi \in C^\infty_c((0, T) \times \mathbb{R}^3, \mathbb{R}^3). \tag{11.21}$$

To get (11.20) consider a sequence $C^\infty_{c\sigma}((0, T) \times \mathbb{R}^3, \mathbb{R}^3) \ni \Phi_n \xrightarrow{n\to\infty} \Phi$ in $C^1_c([0, T], V)$, for $T$ appropriately large s.t. $\mathrm{supp}\Phi \subset (0, T) \times \mathbb{R}^3$. Then, obviously

$$\int_0^T \left(-\langle \nabla u, \nabla \Phi_n \rangle + \langle u, \partial_t \Phi_n \rangle\right) dt' \xrightarrow{n\to\infty} \int_0^T \left(-\langle \nabla u, \nabla \Phi \rangle + \langle u, \partial_t \Phi \rangle\right) dt'$$

and, from $\dot{H}^{3/4}\left(\mathbb{R}^3\right) \hookrightarrow L^4\left(\mathbb{R}^3\right)$ by $\frac{1}{4} = \frac{1}{2} - \frac{3/4}{3}$,

$$\int_0^T |\langle \mathbb{P}\mathrm{div}(u\otimes u), \Phi - \Phi_n\rangle| \le \|u\otimes u\|_{L^1((0,T),L^2)}\|\Phi_n - \Phi\|_{C^0_c([0,T],V)} \le \|u\|^2_{L^2((0,T),L^4)}\|\Phi_n - \Phi\|_{C^0_c([0,T],V)}$$
$$\le \|\nabla u\|^{\frac{3}{2}}_{L^2((0,T),L^2)}\|u\|^{\frac{1}{2}}_{L^2((0,T),L^2)}\|\Phi_n - \Phi\|_{C^0_c([0,T],V)} \xrightarrow{n\to\infty} 0.$$

So, by taking the limit with $n \nearrow \infty$, we obtain (11.20).
Now, looking at (11.21), we can write $\mathbb{P}\mathrm{div}(u\otimes u) = \mathrm{div}(u\otimes u) - (1-\mathbb{P})\mathrm{div}(u\otimes u)$. So, by a direct computation which uses $\mathbb{P}v = v + \overrightarrow{e}_i R_i R_j v_j$, we have

$$\langle \mathbb{P}\mathrm{div}(u\otimes u), \Phi\rangle = \langle \mathrm{div}(u\otimes u), \Phi\rangle + \langle R_iR_j\partial_k(u^ku^j), \Phi_i\rangle = \langle \mathrm{div}(u\otimes u), \Phi\rangle + \langle \partial_i R_j R_k(u^ku^j), \Phi_i\rangle$$
$$= \mathrm{div}(u\otimes u), \Phi\rangle + \langle \nabla p, \Phi\rangle.$$

So, plugging in the previous equation, we get the desired result:

$$\int_{\mathbb{R}_+} \left(\langle u, \triangle\Phi\rangle + \langle u, \partial_t\Phi\rangle - \langle \mathrm{div}(u\otimes u), \Phi\rangle - \langle \nabla p, \Phi\rangle\right) dt' = 0 \text{ for all } \Phi \in C^\infty_c(\mathbb{R}_+\times\mathbb{R}^3, \mathbb{R}^3).$$

*Remark* 11.10. Notice that the related remarks at the bottom of p. 116 [14] are based on an incorrect Helmholz–Weyl decomposition of vector fields in $\mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$. Notice in particular that the solution of exercise 5.2 in p. 429 is wrong.

*Proof of Lemma 6.19.* We start from equation (6.19), which we write as

$$\begin{cases} \dot{u}_\varepsilon - \triangle u_\varepsilon + \rho_\varepsilon * u_\varepsilon \cdot \nabla u_\varepsilon + \nabla p_\varepsilon = 0 \\ u_\varepsilon(0) = u_0 \end{cases}$$
$$\text{where } p_\varepsilon = R_iR_j\left(\rho_\varepsilon * u^i_\varepsilon u^j_\varepsilon\right).$$

Let us consider now for $0 < R_1 < R$

$$\varrho(x) := \begin{cases} 0 \text{ for } |x| \le R_1 \\ \frac{|x|-R_1}{R-R_1} \text{ for } R_1 \le |x| \le R \\ 1 \text{ for } |x| \ge R. \end{cases}$$

Then, applying $\langle \cdot, \varrho u_\varepsilon\rangle$ to the equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \varrho|u_\varepsilon|^2 + \int_{\mathbb{R}^3} \varrho|\nabla u_\varepsilon|^2 = -\int_{\mathbb{R}^3} \partial_i\varrho u^j_\varepsilon \partial_i u^j_\varepsilon u^j_\varepsilon + \int_{\mathbb{R}^3} |u_\varepsilon|^2 \rho_\varepsilon * u_\varepsilon \cdot \nabla\varrho + \int_{\mathbb{R}^3} p_\varepsilon u_\varepsilon \cdot \nabla\varrho.$$

Integrating between $(0,t)$ we have

$$\frac{1}{2}\int_{\mathbb{R}^3} \varrho|u_\varepsilon(t)|^2 \le \frac{1}{2}\int_{\mathbb{R}^3} \varrho|\rho_\varepsilon * u_0|^2 + \int_0^t\int_{\mathbb{R}^3} \left(|\nabla u_\varepsilon|\ |u_\varepsilon| + |u_\varepsilon|^2\ |\rho_\varepsilon * u_\varepsilon| + |p_\varepsilon|\ |u_\varepsilon|\right)|\nabla\varrho|$$
$$\le \frac{1}{2}\int_{|x|\ge R_1} |u_0|^2 + \frac{1}{R-R_1}\int_0^t\int_{\mathbb{R}^3} \left(|\nabla u_\varepsilon|\ |u_\varepsilon| + |u_\varepsilon|^2\ |\rho_\varepsilon * u_\varepsilon| + |p_\varepsilon|\ |u_\varepsilon|\right)$$

and so also

$$\frac{1}{2}\int_{|x|\geq R}|u_\varepsilon(t)|^2 \leq \frac{1}{2}\int_{|x|\geq R_1}|\rho_\varepsilon * u_0|^2 + \frac{1}{R-R_1}\int_0^t\int_{\mathbb{R}^3}\left(|\nabla u_\varepsilon|\ |u_\varepsilon| + |u_\varepsilon|^2\ |\rho_\varepsilon * u_\varepsilon| + |p_\varepsilon|\ |u_\varepsilon|\right).$$

For the nonlinear term, we have

$$\begin{aligned}&\| |\nabla u_\varepsilon|\ |u_\varepsilon| + |u_\varepsilon|^2\ |\rho_\varepsilon * u_\varepsilon| + |p_\varepsilon|\ |u_\varepsilon| \|_{L^1((0,t),L^1_x)} \\ &\leq \left(\|\nabla u_\varepsilon\|_{L^1((0,t),L^2_x)} + \|u_\varepsilon\|^2_{L^2((0,t),L^4_x)} + \|p_\varepsilon\|_{L^1((0,t),L^2_x)}\right)\|u_\varepsilon\|_{L^\infty((0,t),L^2_x)}.\end{aligned}$$

We have $\|p_\varepsilon\|_{L^1((0,t),L^2_x)} \leq C_3\|u_\varepsilon\|^2_{L^2((0,t),L^4_x)}$. Now we bound

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty((0,t),L^2_x)} = \|u_0\|_{L^2_x} \\ &\|\nabla u_\varepsilon\|_{L^1((0,t),L^2_x)} \leq \sqrt{t}\|\nabla u_\varepsilon\|_{L^2((0,t),L^2_x)} \leq \sqrt{t}\|u_0\|_{L^2_x}\end{aligned}$$

and, by Gagliardo Nirenberg,

$$\begin{aligned}&\|u_\varepsilon\|^2_{L^2((0,t),L^4_x)} \lesssim \| \|u_\varepsilon\|^{1/4}_{L^2_x}\|\nabla u_\varepsilon\|^{3/4}_{L^2_x}\|^2_{L^2(0,t)} \leq \|u_\varepsilon\|^{1/2}_{L^\infty((0,t),L^2_x)}\|\nabla u_\varepsilon\|^{3/2}_{L^{3/2}((0,t),L^2_x)} \\ &\leq \|u_0\|^{1/2}_{L^2_x}\left(\sqrt[6]{t}\|\nabla u_\varepsilon\|_{L^2((0,t),L^2_x)}\right)^{3/2} \leq \sqrt[4]{t}\|u_0\|_{L^2_x}.\end{aligned}$$

So, for a dimensional constant $C_3$, we have

$$\int_{|x|\geq R}|u_\varepsilon(t)|^2 \leq \int_{|x|\geq R_1}|\rho_\varepsilon * u_0|^2 + \frac{C_3(\sqrt{t}+\sqrt[4]{t})}{R-R_1}\|u_0\|^2_{L^2_x}\left(1+\|u_0\|_{L^2_x}\right) \text{ for all } \varepsilon\in(0,1).$$

Now we fix $\epsilon > 0$ and, keeping in mind that $t \in [0,T]$ and $\varepsilon \in (0,1)$, we pick $R_1$ such that

$$\int_{|x|\geq R_1}|\rho_\varepsilon * u_0|^2 \leq \int_{|x|\geq R_1-1}|u_0|^2 < \frac{\epsilon^2}{2}$$

and subsequently we pick $R$ such that

$$\frac{C_3(\sqrt{T}+\sqrt[4]{T})}{R-R_1}\|u_0\|^2_{L^2_x}\left(1+\|u_0\|_{L^2_x}\right) < \frac{\epsilon^2}{2}.$$

Then we obtain $R = R(\|u_0\|_{L^2_x}, T, \epsilon)$ such that (6.36) is true.

□

# 12 Local Serrin regularity

In this section we will prove the following result.

**Theorem 12.1.** *Consider $u$, a Leray–Hopf solution of NS in $d = 3$ with $u \in L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, \dot{H}^1)$ and suppose that, for an open subspace $U \subseteq \mathbb{R}^3$, we have*

$$u \in L^r((0,T), L^s(U)) \text{ where } \frac{2}{r} + \frac{3}{s} = 1, \text{ with } r \geq 2 \text{ and } s \geq 3, \tag{12.1}$$

*excluding case $(r,s) = (0,3)$. Then for any open $\Omega \subset \overline{\Omega} \subset\subset U$ and any $t_0 \in (0,T]$ $u \in L^\infty((t_0,T), H^k(\Omega))$ for any $k = 0, 1, ...$ and $u \in C_t^{0,\gamma}([t_0,T], C_x^0(\overline{\Omega}))$ for any $\gamma \in (0, 1/2)$.*

The case $(r,s) = (0,3)$ is also true, but is not discussed here.

Theorem 12.1 will be obtained as a consequence of Theorem 12.6, see below, which requires a definition.

**Definition 12.2.** We say that $u$ is a local weak solution of NS in $(a,b) \times U$ if

1. $u \in L^\infty((a,b), L^2(U))$ and $\nabla u \in L^2((a,b), L^2(U))$ and
2. $u$ satisfies

$$\int_a^b (\langle u, \triangle\Psi\rangle + \langle u, \partial_t \Psi\rangle - \langle \text{div}(u \otimes u), \Psi\rangle)\, dt' = 0 \text{ for all } \Psi \in C^\infty_{c\sigma}((a,b) \times U, \mathbb{R}^3).$$

Notice that weak solutions in $[0,\infty) \times \mathbb{R}^3$ are local weak solutions in $(a,b) \times U$ for any $a \geq 0$. The viceversa is not true.

*Example* 12.3 (Serrin's example). Notice that $u(t,x) = \alpha(t)\nabla\psi(x)$, with $\psi : U \to \mathbb{R}$ harmonic and $\alpha \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is a local weak solution of NS. Obviously $\langle \alpha\nabla\psi, \triangle\Psi\rangle = \langle \alpha\nabla\triangle\psi, \Psi\rangle = 0$. Also

$$\langle \text{div}(u \otimes u), \Psi\rangle = \alpha^2 \langle \partial_j\psi\partial_j\partial_k\psi, \Psi_k\rangle = 2^{-1}\alpha^2 \sum_j \left\langle \partial_k(\partial_j\psi)^2, \Psi_k\right\rangle = -\alpha^2 \left\langle |\nabla\psi|^2, \nabla\cdot\Psi\right\rangle = 0.$$

Finally, by $\nabla \cdot \Psi = 0$,

$$\langle u, \partial_t\Psi\rangle = \alpha\langle \nabla\psi, \partial_t\Psi\rangle = -\alpha\langle \nabla\psi, \partial_t\nabla\cdot\Psi\rangle = 0.$$

*Remark* 12.4. In view of Serrin's example, for Theorem 12.1 it is crucial equation

$$u_t + u \cdot \nabla u - \triangle u = -\nabla p. \tag{11.18}$$

Notice on the other hand that, since $p = R_iR_j(u_iu_j)$ and the Riesz Transforms are non local operators, the regularity in $x$ of $u$ in $U \times (a,b)$ does not lead in an obvious way to regularity of $p$ in $U \times (a,b)$.

*Remark* 12.5. Since equation (11.18) contains a non–local term like the pressure $p$, while the results of this section involve local properties of $u$, it is natural that in the literature the proofs are based on the equation for the vorticity

$$\omega_t - \triangle\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{11.8}$$

Indeed (11.8) has the advantage, over (11.18), of containing only local terms.

In this section we will use pairs $(q',q)$ of indexes, where $q'$ is not the dual index of $q$.

The main technical result of this section is the following.

**Theorem 12.6.** *Consider a local weak solution $u$ in a parabolic cylinder $Q_R(t_0,x_0)$. Then, if*

$$u \in L^{q'}L^q(Q_R(t_0,x_0)) \text{ where } \frac{2}{q'}+\frac{3}{q}\le 1, \text{ for } q'\ge 2 \text{ and } q\ge 3, \tag{12.2}$$

*$u$ is smooth in the $x$ variable in $\overline{Q_{R'}}(t_0,x_0)$ for any $R'\in(0,R)$.*

We will not prove the case $(q',q)=(\infty,3)$, which is more complicated and was proved in [5] some time after the other cases. Notice that, in view of Example 12.3, we cannot get any regularity in $t$. On the other hand, we will see later how to recover the Hölder regularity for the weak solutions of the NS in Theorem 12.1.

Theorem 12.6 is, in the case $(q',q)\ne(\infty,3)$, a consequence of the following theorem. Indeed given any point $(a,s)\in Q_R(t_0,x_0)$ we have for $Q_\rho(s,a)\subset Q_R(t_0,x_0)$

$$\|u\|_{L^{q'}L^q(Q_\rho(s,a))}\le \|u\|_{L^{q'}L^q((s-\rho^2,s)\times B_R(t_0))}\xrightarrow{\rho\to 0}0$$

for $q'<\infty$ when $\frac{2}{q'}+\frac{3}{q}=1$, while if $q'=\infty$ and $q>3$, we can use

$$\|u\|_{L^\infty L^3(Q_\rho(s,a))}\le \|u\|_{L^\infty L^q(Q_\rho(s,a))}|B(a,\rho)|^{\frac{q-3}{3q}}=(4\pi/3)^{\frac{q-3}{3q}}\|u\|_{L^\infty L^q(Q_\rho(s,a))}\rho^{\frac{q-3}{q}}\xrightarrow{\rho\to 0}0.$$

**Theorem 12.7.** *There exists an $\epsilon_{q'q}>0$ such that if $u$ is a local weak solution in a parabolic cylinder $Q_R(t_0,x_0)$ s.t. $u$ satisfies* (12.2)*, with*

$$\|u\|_{L^{q'}L^q(Q_R(t_0,x_0))}<\epsilon_{q'q}, \tag{12.3}$$

*then $u$ is in $L^\infty_t H^k_x(Q_{R'}(t_0,x_0))$ for any $R'\in(0,R)$ and $k\in\mathbb{N}$.*

Considering that the condition of $u$ being smooth in Theorem 12.6 is a local condition, it is natural that, in the case $(q',q)\ne(\infty,3)$, around each point of the cylinder in Theorem 12.6, we can consider a sufficiently small cylinder where (12.3) is satisfied, etc.

We will prove Theorem 12.7 also in the case $(q',q)=(\infty,3)$. The proof will exploit the vorticity Propositions 5.1–5.5.

The proof of Theorem 12.7 is divided in two parts. The first is the following.

**Proposition 12.8.** *Consider a local weak solution $u$ in a parabolic cylinder $Q_R(t_0,x_0)$ Then, if*

$$u \in L^{q'}L^q(Q_R(t_0,x_0)) \text{ where } \frac{2}{q'}+\frac{3}{q}<1, \tag{12.4}$$

*$u$ is smooth in the $x$ variable in $\overline{Q_{R'}(t_0,x_0)}$ for any $R'\in(0,R)$.*

*Proof.* It is enough to prove that $u$ is smooth in the $x$ variable in $\overline{Q}_{R/2}(t_0, x_0)$. To proceed we observe that an analogue of Lemma 11.3 shows that the pair $(u, \omega)$ satisfies the following analogue of (11.9):

$$\int_0^\infty (\langle \omega, \partial_t \Phi\rangle + \langle \omega, \triangle \Phi\rangle + \langle \omega, u \cdot \nabla \Phi\rangle - \langle u, \omega \cdot \nabla \Phi\rangle)\, dt' = 0 \text{ for all } \Phi \in C_c^\infty(Q_R(t_0, x_0), \mathbb{R}^3). \tag{12.5}$$

We define $W = \phi\omega$ with a cutoff $\phi \in C_c^\infty(\mathbb{R}^4, [0, 1])$ with $\mathrm{supp}\phi \cap (\overline{Q}_R(t_0, x_0) \subseteq \overline{Q_{\rho_l R}}(t_0, x_0)$, with $\phi = 1$ in $\overline{Q}_{\rho_i R}(t_0, x_0)$, where $\rho_i$ and $\rho_l$ will be chosen later, they depend on the pair $(q', q)$ and satisfy $1/2 < \rho_i < \rho_l < 1$. Then, in a weak sense, the weak formulation of (12.5) implies a weak form of

$$W_t - \triangle W = (W \cdot \nabla)u - \phi(u \cdot \nabla)\omega + (\phi_t - \triangle\phi)\omega - 2\nabla\phi \cdot \nabla\omega.$$

Writing $-2\nabla\phi \cdot \nabla\omega_i = -2\partial_j(\omega_i\partial_j\phi) + 2\omega_i\triangle\phi$, the above equation can be conveniently written as

$$\partial_t W_i - \triangle W_i = \partial_j(W_j u_i - W_i u_j) - 2\partial_j(\omega_i \partial_j \phi) \tag{12.6}$$
$$+ (\phi_t + \triangle\phi)\omega_i - \partial_j\phi(\omega_j u_i - \omega_i u_j)). \tag{12.7}$$

The proof of Proposition 12.8 is divided in two parts. In the first, we will prove that $\omega \in L^\infty L^\infty(Q_{3R/4}(t_0, x_0))$. Let us assume this and see the conclusion of Proposition 12.8.

The rather standard second part of the proof of Proposition 12.8, starts by noticing that Lemma 11.5 implies, for $k = 0$, $u \in L_t^\infty W_x^{k,\infty}(Q_{R'_k}(t_0, x_0))$ for any $R'_k \in (R/2, 3R/4)$. Having $u, \omega \in L_t^\infty W_x^{k,\infty}(Q_{R'_k}(t_0, x_0))$ we can fix an arbitrary $R''_k \in (R/2, R'_k)$. Then let us consider a cutoff $\phi \in C_c^\infty(\mathbb{R}^4, [0, 1])$ with $\mathrm{supp}\phi \cap \overline{Q}_{3R/4}(t_0, x_0) \subseteq \overline{Q}_{R'_k}(t_0, x_0)$ and with $\phi = 1$ in $\overline{Q}_{R''_k}(t_0, x_0)$. For $W = \phi\omega$ we have the above equation. Applying Propositions 5.4–5.5 in $Q_{R'_k}(t_0, x_0)$ we obtain $W \in L_t^\infty C_x^{k,\alpha}(\overline{Q}_{R''_k}(t_0, x_0))$, that is $\omega \in L_t^\infty C_x^{k,\alpha}(\overline{Q}_{R''_k}(t_0, x_0))$ from $\phi = 1$ in $\overline{Q_{R''_k}}(t_0, x_0)$, for any $\alpha \in (0, 1)$. Then Lemma 11.5 implies (in fact, more regularity than) $u \in L_t^\infty C_x^{k,\alpha}(\overline{Q}_{R'''_k}(t_0, x_0))$ for any $R'''_k \in (R/2, R''_k)$ and for any $\alpha \in (0, 1)$. Now we fix $R_k^{(4)} \in (R/2, R'''_k)$ and a cutoff $\phi \in C_c^\infty(\mathbb{R}^4, [0, 1])$ with $\mathrm{supp}\phi \cap \overline{Q}_{3R/4}(t_0, x_0) \subseteq \overline{Q}_{R'''_k}(t_0, x_0)$ with $\phi = 1$ in $\overline{Q}_{R_k^{(4)}}(t_0, x_0)$. For $W = \phi\omega$ we have the above equation. Applying Propositions 5.4–5.5 in $Q_{R'''_k}(t_0, x_0)$ we obtain $\nabla^{k+1}W \in L_t^\infty L_x^\infty(\overline{Q}_{R_k^{(4)}}(t_0, x_0))$ combined with $W \in L_t^\infty C_t^{k,\alpha}(\overline{Q}_{R^{(4)}}(t_0, x_0))$. Thus $\nabla^{k+1}\omega \in L_t^\infty L_x^\infty(\overline{Q}_{R_k^{(4)}}(t_0, x_0))$ and $\omega \in L_t^\infty C_x^{k,\alpha}(\overline{Q}_{R_k^{(4)}}(t_0, x_0))$. For $R'_{k+1} = R_k^{(4)}$, we can repeat the argument with $k$ replaced by $k + 1$. By induction there is a decreasing sequence $\{R_n\}$ with $R_n > R/2$ with $u \in L_t^\infty C^n(\overline{Q}_{R_n}(t_0, x_0))$ for any $n \in \mathbb{N}$.

We now start the proof of $\omega \in L^\infty L^\infty(Q_{3R/4}(t_0, x_0))$. We start by assuming $\omega \in L^{m'} L^m(Q_R(t_0, x_0))$ for some $(m', m)$. This is certainly true for $(m', m) = (2, 2)$. Obviously,

we assume $(m',m) \neq (\infty,\infty)$, since otherwise there is nothing to prove. As we did above, we consider $W = \phi\omega$ with a cutoff $\phi \in C_c^\infty(\mathbb{R}^4,[0,1])$ with $\text{supp}\phi \cap \overline{Q_R}(t_0,x_0) \subseteq \overline{Q_{\rho_l R}}(t_0,x_0)$ and with $\phi = 1$ in $\overline{Q_{\rho_i R}}(t_0,x_0)$. Notice that $W \in L^{m'}L^m((t_0-R^2,t_0)\times\mathbb{R}^3)$.
Using Propositions 5.1–5.2, we have

$$\|W\|_{L^{r'}L^r} \lesssim \|Wu\|_{L^{a'}L^a} + \|\omega\nabla\phi\|_{L^{m'}L^m} + \|(\phi_t + \triangle\phi)\omega\|_{L^{l'}L^l} + \|\omega u\nabla\phi\|_{L^{l'}L^l} \tag{12.8}$$

where

$$\begin{cases} 1 \le a \le r \le \infty, \quad 1 \le a' \le r' \le \infty \\ \frac{3}{a} + \frac{2}{a'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases}, \quad \begin{cases} 1 \le m \le r \le \infty, \quad 1 \le m' \le r' \le \infty \\ \frac{3}{m} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \end{cases}$$
$$\begin{cases} 1 \le l \le r \le \infty, \quad 1 \le l' \le r' \le \infty \\ \frac{3}{l} + \frac{2}{l'} < \frac{3}{r} + \frac{2}{r'} + 2 \end{cases}. \tag{12.9}$$

Now we have to choose the indexes. Recall that $u \in L^{q'}L^q(Q_{\rho_l R}(t_0,x_0))$, see (12.2). We consider

$$\frac{1}{l} = \frac{1}{a} = \frac{1}{q} + \frac{1}{m} \text{ and } \frac{1}{l'} = \frac{1}{a'} = \frac{1}{q'} + \frac{1}{m'}. \tag{12.10}$$

Here notice that from $\frac{2}{q} + \frac{3}{q'} < 1$ obviously we have $q > 2$ and $q' > 3$, to that from $m \ge 2$ and $m' \ge 2$, we have $l \ge 1$ and $l' \ge 1$.
Inequalities (12.9) become

$$\begin{aligned} &\frac{3}{m} + \frac{3}{q} + \frac{2}{m'} + \frac{2}{q'} < \frac{3}{r} + \frac{2}{r'} + 1, \\ &\frac{3}{m} + \frac{2}{m'} < \frac{3}{r} + \frac{2}{r'} + 1 \\ &\frac{3}{m} + \frac{3}{q} + \frac{2}{m'} + \frac{2}{q'} < \frac{3}{r} + \frac{2}{r'} + 2, \end{aligned}$$

where obviously the 1st implies the other two. Then we have

$$\|W\|_{L^{r'}L^r} \lesssim \|W\|_{L^{m'}L^m}\|u\|_{L^{q'}L^q} + \|u\|_{L^{q'}L^q}\|\omega\|_{L^{m'}L^m} + \|\omega\|_{L^{m'}L^m}. \tag{12.11}$$

In fact we get $\|W\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)} \lesssim \|\omega\|_{L^{m'}L^m(Q_R(t_0,x_0))}$, where

$$\frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} < 1 - \frac{3}{q} - \frac{2}{q'}. \tag{12.12}$$

Since by hypothesis the r.h.s. in (12.12) is strictly positive and $(m',m)$ are given, we can find $m \le r$ and $m' \le r'$, not both equal, so that (12.18) is true. In fact this can be written in a systematic way, setting $\chi := \frac{1}{6}\left(1 - \frac{3}{q} - \frac{2}{q'}\right)$ and defining

$$r = \begin{cases} \frac{m}{1-\chi m} \text{ if } m\chi < 1 \\ \infty \text{ if } m\chi \ge 1 \end{cases} \text{ and } r' = \begin{cases} \frac{m'}{1-\chi m'} \text{ if } m'\chi < 1 \\ \infty \text{ if } m'\chi \ge 1. \end{cases} \tag{12.13}$$

With these choices we have $\frac{3}{m}-\frac{3}{r}\leq 3\chi$ and $\frac{2}{m'}-\frac{2}{r'}\leq 2\chi$, so that $\frac{3}{m}-\frac{3}{r}+\frac{2}{m'}-\frac{2}{r'}\leq 5\chi<1-\frac{3}{q}-\frac{2}{q'}$. Then we have obtained

$$\omega\in L^{m'}L^{m}(Q_R(t_0,x_0))\Longrightarrow \omega\in L^{\frac{m'}{1-\chi m'}}L^{\frac{m}{1-\chi m}}(Q_{\rho_i R}(t_0,x_0)).$$

We repeat this argument until both exponents are $(\infty,\infty)$. Notice that if we repeat the procedure $k$ times, we reach $\omega\in L^{\frac{m'}{1-k\chi m'}}L^{\frac{m}{1-k\chi m}}(Q_{\rho_i^k R}(t_0,x_0))$, since, for $km<1$,

$$\frac{\frac{m}{1-(k-1)\chi m}}{1-\chi\frac{m}{1-(k-1)\chi m}}=\frac{m}{1-(k-1)\chi m-\chi m}=\frac{m}{1-k\chi m}.$$

It is clear that, after a finite number $k$ of iterations, with $k$ dependent on the initial pairs $(m',m)$ and $(q',q)$, the procedure has to stop because, for example, either we get to $\chi\frac{m}{1-k\chi m}\geq 1$ or $1-k\chi m<0$. But $1-k\chi m<0$ cannot occur if $0<\chi\frac{m}{1-(k-1)\chi m}<1$ in the previous iterate. Hence at some point we get to $\chi\frac{m}{1-k\chi m}\geq 1$, so that from that iterate on, we have $r=\infty$. For $r'$ the same argument is true. So, after a finite number of iterate, we obtain the pair $(\infty,\infty)$. We also choose $3/4<\rho_i<\rho_l<1$ so that $\rho_i^k>3/4$ for all the finitely many iterates.

□

Now we consider the 2nd part of the proof of Theorem 12.7.

**Proposition 12.9.** *There exists an $\epsilon_{q'q}>0$ such that if $u$ is a local weak solution in a parabolic cylinder $Q_R(t_0,x_0)$ which satisfies (12.2) with $\frac{2}{q'}+\frac{3}{q}=1$ and if*

$$\|u\|_{L^{q'}L^{q}(Q_R(t_0,x_0))}<\epsilon_{q'q}, \tag{12.14}$$

*then $u$ is smooth in the $x$ variable in $Q_{R'}(t_0,x_0)$ for any $R'\in(0,R)$.*

The proof consists in finding $\beta'>2$ s.t. $u\in L^{\beta'}L^{\infty}(Q_{R'}(t_0,x_0))$ for any $R'\in(0,R)$. Then we can apply Proposition 12.8.
Notice that we can normalize and consider $R=1$, thanks to scaling. It would be reasonable to proceed as in the proof of Proposition 12.8, starting with $\omega\in L^{m'}L^{m}(Q_R(t_0,x_0))$ and then reaching $\omega\in L^{\beta'}L^{\infty}(Q_{R'}(t_0,x_0))$, and then to apply Lemma 11.5.

So we could consider (12.8)

$$\|W\|_{L^{r'}L^{r}}\lesssim\|Wu\|_{L^{a'}L^{a}}+\|\omega\nabla\phi\|_{L^{m'}L^{m}}+\|(\phi_t+\triangle\phi)\omega\|_{L^{m'}L^{m}}+\|\omega u\nabla\phi\|_{L^{l'}L^{l}} \tag{12.8}$$

with

$$\begin{cases}1\leq a\leq r\leq\infty, & 1\leq a'\leq r'<\infty\\ \frac{3}{a}+\frac{2}{a'}=\frac{3}{r}+\frac{2}{r'}+1=\frac{3}{r}+\frac{2}{r'}+\frac{3}{q}+\frac{2}{q'}\end{cases},$$
$$\begin{cases}1\leq m\leq r\leq\infty, & 1\leq m'\leq r'<\infty\\ \frac{3}{m}+\frac{2}{m'}\leq\frac{3}{r}+\frac{2}{r'}+1\end{cases}$$
$$\begin{cases}1\leq l\leq r\leq\infty, & 1\leq l'\leq r'<\infty\\ \frac{3}{l}+\frac{2}{l'}\leq\frac{3}{r}+\frac{2}{r'}+2\end{cases}. \tag{12.15}$$

Next, in analogy to (12.10) we could consider

$$\frac{1}{l} = \frac{1}{q} + \frac{1}{m} \text{ and } \frac{1}{l'} = \frac{1}{q'} + \frac{1}{m'}, \tag{12.16}$$

while we will take

$$\frac{1}{a} = \frac{1}{q} + \frac{1}{r} \text{ and } \frac{1}{a'} = \frac{1}{q'} + \frac{1}{r'}, \tag{12.17}$$

(notice that here $\frac{1}{r} + \frac{1}{q} \le 1$ (because $r \ge 3$ and $q \ge 3$) and $\frac{1}{r'} + \frac{1}{q'} \le 1$, so $a \ge 1$ and $a' \ge 1$). Here the point is that if we chose exactly (12.10), we would be forced, from

$$0 \le \frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} \le 1 - \frac{3}{q} - \frac{2}{q'} = 0$$

and from $m \le r$ and $m' \le r'$, to have exactly $(m, m') = (r, r')$. So (12.16)–(12.17), gives us a little more of flexibility. Indeed (12.15) reduce to

$$\begin{cases} 1 \le m \le r \le \infty, \quad 1 \le m' \le r' < \infty \\ \frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} \le 1 \end{cases} \tag{12.18}$$

and we can certainly pick $m \le r$ and $m' \le r'$ appropriate and not both equalities if $(m', m)$ is not of the form $(\beta', \infty)$ with $\beta' > 0$.
Then we obtain, in $Q_R(t_0, x_0)$,

$$\|W\|_{L^{r'}L^r} \lesssim \|W\|_{L^{r'}L^r}\|u\|_{L^{q'}L^q} + \|u\|_{L^{q'}L^q}\|\omega\|_{L^{m'}L^m} + \|\omega\|_{L^{m'}L^m} \lesssim \|W\|_{L^{r'}L^r}\|u\|_{L^{q'}L^q} + \|\omega\|_{L^{m'}L^m}. \tag{12.19}$$

Then, from $\|u\|_{L^{q'}L^q} < \epsilon_{q'q}$, for $\epsilon_{q'q}$ small enough we would absorb the $\|W\|_{L^{r'}L^r}\|u\|_{L^{q'}L^q}$ into the l.h.s., obtaining

$$\|W\|_{L^{r'}L^r(Q_R(t_0,x_0))} \lesssim \|\omega\|_{L^{m'}L^m(Q_R(t_0,x_0))}. \tag{12.20}$$

Then we could improve until we get to the desired $(\beta', \infty)$. In fact, for $\chi = 1/6$, we can proceed like in (12.13) obtaining $\frac{3}{m} - \frac{3}{r} \le 3\chi$ and $\frac{2}{m'} - \frac{2}{r'} \le 2\chi$, so that $\frac{3}{m} - \frac{3}{r} + \frac{2}{m'} - \frac{2}{r'} \le 5\chi < 1$. After a finite number of iterates, we would get to $(\beta', \infty)$ and stop.
However, the above argument is formal only, because it assumes implicitly that $\|W\|_{L^{r'}L^r} < \infty$. To perform rigorously the argument, we consider a mollification, both in space and time. We still can consider the equation (12.6)–(12.7) for $W$.
Now consider

$$\begin{aligned} \partial_t W_i^\epsilon - \triangle W_i^\epsilon &= \partial_j(W_j^\epsilon u_i^\epsilon - W_i^\epsilon u_j^\epsilon) - 2\partial_j(\omega_i \partial_j \phi) \\ &\quad + (\phi_t + \triangle\phi)\omega_i^\epsilon - \partial_j\phi(\omega_j^\epsilon u_i^\epsilon - \omega_i^\epsilon u_j^\epsilon)), \end{aligned} \tag{12.21}$$

where $W_i^\epsilon$ is an unknown, we set $(u^\epsilon, \omega^\epsilon) := \rho_\epsilon * (u, \omega)$ and $W_i^\epsilon(t_0 - R^2) = 0$, where we extend $(\omega, u) = 0$ in $\mathbb{R}^4 \backslash Q_R(t_0, x_0)$.

Now the previous argument works, and we obtain for a fixed $c$ (notice that in the equation, $\omega^\epsilon$ appears with factors involving $\phi$ which live in $Q_R(t_0,x_0)$)

$$\|W^\epsilon\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)} \le c\|\omega^\epsilon\|_{L^{m'}L^m((t_0-R^2,t_0)\times\mathbb{R}^3)}.$$

There exists a sequence $\epsilon_n \xrightarrow{n\to\infty} 0$, such that $W^{\epsilon_n} \rightharpoonup \overline{W}$ in $L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)$), and we have

$$\|\overline{W}\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)} \le c\|\omega\|_{L^{m'}L^m(Q_R(t_0,x_0))}. \tag{12.22}$$

Now we have to establish that $\overline{W}=W$ in $L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)$ to obtain

$$\|W\|_{L^{r'}L^r(Q_R(t_0,x_0))} \le c\|\omega\|_{L^{m'}L^m(Q_R(t_0,x_0))}.$$

and so, restricting the domain in the left

$$\|W\|_{L^{r'}L^r(Q_{\rho_i R}(t_0,x_0))} \le c\|\omega\|_{L^{m'}L^m(Q_R(t_0,x_0))}.$$

Once we do this, we conclude that the formal argument leading to (12.20) is correct.

The first step to prove $\overline{W}=W$, is to show that $\overline{W}$ is a weak solution of (12.6)–(12.7). Taking a test function $\psi\in C_c^\infty\left(\left[t_0-R^2,t_0\right)\times\mathbb{R}^3,\mathbb{R}^3\right)$, from (12.21) we have

$$0=\int_{t_0-R^2}^{t_0}\left(\langle W_i^{\epsilon_n},\partial_t\psi_i\rangle+\langle W_i^{\epsilon_n},\triangle\psi_i\rangle\right)dt'-\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i^{\epsilon_n}-W_i^{\epsilon_n}u_j^{\epsilon_n},\partial_j\psi_i\right\rangle dt'$$
$$+2\int_{t_0-R^2}^{t_0}\langle\omega_i^{\epsilon_n}\partial_j\phi,\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\langle(\phi_t+\triangle\phi)\omega_i^{\epsilon_n},\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\left\langle\partial_j\phi(\omega_j^{\epsilon_n}u_i^{\epsilon_n}-\omega_i^{\epsilon_n}u_j^{\epsilon_n}),\psi_i\right\rangle dt'.$$

Taking the limit $n\to\infty$ we get

$$0=\int_{t_0-R^2}^{t_0}\left(\langle\overline{W}_i,\partial_t\psi_i\rangle+\langle\overline{W}_i,\triangle\psi_i\rangle\right)dt'-\lim_{n\to+\infty}\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i^{\epsilon_n}-W_i^{\epsilon_n}u_j^{\epsilon_n},\partial_j\psi_i\right\rangle dt'$$
$$+2\int_{t_0-R^2}^{t_0}\langle\omega_i\partial_j\phi,\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\langle(\phi_t+\triangle\phi)\omega_i,\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\langle\partial_j\phi(\omega_ju_i-\omega_iu_j),\psi_i\rangle\,dt'.$$

Now we show that

$$\lim_{n\to+\infty}\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i^{\epsilon_n}-W_i^{\epsilon_n}u_j^{\epsilon_n},\partial_j\psi_i\right\rangle dt'=\int_{t_0-R^2}^{t_0}\langle W_ju_i-W_iu_j,\partial_j\psi_i\rangle\,dt'. \tag{12.23}$$

We have

$$\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i^{\epsilon_n},\partial_j\psi_i\right\rangle dt'=\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i,\partial_j\psi_i\right\rangle dt+\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}(u_i^{\epsilon_n}-u_i),\partial_j\psi_i\right\rangle dt'.$$

But now

$$\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}u_i,\partial_j\psi_i\right\rangle dt'\xrightarrow{n\to\infty}\int_{t_0-R^2}^{t_0}\langle\overline{W}_ju_i,\partial_j\psi_i\rangle\,dt'$$

while

$$\left|\int_{t_0-R^2}^{t_0}\left\langle W_j^{\epsilon_n}(u_i^{\epsilon_n}-u_i),\partial_j\psi_i\right\rangle dt'\right| \leq C\|W_j^{\epsilon_n}\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)}\|u_i^{\epsilon_n}-u_i\|_{L^{q'}L^q((t_0-R^2,t_0)\times\mathbb{R}^3)}$$
$$\leq C'\|\omega^{\epsilon_n}\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)}\|u_i^{\epsilon_n}-u_i\|_{L^{q'}L^q((t_0-R^2,t_0)\times\mathbb{R}^3)}$$
$$\leq C'\|\omega\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)}\|u_i^{\epsilon_n}-u_i\|_{L^{q'}L^q((t_0-R^2,t_0)\times\mathbb{R}^3)}\xrightarrow{n\to\infty}0.$$

Notice that here $\frac{1}{r}+\frac{1}{q}\leq 1$ (because $r\geq 2$ and $q\geq 3$) and $\frac{1}{r'}+\frac{1}{q'}\leq 1$ (because $r'\geq 2$ and $q'\geq 2$) justify the above use of Hölder inequality. Proceeding in this way we obtain the proof of limit (12.23).
So we conclude that for any $\psi\in C_c^\infty\left(\left[t_0-R^2,t_0\right)\times\mathbb{R}^3,\mathbb{R}^3\right)$

$$0=\int_{t_0-R^2}^{t_0}\left(\left\langle\overline{W}_i,\partial_t\psi_i\right\rangle+\left\langle\overline{W}_i,\triangle\psi_i\right\rangle\right)dt'-\int_{t_0-R^2}^{t_0}\left\langle\overline{W}_j,u_i-\overline{W}_iu_j,\partial_j\psi_i\right\rangle dt'$$
$$+2\int_{t_0-R^2}^{t_0}\langle\omega_i\partial_j\phi,\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\langle(\phi_t+\triangle\phi)\omega_i,\partial_j\psi_i\rangle\,dt'+\int_{t_0-R^2}^{t_0}\langle\partial_j\phi(\omega_ju_i-\omega_iu_j),\psi_i\rangle\,dt'.$$

This implies that $\overline{W}\in L^{r'}L^r(\left[t_0-R^2,t_0\right)\times\mathbb{R}^3,\mathbb{R}^3)$ is a distributional solution of (12.6)–(12.7) with initial datum $\overline{W}(t_0-R^2)=0$, see Takahashi [20].
Then, taking the difference between the equations of $\overline{W}$ and $W$, we have

$$\begin{aligned}&\partial_t(W_i-\overline{W}_i)-\triangle(W_i-\overline{W}_i)=\partial_jf_j\\&\text{with } f_j=(W_j-\overline{W}_j)u_i-(W_i-\overline{W}_i)u_j\\&\text{and initial condition } (W-\overline{W})(t_0-R^2)=0\end{aligned}\tag{12.24}$$

in a weak sense, that is for any $\psi\in C_c^\infty\left(\left[t_0-R^2,t_0\right)\times\mathbb{R}^3,\mathbb{R}^3\right)$

$$0=\int_{t_0-R^2}^{t_0}\left(\left\langle W_i-\overline{W}_i,\partial_t\psi_i\right\rangle+\left\langle W_i-\overline{W}_i,\triangle\psi_i\right\rangle\right)dt'-\int_{t_0-R^2}^{t_0}\left\langle(W_j-\overline{W}_j)u_i-(W_i-\overline{W}_i)u_j,\partial_j\psi_i\right\rangle dt'.$$

We claim that

$$W_i-\overline{W}_i=\int_{t_0-R^2}^{t}e^{(t-t')\triangle}\partial_j((W_j-\overline{W}_j)u_i-(W_i-\overline{W}_i)u_j)dt'\tag{12.25}$$

Indeed both sides are solutions in $L^2((t_0-R^2,t_0)\times\mathbb{R}^3,\mathbb{R})$ of the equation $(\partial_t-\triangle)w=\partial_jf_j$ with initial condition $w(t_0-R^2)=0$. So their difference, is a solution, which we again denote by $w$, with $w\in L^2((t_0-R^2,t_0)\times\mathbb{R}^3,\mathbb{R})$ with

$$(\partial_t-\triangle)w=0 \text{ with } w(t_0-R^2)=0.$$

By scaling and translation we get a solution $w\in L^2((0,1)\times\mathbb{R}^3,\mathbb{R})$

$$(\partial_t-\triangle)w=0 \text{ with } w(0)=0,$$

which satisfies

$$\int_0^1 \langle w, (\partial_t + \triangle)\varphi\rangle \, dt' = 0 \text{ for all } \varphi \in \mathcal{S}([0,1]\times\mathbb{R}^3, \mathbb{R}).$$

But for any $F \in \mathcal{S}([0,1]\times\mathbb{R}^3,\mathbb{R})$ with $\mathcal{F}_x F \in C_c^\infty([0,1]\times\mathbb{R}^3,\mathbb{R})$ it is possible to define such a $\varphi$ with $(\partial_t + \triangle)\varphi = F$ proceeding as right under (1.10). Then by density of such $F$ 's in $L^2((0,1)\times\mathbb{R}^3,\mathbb{R})$, we conclude $w = 0$.
Having established formula (12.25), we can apply Propositions 5.1–5.2 to $\overline{W} - W$. For

$$\frac{3}{l} + \frac{2}{l'} \le \frac{3}{r} + \frac{2}{r'} + 1 = \frac{3}{r} + \frac{2}{r'} + \frac{3}{q} + \frac{2}{q'},$$

we have

$$\begin{aligned} &\|W - \overline{W}\|_{L^{r'}L^r(Q_R(t_0,x_0))} \lesssim \|(W - \overline{W})u\|_{L^{l'}L^l(Q_R(t_0,x_0))} \\ &\lesssim \|W - \overline{W}\|_{L^{r'}L^r(Q_R(t_0,x_0))}\|u\|_{L^{q'}L^q(Q_R(t_0,x_0))} \le \epsilon_{q'q}\|W - \overline{W}\|_{L^{r'}L^r(Q_R(t_0,x_0))}, \end{aligned}$$

where we are free to choose a $L^{r'}L^r(Q_R(t_0,x_0))$ s.t. both $W$ and $\overline{W}$ belong to it. We exploited the fact that $\|u\|_{L^{q'}L^q((t_0-R^2,t_0)\times\mathbb{R}^3)} = \|u\|_{L^{q'}L^q(Q_R(t_0,x_0))}$ and, in the left hand side, that $\|W - \overline{W}\|_{L^{r'}L^r(Q_R(t_0,x_0))} \le \|W - \overline{W}\|_{L^{r'}L^r((t_0-R^2,t_0)\times\mathbb{R}^3)}$.
Now we exploit that $\epsilon_{q'q}$ is small, to conclude that $W = \overline{W}$ in $Q_R(t_0,x_0)$. This completes the proof of Proposition 12.9.

□

*End of the proof of Theorem 12.1.* By Theorem 12.6, in particular by its proof, we know that $\triangle u \in L^\infty[t_0,T], L^\infty(\Omega))$ for $\Omega \subset \overline{\Omega} \subset\subset U$ and for any $t_0 \in (0,T)$. Next, we claim that

$$u\cdot\nabla u + \nabla p \in L^{\frac{2r}{4r-3}}((0,T), L^r(\mathbb{R}^3)) \text{ for all } 1 < r \le 3/2. \tag{12.26}$$

Assuming (12.26), it follows from (11.18) that $u \in W^{1,\frac{2r}{4r-3}}(t_0,T), L^r(\Omega))$. We know that elements of $W^{1,\frac{2r}{4r-3}}((t_0,T), L^r(\Omega))$ are classes of functions and that, by Sobolev's inequality, one of the elements of this class is in $C^{0,\alpha}([t_0,T], L^r(\Omega))$ for $\alpha = 1 - \frac{4r-3}{2r} = \frac{3-2r}{2r}$. In fact, by $u \in C^0([0,T], L^2_w(\mathbb{R}^3))$, it is easy to conclude that $u \in C^{0,\alpha}([t_0,T], L^r(\Omega))$, so that

$$\|u(t) - u(s)\|_{L^r(\Omega)} \le c|t-s|^\alpha \text{ for } t_0 \le s < t \le T. \tag{12.27}$$

Next, by the variation of Agmon's inequality in (2.31), for almost any pair $(t,s)$ in $(t_0,T)$ we have

$$\begin{aligned} &\|u(t)-u(s)\|_{L^\infty(\Omega)} \le C_{\Omega,k,r}\|u(t)-u(s)\|^\theta_{L^r(\Omega)}\|u(t)-u(s)\|^{1-\theta}_{H^k(\Omega)} \text{ with } \theta = \frac{r\left(k-\frac{3}{2}\right)}{kr + \frac{3}{2}(2-r)} \\ &\le C'_{k,r}|t-s|^{\alpha\theta} = C'_{k,r}|t-s|^{\frac{3-2r}{2}\frac{\left(k-\frac{3}{2}\right)}{kr+\frac{3}{2}(2-r)}} = C'_{k,r}|t-s|^{\frac{3-2r}{2}\frac{\left(k-\frac{3}{2}\right)}{\left(k-\frac{3}{2}\right)r+3}}. \end{aligned} \tag{12.28}$$

Then, for any $\gamma < 1/2$ we can find $r \in (1, 3/2)$ and $k \in \mathbb{N}$ s.t. $\gamma = \alpha\theta$ so that

$$\|u(t) - u(s)\|_{L^\infty(\Omega)} \le C_\gamma |t-s|^\gamma \text{ for almost any pair } (t,s) \text{ in } (t_0, T). \tag{12.29}$$

Notice that (12.27) and (12.29), together imply that (12.29) must be true for all pairs $(t, s)$ in $(t_0, T)$ and on $\Omega$. Hence we have proved that $u \in C_t^{0,\gamma}([t_0, T), C_x^0(\Omega))$. In fact, this extends to an element of $C_t^{0,\gamma}([t_0, T], C_x^0(\overline{\Omega}))$, and by the continuity $u \in C^0([0, T], L_w^2(\mathbb{R}^3))$ we conclude that the extension in $C_t^{0,\gamma}([t_0, T], C_x^0(\overline{\Omega}))$ is exactly $u$. With this the proof of Theorem 12.1 is completed except for (12.26).

To prove (12.26) notice first of all that a weak solution satisfies

$$u \in L^s((0, T), L^r(\mathbb{R}^3)) \text{ for all } \frac{2}{s} + \frac{3}{r} = 3/2. \tag{12.30}$$

Indeed the case $(s, r) = (\infty, 2)$ follows from $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3))$ and case $(s, r) = (2, 6)$ from $\nabla u \in L^2(\mathbb{R}_+, L^2(\mathbb{R}^3))$ and Sobolev's embedding. The intermediate cases are obtained by Hölder inequality. Next, by Hölder inequality we get

$$\|u \cdot \nabla u\|_r \le \|\nabla u\|_2 \|u\|_{\frac{2r}{2-r}},$$

where $\frac{2r}{2-r} \le 6 \Leftrightarrow r \le 3/2$. Now, since the pair $\left(\frac{2r}{3r-3}, \frac{2r}{2-r}\right)$ satisfies the condition in (12.30), we obtain

$$\|u \cdot \nabla u\|_{L^{\frac{2r}{4r-3}}((0,T), L^r} \le \|\nabla u\|_{L^2(\mathbb{R}_+, L^2)} \|u\|_{L^{\frac{2r}{3r-3}}((0,T), L^{\frac{2r}{2-r}})}.$$

The same is true for $\mathbb{P}(u \cdot \nabla u)$ and for $\nabla p = (1 - \mathbb{P})(u \cdot \nabla u)$, proving (12.26). This proves $u \in C_t^{0,\gamma}([t_0, T], C_x^0(\overline{\Omega}))$ for any $\gamma \in (0, 1/2)$ and any $t_0 \in (0, T)$ and any open $\Omega \subset \overline{\Omega} \subset\subset U$. $\square$

*Remark* 12.10. Notice that it is easy to prove

$$\|\nabla f\|_{L^\infty(\Omega)} \le C \|f\|_{L^r(\Omega)}^\theta \|f\|_{H^k(\Omega)}^{1-\theta}$$

for appropriate $\theta$ and in fact more generally

$$\|\nabla^l f\|_{L^\infty(\Omega)} \le C \|f\|_{L^r(\Omega)}^{\theta_l} \|f\|_{H^k(\Omega)}^{1-\theta_l}$$

for appropriate $\theta_l$ with $l \le L$ and $k - L$ sufficiently large. Then, one can repeat the argument and prove $u \in C_t^{0,\gamma}([t_0, T], C_x^{L-2}(\overline{\Omega}))$ for $L$ arbitrary and appropriate $\gamma \in (0, 1)$ and for any $t_0 \in (0, T)$ and any open $\Omega \subset \overline{\Omega} \subset\subset U$. This yields the result stated in Remark **??**.

# 13 Local energy inequality

We will later need *suitable* weak solutions.

**Proposition 13.1** (Global suitable weak solutions)**.** *Consider $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and a Leray–Hopf solution $u$ proved to exists in Sect. 6. Then $u$ satisfies the following Local Energy Inequality:*

$$2\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \varphi dxds \leq \int_0^T \int_{\mathbb{R}^3} |u|^2(\varphi_t + \triangle\varphi)dxds \tag{13.1}$$
$$+ \int_0^T \int_{\mathbb{R}^3} (|u|^2 + 2p)(u \cdot \nabla)\varphi dxds \text{ for all } \varphi \in C_c^\infty((0,T) \times \mathbb{R}^3, [0,+\infty)),$$

*where $p$ is defined by* (11.16)*.*

*Proof.* Consider the sequence

$$\begin{cases} (\partial_t - \triangle)u_n + \rho_{\epsilon_n} * u_n \cdot \nabla u_n = -\nabla R_i R_j(\rho_{\epsilon_n} * u_n^i u_n^j) \\ u_n(0,x) = u_0(x). \end{cases}$$

We apply to the above equation $\langle \cdot, \varphi u_n \rangle$. Then, for $p_n := R_i R_j(\rho_{\epsilon_n} * u_n^i u_n^j)$,

$$\frac{1}{2}\frac{d}{dt} \langle u_n, \varphi u_n \rangle - \frac{1}{2} \left\langle |u_n|^2, \partial_t \varphi \right\rangle - \langle \triangle u_n, \varphi u_n \rangle + \langle \rho_{\epsilon_n} * u_n \cdot \nabla u_n, \varphi u_n \rangle = - \langle \nabla p_n, \varphi u_n \rangle .$$

We have

$$\begin{aligned} - \langle \triangle u_n, \varphi u_n \rangle &= \left\langle |\nabla u_n|^2, \varphi \right\rangle + \langle \partial_j u_n, u_n \partial_j \varphi \rangle = \left\langle |\nabla u_n|^2, \varphi \right\rangle + 2^{-1} \left\langle \partial_j |u_n|^2, \partial_j \varphi \right\rangle \\ &= \left\langle |\nabla u_n|^2, \varphi \right\rangle - 2^{-1} \left\langle |u_n|^2, \triangle \varphi \right\rangle , \end{aligned}$$

$$\langle \nabla p_n, \varphi u_n \rangle = \left\langle \partial_j p_n, \varphi u_n^j \right\rangle = - \langle p_n, (u_n \cdot \nabla)\varphi \rangle$$

and

$$\langle \rho_{\epsilon_n} * u_n \cdot \nabla u_n, \varphi u_n \rangle = 2^{-1} \left\langle \rho_{\epsilon_n} * u_n \cdot \nabla |u_n|^2, \varphi \right\rangle = -2^{-1} \left\langle |u_n|^2, \rho_{\epsilon_n} * u_n \cdot \nabla \varphi \right\rangle$$

So, integrating, we obtain

$$\begin{aligned} 2\int_0^T \left\langle |\nabla u_n|^2, \varphi \right\rangle dt &= 2\int_0^T \left\langle |u_n|^2, \partial_t \varphi + \triangle \varphi \right\rangle dt + \int_0^T \left\langle |u_n|^2 + 2p_n, u_n \cdot \nabla \varphi \right\rangle dt \\ &+ \int_0^T \left\langle |u_n|^2, (\rho_{\epsilon_n} * u_n - u_n) \cdot \nabla \varphi \right\rangle dt \end{aligned}$$

which, up to the term the last line, is formula (13.1) for the solutions of the truncated problems. So now we will take the limit for $n \nearrow \infty$ in this equality.

We have

$$\int_0^T \left\langle |u_n|^2, \partial_t \varphi + \triangle \varphi \right\rangle dt \xrightarrow{n\to\infty} \int_{\mathbb{R}_+} \left\langle |u|^2, \partial_t \varphi + \triangle \varphi \right\rangle dt$$

because $u_n \xrightarrow{n\to\infty} u$ in $L^2((0,T)\times K,\mathbb{R}^3)$ for any compact set $K\subset\subset\mathbb{R}^3$ and any $T>0$. We have

$$\int_0^T \left\langle |\nabla u|^2,\varphi\right\rangle dt \le \liminf_{n\to\infty}\int_0^T \left\langle |\nabla u_n|^2,\varphi\right\rangle dt$$

by $\nabla u_n \overset{n\to\infty}{\rightharpoonup} \nabla u$ in $L^2(\mathbb{R}_+\times\mathbb{R}^3)$, which in turn implies $\nabla u_n\sqrt{\varphi} \overset{n\to\infty}{\rightharpoonup} \nabla u\sqrt{\varphi}$ in $L^2(\mathbb{R}_+\times\mathbb{R}^3)$, and by Fathou's Lemma. Next, we claim

$$\lim_{n\to+\infty}\int_0^T \left(\left\langle |u_n|^2,(u_n\cdot\nabla)\varphi\right\rangle dt - \left\langle |u|^2,(u\cdot\nabla)\varphi\right\rangle\right) dt = 0. \tag{13.2}$$

Indeed, the difference of the two terms is a sum of various terms. We bound a typical one:

$$\left|\int_0^T \langle u_n-u,u_n(u_n\cdot\nabla)\varphi\rangle\, dt\right| \lesssim \|u_n-u\|_{L^2_tL^4_x(\Omega)}\|u_n\|_{L^\infty_tL^2_x(\Omega)}\|u_n\|_{L^2_tL^4_x(\Omega)} \le C_\Omega\|u_n-u\|_{L^2_tL^4_x(\Omega)}$$

for $\Omega$ =supp$\varphi$ and where $\|u_n\|_{L^\infty_tL^2_x(\Omega)}\|u_n\|_{L^2_tL^4_x(\Omega)} \le C_\Omega$ by the energy equality (4.5), satisfied by the $u_n$. By (6.32) we have $\|u_n-u\|_{L^2_tL^4_x(\Omega)} \xrightarrow{n\to\infty} 0$ and so, treating analogously the other similar terms, we get the desired limit (13.2) Similarly, for the pressure we have

$$\lim_{n\to+\infty}\int_0^T \left(\langle p_n,(u_n\cdot\nabla)\varphi\rangle\, dt - \langle p,(u\cdot\nabla)\varphi\rangle\right) dt = 0. \tag{13.3}$$

Next we show

$$\lim_{n\to+\infty}\int_0^T \left\langle |u_n|^2,(\rho_{\epsilon_n}*u_n-u_n)\cdot\nabla\varphi\right\rangle dt = 0.$$

Like above, we have

$$\begin{aligned}&\left|\int_0^T \left\langle |u_n|^2,(\rho_{\epsilon_n}*u_n-u_n)\cdot\nabla\varphi\right\rangle dt\right| \lesssim \|\rho_{\epsilon_n}*u_n-u_n\|_{L^2_tL^4_x(\Omega)}\|u_n\|_{L^\infty_tL^2_x(\Omega)}\|u_n\|_{L^2_tL^4_x(\Omega)}\\ &\lesssim \|\rho_{\epsilon_n}*u_n-u_n\|_{L^2_tL^4_x(\Omega)} \le \|\rho_{\epsilon_n}*u-u\|_{L^2_tL^4_x(\Omega)} + \|\left(\rho_{\epsilon_n}*-\mathrm{id}\right)(u-u_n)\|_{L^2_tL^4_x(\Omega)}\\ &\lesssim \|\rho_{\epsilon_n}*u-u\|_{L^2_tL^4_x(\Omega)} + \|u-u_n\|_{L^2_tL^4_x(\Omega)} \xrightarrow{n\to\infty} 0\end{aligned}$$

□

**Proposition 13.2** (Alternative local energy inequality)**.** *Suppose that u s.t.* $u\in L^\infty((a,b),L^2(U))$ *and* $\nabla u\in L^2((a,b),L^2(U))$ *satisfies also the Local Energy Inequality*

$$\begin{aligned}&2\int_a^b\int_{\mathbb{R}^3}|\nabla u|^2\varphi dxds \le \int_a^b\int_{\mathbb{R}^3}|u|^2(\varphi_s+\triangle\varphi)dxds \\ &+\int_a^b\int_{\mathbb{R}^3}(|u|^2+2p)(u\cdot\nabla)\varphi dxds \textit{ for all } \varphi\in C_c^\infty((a,b)\times U,[0,+\infty)),\end{aligned} \tag{13.4}$$

*where $p$ is defined by (11.16). Then $u$ satisfies for almost all $t \in (a,b)$ also*

$$\int_{\mathbb{R}^3} |u(t)|^2\phi(t)dx + 2\int_a^t\int_{\mathbb{R}^3} |\nabla u|^2\phi dxds \le \int_a^t\int_{\mathbb{R}^3} |u|^2(\phi_s + \triangle\phi)dxds \qquad (13.5)$$
$$+ \int_a^t\int_{\mathbb{R}^3} (|u|^2 + 2p)(u\cdot\nabla)\phi dxds \text{ for all } \phi \in C_c^\infty((a,b)\times U,[0,+\infty)).$$

*Proof.* We start from (13.4) and we consider $\varphi_\epsilon(s,x) = \phi(s,x)\chi\left(\frac{t-s}{\epsilon}\right)$ where $\chi \in C^\infty(\mathbb{R},[0,1])$ satisfies $\chi = 0$ in $\mathbb{R}_-$ and $\chi = 1$ in $[1,\infty)$. Notice that

$$\chi'\left(\frac{t-s}{\epsilon}\right) = 0 \text{ for } s \le t-\epsilon \text{ and } s\ge t \text{ and } \int_{t-\epsilon}^t \epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)ds = -\chi\left(\frac{t-s}{\epsilon}\right)]_{s=t-\epsilon}^t = \chi(1)-\chi(0) = 1.$$

We have

$$\partial_s\varphi_\epsilon(s,x) = \partial_s\phi(s,x)\chi\left(\frac{t-s}{\epsilon}\right) - \phi(s,x)\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right).$$

So when we enter this information in (13.4) with $\varphi = \varphi_\epsilon$, we obtain

$$\int_a^b\int_{\mathbb{R}^3} |u|^2\phi(s)\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)ds + 2\int_a^b\int_{\mathbb{R}^3}|\nabla u|^2\phi\chi\left(\frac{t-s}{\epsilon}\right)dxds \le \int_a^b\int_{\mathbb{R}^3}|u|^2\chi\left(\frac{t-s}{\epsilon}\right)(\phi_s+\triangle\phi)\,dxds$$
$$+\int_a^b\int_{\mathbb{R}^3}(|\nabla u|^2+2p)\chi\left(\frac{t-s}{\epsilon}\right)(u\cdot\nabla)\phi dxds.$$

Taking limit $\epsilon \searrow 0$ we get

$$\lim_{\epsilon\searrow 0}\int_a^b\int_{\mathbb{R}^3}|u|^2\phi(s)\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)dsdx + 2\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\phi dxds \le \int_0^t\int_{\mathbb{R}^3}|u|^2\left(\phi_s+\triangle\phi\right)dxds$$
$$+\int_0^t\int_{\mathbb{R}^3}(|\nabla u|^2+2p)(u\cdot\nabla)\phi dxds$$

where we have applied dominated convergence, leaving aside the most crucial limit. We have

$$\int_a^b\int_{\mathbb{R}^3}|u|^2\phi(s)\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)dsdx = \int_{t-\epsilon}^t ds\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)\int_{\mathbb{R}^3}|u(s,x)|^2\phi(s,x)dx.$$

Now, we have

$$\int_{t-\epsilon}^t ds\epsilon^{-1}\chi'\left(\frac{t-s}{\epsilon}\right)\int_{\mathbb{R}^3}|u(s,x)|^2\phi(s,x)dx \xrightarrow{\epsilon\searrow 0}\int_{\mathbb{R}^3}|u(t,x)|^2\phi(t,x)dx \text{ in } L^p(\mathbb{R})$$

for any $1\le p<\infty$, by $u\in L^\infty((a,b),L^2(U))$, $\phi\in C_c^\infty((a,b)\times U,[0,+\infty))$ and, finally, by Theorem 1.6. Then, there is a sequence $\epsilon_n\searrow 0$, s.t. for a.e. $t$ we have

$$\int_{t-\epsilon}^t ds\epsilon_n^{-1}\chi'\left(\frac{t-s}{\epsilon_n}\right)\int_{\mathbb{R}^3}|u(s,x)|^2\phi(s,x)dx \xrightarrow{n\nearrow\infty}\int_{\mathbb{R}^3}|u(t,x)|^2\phi(t,x)dx \text{ for a.e. } t\in\mathbb{R},$$

see in the proof of Theorem A.19.

□

## 14 A first result of Caffarelli, Kohn and Nirenberg

**Definition 14.1** (Suitable pairs)**.** A pair $(u, p)$ is suitable in $(a, b) \times U$ if:

1. $u \in L^\infty((a, b), L^2(U, \mathbb{R}^3))$ and $\nabla u \in L^2((a, b) \times U)$ and is divergence free, and $p \in L^{3/2}((a, b) \times U, \mathbb{R})$;
2. $-\triangle p = \partial_i \partial_j (u_i u_j)$;
3. $u$ satisfies for all $t \in (a, b)$ the local energy inequality

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(t) dx + 2 \int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \le \int_a^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \triangle \phi) dx ds \tag{14.1}$$
$$+ \int_a^t \int_{\mathbb{R}^3} (u|^2 + 2p)(u \cdot \nabla) \phi dx ds \text{ for all } \phi \in C_c^\infty((a, b) \times U, [0, +\infty)).$$

In this section we will prove the following theorem.

**Theorem 14.2.** *There exists absolute constants $\epsilon_0^* > 0$ and $c_M > 0$ s.t. if $(u, p)$ is a suitable weak solution of the NS with*

$$R^{-2} \int_{Q_R(t_0, x_0)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dt dx < \epsilon_0 \tag{14.2}$$

*for an $R > 0$ and for a $\epsilon_0 \in (0, \epsilon_0^*]$, then $\|u\|_{L^\infty(Q_{R/2}(t_0, x_0))} \le c_M \epsilon_0^{\frac{1}{3}}$.*

Notice that, in view of Theorem 12.1, $u$ would be smooth in $x$ and Hölder continuous in $t$ inside $Q_{R/2}(t_0, x_0)$. The proof of Theorem 14.2 is rather articulated. Before proving it we will discuss a consequence. Notice that Theorem 14.2 says that for an $R > 0$ and for a $\epsilon_0 \in (0, \epsilon_0^*]$

$$R^{-2} \int_{Q_R\left(t_0 + \frac{R^2}{8}, x_0\right)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dt dx < \epsilon_0 \Longrightarrow \|u\|_{L^\infty\left(Q_{R/2}\left(t_0 + \frac{R^2}{8}, x_0\right)\right)} \le c_M \epsilon_0^{\frac{1}{3}}. \tag{14.3}$$

Notice that $Q_{R/2}\left(t_0 + \frac{R^2}{8}, x_0\right) = \left(t_0 + \frac{R^2}{8} - \frac{R^2}{4}, t_0 + \frac{R^2}{8}\right) \times B_{R/2}(x_0)$ is a neighborhood of $(t_0, x_0)$

**Definition 14.3.** A point $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$ is called a regular point of a weak solution $u$ if there exists a neighborhood of $(t, x)$ in $\mathbb{R}_+ \times \mathbb{R}^3$ such that $u \in L^\infty(U, \mathbb{R}^3)$. If not regular, a point $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$ is called singular.

A simple consequence of Theorem 14.2 is the following result.

**Proposition 14.4.** *Given a suitable Leray–Hopf weak solution $u$, then the set of singular points $S$ of $u$ is bounded in in $[0, \infty) \times \mathbb{R}^3$.*

*Proof.* We already know that there exists $T$ s.t. $u \in C^\infty((T,\infty)\times\mathbb{R}^3,\mathbb{R}^3)$. Now we will show that there is an $R_0 > 0$ s.t. $S \subset [0,T]\times B(0,R_0)$. Theorem 14.2 implies that if $(t,x)\in S$, then

$$R^{-2}\int_{Q_R(t+\eta R^2,x)}\left(|u|^3+|p|^{\frac{3}{2}}\right) \geq \epsilon_0 \text{ for all } R>0 \text{ and for } \eta\in(0,1) \text{ such that } t+\eta R^2-R^2>0.$$

From

$$\|u\|_{L^3(Q_R(t+\eta R^2,x))} \leq \|u\|_{L^{\frac{10}{3}}(Q_R(t+\eta R^2,x))}\left|Q_R\left(t+\eta R^2,x\right)\right|^{\frac{1}{30}} = \|u\|_{L^{\frac{10}{3}}(Q_R(t+\eta R^2,x))}\left(\frac{4\pi}{3}\right)^{\frac{1}{30}}2^{-\frac{1}{6}}R^{\frac{1}{6}}$$

and

$$\|p\|_{L^{\frac{3}{2}}(Q_R(t+\eta R^2,x))} \leq \|p\|_{L^{\frac{5}{3}}(Q_R(t+\eta R^2,x))}\left|Q_R\left(t+\eta R^2,x\right)\right|^{\frac{1}{15}} = \|p\|_{L^{\frac{10}{3}}(Q_R(t+\eta R^2,x))}\left(\frac{4\pi}{3}\right)^{\frac{1}{15}}2^{-\frac{1}{3}}R^{\frac{1}{3}},$$

we get

$$\int_{Q_R(t+\eta R^2,x)}\left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \geq C\epsilon_0^{\frac{10}{9}}R^{-\frac{5}{3}} \text{ for all } R>0. \tag{14.4}$$

But we also know that $u \in L^{\frac{10}{3}}((0,T)\times\mathbb{R}^3,\mathbb{R}^3)$ and $p \in L^{\frac{5}{3}}((0,T)\times\mathbb{R}^3,\mathbb{R}^3)$. If $S$ is unbounded, then for any $R$ there is a sequence $(t_n,x_n)$ in $[0,2T]\times\mathbb{R}^3$ and corresponding $\eta_n\in(0,1)$ where $\left(t_n+\eta_n R^2,x_n\right)\cap\left(t_m+\eta_n R^2,x_m\right)=\emptyset$, we have $Q_R\left(t_n+\eta_n R^2,x_n\right)\subset[0,2T]\times\mathbb{R}^3$ for any $n$ and, for any fixed $R>0$ with $t_n - 7\frac{7}{8}R^2 > 0$,

$$\int_{Q_R(t_n+\eta_n R^2,x_n)}\left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \geq C\epsilon_0^{\frac{10}{9}}R^{-\frac{5}{3}}.$$

But then we get a contradiction

$$\infty > \int_{[0,2T]\times\mathbb{R}^3}\left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \geq \sum_n \int_{Q_R(t_n+\eta_n R^2,x_n)}\left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) = +\infty.$$

□

Another corollary is the following.

**Proposition 14.5.** *Given a suitable Leray–Hopf weak solution $u$ and any compact subspace $K \subset\subset \mathbb{R}_+\times\mathbb{R}^3$, then the set of singular points $S$ satisfies* $\dim_B(S\cap K)\leq 5/3$.

*Proof.* Suppose this is false, so that we have that $\dim_B(S\cap K) > 5/3$ in a case, and let $\dim_B(S\cap K) > d > 5/3$. Then, by Lemma 7.11 there is a sequence $\epsilon_j \to 0$ s.t. $M(S\cap K,\epsilon_j)\geq \epsilon_j^{-d}$, where $M(S\cap K,\epsilon_j)$ the largest number of disjoint open balls of radius $\epsilon_j$ with centers at points of $S\cap K$. Now for $\epsilon\in(0,1)$ we have

$$B_\epsilon(t,x) \supset Q^*_{\epsilon/2}(t,x) = Q_{\epsilon/2}(t+\epsilon^2/8,x) = \left(t-\frac{\epsilon^2}{4},t+\frac{\epsilon^2}{4}\right)\times B_{\epsilon/2}(x),$$

indeed, for any $(s,y) \in Q^*_{\epsilon/2}(t,x)$ we have for

$$\sqrt{(t-s)^2+(y-x)^2} < \sqrt{\frac{\epsilon^4}{16}+\frac{\epsilon^2}{4}} \le \sqrt{\frac{\epsilon^4}{16}+\frac{\epsilon^2}{4}} < 2^{-\frac{1}{2}}\epsilon.$$

For any $j$, fix $M_j := M(S\cap K, \epsilon_j)$ open balls of radius $\epsilon_j$ with centers at points of $S\cap K$. Then, we get a contradiction:

$$\infty > \int_{[0,2T]\times\mathbb{R}^3} \left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \ge \sum_{l=1}^{M_j} \int_{Q_{\epsilon_j/2}(t_l+\epsilon^2/8,x_l)} \left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \ge c\epsilon_j^{-d}\epsilon_j^{\frac{5}{3}} \xrightarrow{j\nearrow\infty} \infty.$$

□

We now turn to the proof of Theorem 14.2. Following [14] we proceed by outlining twice the argument, with increasing precision, before giving a full proof in the third try. First of notice that, by scaling invariance of the NS and of the estimate (14.6), it is enough to take $R=1$. Furthermore, we can take $t_0=0$ and $x_0=0$.

### 14.1 First outline

We oversimplify and we assume that there is no pressure in the local energy inequality (13.5), so that the latter is for $s\in(a,b)$ of the form

$$\int_{\mathbb{R}^3} |u(s)|^2\phi(s)dx + 2\int_a^s\int_{\mathbb{R}^3} |\nabla u|^2\phi \le \int_a^s\int_{\mathbb{R}^3} |u|^2(\partial_t+\triangle)\phi \qquad (14.5)$$
$$+\int_a^s\int_{\mathbb{R}^3} |u|^2(u\cdot\nabla)\phi \text{ for all } \phi\in C_c^\infty((a,b)\times U,[0,+\infty)).$$

Then using (14.5) it is possible to prove rigorously the following.

**Proposition 14.6.** *There exists absolute constants $\epsilon_0^*>0$ s.t. if $u$ satisfies* (14.5) *and*

$$R^{-2}\int_{Q_R(t_0,x_0)} |u|^3 dtdx < \epsilon_0 \qquad (14.6)$$

*for an $R>0$ and for a $\epsilon_0\in(0,\epsilon_0^*]$, then $\|u\|_{L^\infty(Q_{R/2}(t_0,x_0))} \le \epsilon_0^{\frac{2}{9}}$.*

First we give a heuristic argument picking $R=1$ and $(t_0,x_0)=(0,0)$. We will prove that for any $(s,a)\in Q_{1/2}(0,0)$ we have

$$2^{5n}\int_{Q_{2^{-n}}(s,a)} |u|^3 dtdx < \epsilon_0^{\frac{2}{3}} \text{ for all } n\in\mathbb{N} \qquad (14.7)$$

Then by Lebesgue's Differentiation Theorem, this will imply $\|u\|_{L^\infty(Q_{R/2}(t_0,x_0))} \le \epsilon_0^{\frac{2}{9}}$ for a.a. $(t_0,x_0)$.

We will consider an appropriate sequence of cutoffs $\phi_n$. They are chosen so that $(\partial_t + \triangle)\phi_n \approx 0$. Here let us assume $(\partial_t + \triangle)\phi_n = 0$. In fact the $\phi_n$'s will be almost fundamental solutions of the backwards heath equation, but not quite. They will satisfy estimates of the form

$$\begin{aligned} &\phi_n \sim 2^n \text{ in } Q_{2^{-n}}(s,a) \text{ and} \\ &|\nabla\phi_n| \leq \begin{cases} C2^{2n} \text{ in } Q_{2^{-n}}(s,a) \\ C2^{-2n}2^{4k} \text{ in } Q_{2^{-k}}(s,a)\backslash Q_{2^{-(k+1)}}(s,a). \end{cases} \end{aligned} \tag{14.8}$$

We assume by induction that

$$2^{2k}\int_{Q_{2^{-k}}(s,a)} |u|^3 dtdx < \epsilon_0^{\frac{2}{3}} 2^{-3k} \text{ for all } k \leq n. \tag{14.9}$$

Using (14.5) we have for $t \in (s - 2^{-2n}, s)$

$$\begin{aligned} &\int_{B_{2^{-n}}(a)} |u(t)|^2 2^n dx + 2\iint_{Q_{2^{-n}}(s,a)\cap\{t'<t\}} |\nabla u|^2 2^n dxdt' \leq \iint_{Q_{2^{-1}}(s,a)} |u|^3 |\nabla\phi_n| dxdt' \\ &= \sum_{k=1}^{n-1} \iint_{Q_{2^{-k}}(s,a)\backslash Q_{2^{-(k+1)}}(s,a)} |u|^3|\nabla\phi_n|dxdt' + \iint_{Q_{2^{-n}}(s,a)} |u|^3|\nabla\phi_n|dxdt' \end{aligned}$$

where we decomposed the domain of integration on the r.h.s.

$$Q_{2^{-1}}(s,a) = (Q_{2^{-1}}(s,a)\backslash Q_{2^{-2}}(s,a)) \cup (Q_{2^{-2}}(s,a)\backslash Q_{2^{-3}}(s,a)) \cup ... \cup (Q_{2^{-(n-1)}}(s,a)\backslash Q_{2^{-n}}(s,a)) \cup Q_{2^{-n}}(s,a).$$

Now, using (14.8) we obtain for $t \in (s - 2^{-2n}, s)$

$$\begin{aligned} &2^n \int_{B_{2^{-n}}(a)} |u(t)|^2 dx + 2^n 2 \iint_{Q_{2^{-n}}(s,a)\cap\{t'<t\}} |\nabla u|^2 dxdt' \leq \\ &\leq C\sum_{k=1}^{n-1} 2^{-2n}2^{4k} \iint_{Q_{2^{-k}}(s,a)} |u|^3 dxdt' + C2^{2n}\iint_{Q_{2^{-n}}(s,a)} |u|^3 dxdt' \\ &\leq C2^{-2n}\sum_{k=1}^{n-1} 2^{-k}\epsilon_0^{\frac{2}{3}} + C2^{-3n}\epsilon_0^{\frac{2}{3}} = C2^{-2n}\epsilon_0^{\frac{2}{3}}\sum_{k=1}^{n} 2^{-k} < C2^{-2n}\epsilon_0^{\frac{2}{3}}. \end{aligned}$$

From this, for $t \in (s - 2^{-2n-2}, s)$ we get

$$2^{n+1}\int_{B_{2^{-n-1}}(a)} |u(t)|^2 dx + 2^{n+1}2\iint_{Q_{2^{-n-1}}(s,a)} |\nabla u|^2 dxdt < 2^3 C 2^{-2(n+1)}\epsilon_0^{\frac{2}{3}}. \tag{14.10}$$

So far we have shown

$$(14.7) \text{ for } n\text{"} \implies \text{"}(14.10)$$

using heuristically inequality (14.5). Now we show rigorously

$$(14.10) \implies (14.7) \text{ for } n+1,$$

using Sobolev's Embedding and, specifically, the following lemma.

**Lemma 14.7.** *There exists a constant $C_0 > 0$ such that for any $s \in \mathbb{R}$, $r > 0$ and $a \in \mathbb{R}^3$ and any $u$ s.t.*

$$u \in L^\infty\left(\left(s-r^2, s\right), L^2(B_r(a))\right) \text{ and } \nabla u \in L^2\left(Q_r(s,a)\right),$$

*then*

$$r^{-2}\int_{Q_r(s,a)} |u|^3 dx \le C_0 \left[ r^{-1} \sup_{s-r^2<t<s} \int_{B_r(a)} |u(t)|^2 dx + r^{-1} \int_{Q_r(s,a)} |\nabla u|^2 dt dx \right]^{\frac{3}{2}}. \quad (14.11)$$

*Proof.* By scaling, it is sufficient to consider $r = 1$, and by translation invariance we can consider $(s,a) = (0,0)$. By Hölder inequality $\|u\|_{L^3(B_1)} \le \|u\|_{L^6(B_1)}^{\frac{1}{2}} \|u\|_{L^2(B_1)}^{\frac{1}{2}}$ and by Sobolev's inequality $\|u\|_{L^6(B_1)} \le c_0\|u\|_{L^2(B_1)} + c_0\|\nabla u\|_{L^2(B_1)}$. Then $\|u\|_{L^3(B_1)} \le c_0^{\frac{1}{2}} \left( \|u\|_{L^2(B_1)} + \|\nabla u\|_{L^2(B_1)}^{\frac{1}{2}} \|u\|_{L^2(B_1)}^{\frac{1}{2}} \right)$ and so, by $(\alpha+\beta)^q \le 2^{q-1}(\alpha^q + \beta^q)$ for $q \ge 1$ and for $\alpha, \beta \in \mathbb{R}_+$ (this by the convexity of $t \to t^q$)

$$\begin{aligned} \int_{B_1} |u|^3 dx &\le c_0^{\frac{3}{2}} \left( \|u\|_{L^2(B_1)} + \|\nabla u\|_{L^2(B_1)}^{\frac{1}{2}} \|u\|_{L^2(B_1)}^{\frac{1}{2}} \right)^3 \\ &\le 4c_0^{\frac{3}{2}} \left( \|u\|_{L^2(B_1)}^3 + \|\nabla u\|_{L^2(B_1)}^{\frac{3}{2}} \|u\|_{L^2(B_1)}^{\frac{3}{2}} \right). \end{aligned}$$

Then, by Hölder,

$$\begin{aligned} &\int_{Q_1} |u|^3 dx dt \le 4c_0^{\frac{3}{2}} \int_{-1}^0 \|\nabla u\|_{L^2(B_1)}^{\frac{3}{2}} \|u\|_{L^2(B_1)}^{\frac{3}{2}} dt + 4c_0^{\frac{3}{2}} \int_{-1}^0 \|u\|_{L^2(B_1)}^3 dt \\ &\le 4c_0^{\frac{3}{2}} \left\| \|\nabla u\|_{L^2(B_1)}^{\frac{3}{2}} \right\|_{L^{\frac{4}{3}}(-1,0)} \left\| \|u\|_{L^2(B_1)}^{\frac{3}{2}} \right\|_{L^4(-1,0)} + 4c_0^{\frac{3}{2}} \int_{-1}^0 \|u\|_{L^2(B_1)}^3 dt \\ &= 4c_0^{\frac{3}{2}} \left\| \|\nabla u\|_{L^2(B_1)} \right\|_{L^2(-1,0)}^{\frac{3}{2}} \left\| \|u\|_{L^2(B_1)} \right\|_{L^6(-1,0)}^{\frac{3}{2}} + 4c_0^{\frac{3}{2}} \int_{-1}^0 \|u\|_{L^2(B_1)}^3 dt \\ &= 4c_0^{\frac{3}{2}} \|\nabla u\|_{L^2(Q_1)}^{\frac{3}{2}} \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)} \right)^{\frac{3}{2}} + 4c_0^{\frac{3}{2}} \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)} \right)^3 \\ &= 4c_0^{\frac{3}{2}} \|\nabla u\|_{L^2(Q_1)}^{\frac{3}{2}} \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)}^2 \right)^{\frac{3}{4}} + 4c_0^{\frac{3}{2}} \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)}^2 \right)^{\frac{3}{2}} \\ &\le 2c_0^{\frac{3}{2}} \|\nabla u\|_{L^2(Q_1)}^3 + 6c_0^{\frac{3}{2}} \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)}^2 \right)^{\frac{3}{2}} \le 6c_0^{\frac{3}{2}} \left[ \|\nabla u\|_{L^2(Q_1)}^3 + \left( \sup_{-1<t<0} \|u\|_{L^2(B_1)}^2 \right)^{\frac{3}{2}} \right] \\ &\le 6c_0^{\frac{3}{2}} \left[ \|\nabla u\|_{L^2(Q_1)}^2 + \sup_{-1<t<0} \|u\|_{L^2(B_1)}^2 \right]^{\frac{3}{2}}, \end{aligned}$$

where in the last step we use $\alpha^q + \beta^q \le (\alpha+\beta)^q$ for $q \ge 1$ and for $\alpha, \beta \in \mathbb{R}_+$, which follows from $\left(\frac{\alpha}{\alpha+\beta}\right)^q + \left(\frac{\beta}{\alpha+\beta}\right)^q \le \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1$ for $q \ge 1$.

Then we are done, for $C_0 = 6c_0^{\frac{3}{2}}$.

□

Then, applying the lemma, for $\epsilon_0 < (2^{\frac{9}{2}} C_0 C^{\frac{3}{2}})^{-3}$,

$$2^{2n+2} \iint_{Q_{2^{-n-1}}(s,a)} |u|^3 \leq C_0 \left[ 2^3 C 2^{-2(n+1)} \epsilon_0^{\frac{2}{3}} \right]^{\frac{3}{2}} = 2^{\frac{9}{2}} C_0 C^{\frac{3}{2}} 2^{-3(n+1)} \epsilon_0 < 2^{-3(n+1)} \epsilon_0^{\frac{2}{3}} \tag{14.12}$$

where $C$ is a fixed constant, dependent on the $\phi_n$. Notice that (14.12) yields the induction (14.9) with $n$ replaced by $n+1$. Notice that here the nonlinear structure is crucial, specifically the fact that we have taken the 3/2 power of (14.10).

## 14.2 Proof of Proposition 14.6

It is worth, first of all, to see the definition of the cutoffs $\phi_n$, in order to make sense of the bounds in (14.11).

**Lemma 14.8.** *There exists a constant $C_1 > 1$ and for any fixed $(s,a) \in \mathbb{R}^4$ a sequence $\phi_n \in C_c^\infty\left((s - 1/9, s + 2^{-(n+1)}) \times B_{1/3}(a)\right)$ such that for all $n \geq 2$ we have the following facts:*

*(i)* $C_1^{-1} 2^n \leq \phi_n \leq C_1 2^n$ *and* $|\nabla \phi_n| \leq C_1 2^{2n}$ *in* $Q_{2^{-n}}(s,a)$;

*(ii)* $\phi_n \leq C_1 2^{-2n} 2^{3k}$ *and* $|\nabla \phi_n| \leq C_1 2^{-2n} 2^{4k}$ *in* $Q_{2^{-(k-1)}}(s,a) \backslash Q_{2^{-k}}(s,a)$;

*(iii)* *supp*$\phi_n \cap ((-\infty, s] \times \mathbb{R}^3) \subset \overline{Q_{1/3}(s,a)}$;

*(iv)* $|(\partial_t + \triangle)\phi_n| \leq C_1 2^{-2n}$ *in* $(-\infty, s] \times \mathbb{R}^3$.

*Proof.* It is enough to consider $(s,a) = (0,0)$. Then

$$\phi_n(t,x) = 2^{-2n} \theta_n(t,x) = 2^{-2n} \chi_n(t,x) \psi_n(t,x). \tag{14.13}$$

Here we choose $\psi_n$ such that

$$(\partial_t + \triangle)\psi_n(t,x) = 0 \text{ for } t < 2^{-2n} \text{ and with initial value } \psi_n(2^{-2n}, x) = \delta(x). \tag{14.14}$$

Recall that $K_t(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$ satisfies $(\partial_t - \triangle)K_t(x) = 0$ for $t > 0$ and $K_t(x)|_{t=0} = \delta(x)$. Then $K_{-t}(x)$ solves the analogue of problem (14.14) but with the condition $K_{-t}(x)_{t=0} = \delta(x)$. Finally, by translation invariance we find

$$\psi_n(t,x) = K_{2^{-2n}-t}(x) = (4\pi(2^{-2n} - t))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}. \tag{14.15}$$

Notice that the constant factor $(4\pi)^{-\frac{3}{2}}$ is not important in the discussion. We have

$$\psi_n(t,x) = (4\pi(2^{-2n}-t))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}} \le (4\pi(2^{-2n}-t))^{-\frac{3}{2}}$$
$$\le (4\pi 2^{-2n})^{-\frac{3}{2}} = (4\pi)^{-\frac{3}{2}} 2^{3n} \text{ in } Q_{2^{-n}} = (-2^{-2n},0)\times B_{2^{-n}} \tag{14.16}$$
$$\psi_n(t,x) = (4\pi(2^{-2n}+|t|))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}}$$
$$\ge (4\pi(2^{-2n}+2^{-2n}))^{-\frac{3}{2}} e^{-\frac{2^{-2n}}{4(2^{-2n}+2^{-2n})}} = (8\pi)^{-\frac{3}{2}} e^{-\frac{1}{8}} 2^{3n} \text{ in } Q_{2^{-n}}. \tag{14.17}$$

Next,

$$\nabla\psi_n(t,x) = -2^{-1}\pi^{-\frac{3}{2}}(4(2^{-2n}-t))^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}\ x$$

so that

$$|\nabla\psi_n(t,x)| \le 2^{-6}\pi^{-\frac{3}{2}} 2^{5n} 2^{-n} = 2^{-6}\pi^{-\frac{3}{2}} 2^{4n} \text{ in } Q_{2^{-n}}. \tag{14.18}$$

Keeping in mind the factor $2^{-2n}$ in (14.13), (14.16)–(14.18) explain (i). We will see of course the full estimate of $\phi_n$ shortly.
Next, still focusing on $\nabla\psi_n(t,x)$ only, observe that

$$Q_{2^{-(k-1)}}\backslash Q_{2^{-k}} = (-2^{-2(k-1)},-2^{-2k})\times B_{2^{-(k-1)}} \cup [-2^{-2k},0)\times (B_{2^{-(k-1)}}\backslash B_{2^{-k}}).$$

Now, in $(-2^{-2(k-1)},-2^{-2k})\times B_{2^{-(k-1)}}$ we have

$$\psi_n(t,x) = (4\pi(2^{-2n}+|t|))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} \le (4\pi(2^{-2n}+2^{-2k}))^{-\frac{3}{2}}$$
$$\le (4\pi 2^{-2k})^{-\frac{3}{2}} = (4\pi)^{-\frac{3}{2}} 2^{3k}, \tag{14.19}$$

while in $[-2^{-2k},0)\times (B_{2^{-(k-1)}}\backslash B_{2^{-k}})$ we have

$$\psi_n(t,x) = (4\pi(2^{-2n}+|t|))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} \le (4\pi(2^{-2n}+|t|))^{-\frac{3}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}}$$
$$= (4\pi)^{-\frac{3}{2}}\, 2^{3k} \left(\frac{2^{-2k}}{2^{-2n}+|t|}\right)^{\frac{3}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}} \le (4\pi)^{-\frac{3}{2}}\, 2^{3k} \sup_{\alpha\ge 0} \alpha^{\frac{3}{2}} e^{-\frac{\alpha}{4}}. \tag{14.20}$$

Turning to $\nabla\psi_n(t,x)$, in $(-2^{-2(k-1)},-2^{-2k})\times B_{2^{-(k-1)}}$ we have

$$|\nabla\psi_n(t,x)| = 2^{-1}\pi^{-\frac{3}{2}}(4(2^{-2n}+|t|))^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} |x| \le \pi^{-\frac{3}{2}}(8 2^{-2k})^{-\frac{5}{2}} 2^{-k}$$
$$= \pi^{-\frac{3}{2}}(8)^{-\frac{5}{2}} 2^{4k}, \tag{14.21}$$

and in $[-2^{-2k},0)\times (B_{2^{-(k-1)}}\backslash B_{2^{-k}})$ we have

$$|\nabla\psi_n(t,x)| = 2^{-1}\pi^{-\frac{3}{2}}(4(2^{-2n}+|t|))^{-\frac{5}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}} |x| \le 2^{-1}\pi^{-\frac{3}{2}}(4(2^{-2n}+|t|))^{-\frac{5}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}} 2^{-(k-1)}$$
$$= 4^{-\frac{5}{2}}\pi^{-\frac{3}{2}} 2^{4k} \left(\frac{2^{-2k}}{2^{-2n}+|t|}\right)^{\frac{5}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}} \le 4^{-\frac{5}{2}}\pi^{-\frac{3}{2}} 2^{4k} \sup_{\alpha\ge 0} \alpha^{\frac{5}{2}} e^{-\frac{\alpha}{4}}. \tag{14.22}$$

Keeping in mind the factor $2^{-2n}$ in (14.13), (14.20)–(14.22) explain (ii).

If we chose $\chi_n(t,x) \equiv 1$ in (14.13), that is if we chose $\phi_n(t,x) = 2^{-2n}\psi_n(t,x)$, we would then have (i),(ii) and (vi), but obviously we would not have (iii). We define

$$\chi_n(t,x) = X(x)T_n(t) \text{ where}$$
$$X(x) = \begin{cases} 1 \text{ in } B_{1/4} \\ 0 \text{ outside } B_{1/3} \end{cases} \text{ and } T_n(t) = \begin{cases} 1 \text{ for } t \in (-1/16, 0) \\ 0 \text{ for } t < -1/9 \text{ and } t > 2^{-2n-1}, \end{cases} \tag{14.23}$$

with $T_n|_{(-\infty,0]} = T$ independent of $n$, and they are $X \in C_c^\infty(\mathbb{R}^3, [0,1])$ and $T_n \in C_c^\infty(\mathbb{R}, [0,1])$. Now $\text{supp}\chi_n \subseteq \overline{Q_{1/3}} \cup [0, 2^{-2n-1}] \times B_{1/3}$, and so clearly the $\phi_n$ in (14.13) satisfies (iii). Notice now that

$$\nabla\theta_n(t,x) = \psi_n(t,x)\nabla\chi_n(t,x) + \chi_n(t,x)\nabla\psi_n(t,x).$$

Then $|\chi_n\nabla\psi_n| \le |\nabla\psi_n|$ and the previous estimates apply, while $|\psi_n\nabla\chi_n| \le |\psi_n\nabla X| \le c\psi_n$ is smaller. Hence our $\phi_n$ in (14.13) satisfies (i) and (ii).
Finally, we have

$$(\partial_t + \triangle)\phi_n(t,x) = 2^{-2n}\psi_n(t,x)(\partial_t + \triangle)\chi_n(t,x) + 2\ 2^{-2n}\nabla\chi_n(t,x)\cdot\nabla\psi_n(t,x). \tag{14.24}$$

Here is important to observe that $\chi_n = 1$ in $Q_{1/4}$ and $\chi_n = 0$ in $(-\infty, 0] \times \mathbb{R}^3$ outside $Q_{1/3}$. This means that the terms in (14.24) need to be bounded only in $Q_{1/3}\backslash Q_{1/4} \subset Q_{1/2}\backslash Q_{1/4}$, where $\psi_n \le (4\pi)^{-\frac{3}{2}}2^6$, by (14.19), and where $|\nabla\psi_n| \le c_0 2^8$, by (14.20)–(14.21). From $|\nabla\chi_n| \le |\nabla X| \le c$, it follows that the 2nd term in the r.h.s. of (14.24) satisfies the desired estimate. The same is true for the 1st, since

$$|(\partial_t + \triangle)\chi_n(t,x)| \le |T_n'| + |\triangle X| \le |T'| + |\triangle X| \le c_1$$

and so

$$|\psi_n(\partial_t + \triangle)\chi_n| \le c_1|\psi_n|_{Q_{1/2}\backslash Q_{1/4}}| \le c_1(8\pi)^{-\frac{3}{2}}2^6.$$

□

*Proof of Proposition 14.6.* We proceed proving by induction

$A_n'$

$$2^{5n}\int_{Q_{2^{-n}}(s,a)} |u|^3 dtdx < \epsilon_0^{\frac{2}{3}} \text{ for all } n \ge 4 \tag{14.25}$$

$B_n$

$$2^n \sup_{s-2^{-2n}<t<s} \int_{B_{2^{-n}}(a)} |u(t)|^2 dx + 2^n \iint_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \le C_B 2^{-2n}\epsilon_0^{\frac{2}{3}}. \tag{14.26}$$

We already saw how $B_n \Longrightarrow A'_n$ by Lemma 14.7. Now we prove $A'_4$ and how $A'_4, ..., A'_n \Longrightarrow B_{n+1}$.
We start with $A'_n$ for $n \leq 4$. We have the following, which uses the hypothesis (14.6) for $R = 1$ and $(t_0, x_0) = (0, 0)$ and which proves $A'_n$ for $n \leq 4$: for any $(s, a) \in Q_{1/2}(0, 0)$

$$2^{5\cdot n} \int_{Q_{2^{-n}}(s,a)} |u|^3 dtdx \leq 2^{20} \int_{Q_1(0,0)} |u|^3 dtdx \leq 2^{20}\epsilon_0 < \epsilon_0^{\frac{2}{3}} \text{ for } 2^{20}\epsilon_0^{\frac{1}{3}} < 1, \text{ that is for } \epsilon_0 < 2^{-15}.$$

Now we show that $A'_1, ..., A'_n \Longrightarrow B_{n+1}$. We consider, for $t \leq 0$,

$$\int_{B_1} |u(t)|^2 \phi_n(t) + 2\int_{-1}^{t} \int_{B_1} |\nabla u|^2 \phi_n \leq \int_{-1/9}^{t} \int_{B_{1/3}} |u|^2 (\partial_t + \triangle)\phi_n$$
$$+ \int_{-1/9}^{t} \int_{B_{1/3}} |u|^2 (u \cdot \nabla)\phi_n,$$

where we exploited that supp$\phi_n(x) \subset$supp$X \subset \overline{B}_{1/3}$ and supp$\phi_n(t) \subset$supp$T_n \subset (-1/9, 2^{-2n-1})$.
Let us focus now on one term of the l.h.s. at a time.
For $s - 2^{-2n} < t < s$ and restricting to $a = 0$, we have

$$C_1^{-1} 2^n \int_{B_{2^{-n}}} |u(t)|^2 \leq \int_{B_{2^{-n}}} |u(t)|^2 \phi_n(t) \leq \int_{-1/9}^{t} \int_{B_{1/3}} |u|^2 (\partial_t + \triangle)\phi_n + \int_{-1/9}^{t} \int_{B_{1/3}} |u|^2 (u \cdot \nabla)\phi_n$$
$$\leq C_1 2^{-2n} \int_{-1/9}^{s} \int_{B_{1/3}} |u|^2 + \int_{-1/9}^{s} \int_{B_{1/3}} |u|^3 |\nabla \phi_n|.$$

and similarly

$$2C_1^{-1} 2^n \int_{s-2^{-2n}}^{t} \int_{B_{2^{-n}}} |\nabla u|^2 \leq 2\int_{-1/9}^{t} \int_{B_1} |\nabla u|^2 \phi_n \leq C_1 2^{-2n} \int_{-1/9}^{s} \int_{B_{1/3}} |u|^2 + \int_{-1/9}^{s} \int_{B_{1/3}} |u|^3 |\nabla \phi_n|,$$

so that

$$C_1^{-1} 2^n \sup_{s-2^{-2n}<t<s} \int_{B_{2^{-n}}} |u(t)|^2 + C_1^{-1} 2^n \int_{s-2^{-2n}}^{t} \int_{B_{2^{-n}}} |\nabla u|^2$$
$$\leq \frac{3}{2} C_1 2^{-2n} \int_{-1/9}^{s} \int_{B_{1/3}} |u|^2 + \frac{3}{2} \int_{-1/9}^{s} \int_{B_{1/3}} |u|^3 |\nabla \phi_n|. \tag{14.27}$$

Now we examine the 1st term in the r.h.s. of (14.27), for which by $s \leq 0$ we have

$$\frac{3}{2} C_1 2^{-2n} \int_{-1/9}^{s} \int_{B_{1/3}} |u|^2 = \frac{3}{2} C_1 2^{-2n} \iint_{Q_{1/3}(s,0)} |u|^2$$
$$\leq \frac{3}{2} C_1 2^{-2n} |Q_{1/3}|^{1/3} \|u\|^2_{L^3(Q_{1/3}(s,0))} \leq \frac{3}{2} C_1 2^{-2n} |Q_{1/3}|^{1/3} \|u\|^2_{L^3(Q_1(0,0))} \tag{14.28}$$
$$< C_1 2^{-2n} \frac{3}{2} \left(3^{-5} \frac{4\pi}{3}\right)^{1/3} \epsilon_0^{2/3} = C_1 2^{-2n} \frac{3}{2 \cdot 3^2} (4\pi)^{1/3} \epsilon_0^{2/3} < C_1 2^{-2n} \epsilon_0^{2/3} < C_1 2^{-2n} \epsilon_0^{2/3},$$

where we used $\|u\|_{L^3(Q_1(0,0))} < \epsilon_0^{1/3}$ from hypothesis (14.6) for $R = 1$ and $(t_0, x_0) = (0, 0)$ and $\sqrt[3]{4\pi}/6 < 1$.
We consider now the 2nd term in the r.h.s. of (14.27). We have, by $s \le 0$

$$\begin{aligned}
&\int_{-1/9}^{s} \int_{B_{1/3}} |u|^3 |\nabla \phi_n| \le \iint_{Q_{1/3}(s,0)} |u|^3 |\nabla \phi_n| \le \iint_{Q_{1/2}(s,0)} |u|^3 |\nabla \phi_n| \\
&= \sum_{k=2}^{n} \iint_{Q_{2^{-(k-1)}}(s,0) \setminus Q_{2^{-k}}(s,0)} |u|^3 |\nabla \phi_n| + \iint_{Q_{2^{-n}}(s,0)} |u|^3 |\nabla \phi_n| \\
&\le \sum_{k=2}^{n} C_1 2^{-2n} 2^{4k} \iint_{Q_{2^{-(k-1)}(s,0)} \setminus Q_{2^{-k}}(s,0)} |u|^3 + C_1 2^{2n} \iint_{Q_{2^{-n}}(s,0)} |u|^3 \\
&\le \sum_{k=2}^{n} C_1 2^{-2n} 2^{4k} \iint_{Q_{2^{-(k-1)}(s,0)}} |u|^3 + C_1 2^{2n} \iint_{Q_{2^{-n}}(s,0)} |u|^3 = \sum_{k=1}^{n} C_1 2^{-2n} 2^{4k} \iint_{Q_{2^{-k}}(s,0)} |u|^3 \\
&\le C_1 2^{-2n} \epsilon_0^{\frac{2}{3}} \sum_{k=1}^{n} 2^{-k} = C_1 2^{-2n} \epsilon_0^{\frac{2}{3}} 2^{-1} \sum_{k=0}^{n-1} 2^{-k} < C_1 2^{-2n} \epsilon_0^{\frac{2}{3}} 2^{-1} 2 = C_1 2^{-2n} \epsilon_0^{\frac{2}{3}}.
\end{aligned}$$

So, returning to (14.27), we have proved

$$2^n \sup_{s-2^{-2n} < t < s} \int_{B_{2^{-n}}} |u(t)|^2 + 2^n \int_{-1}^{s} \int_{B_{2^{-n}}} |\nabla u|^2 \le 3 C_1^2 2^{-2n} \epsilon_0^{\frac{2}{3}}.$$

Then

$$2^{n+1} \sup_{s-2^{-2(n+1)} < t < s} \int_{B_{2^{-n-1}}} |u(t)|^2 + 2^{n+1} \iint_{Q_{2^{-n-1}}(s,0)} |\nabla u|^2 \le 2^3 3 C_1^2 2^{-2(n+1)} \epsilon_0^{\frac{2}{3}}$$

and this proves the induction argument for $C_B = 24 C_1^2$.

□

## 14.3 Proof of Theorem 14.2

In the proof of Theorem 14.2, the presence of the pressure complicates the discussion. As before, we normalize to the case $Q_1(0, 0)$. We proceed by induction proving the following:

$A_n$

$$2^{2n} \int_{Q_{2^{-n}}(s,a)} |u|^3 + 2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(s,a)} |p - (p)_{B_{2^{-n}}(a)}|^{\frac{3}{2}} < 2^{-3n} \epsilon_0^{\frac{2}{3}} \text{ for all } n \in \mathbb{N}; \quad (14.29)$$

$B_n$

$$2^n \sup_{s-2^{-2n} < t < s} \int_{B_{2^{-n}}(a)} |u(t)|^2 dx + 2^n \iint_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \le C_B 2^{-2n} \epsilon_0^{\frac{2}{3}}. \quad (14.30)$$

We prove $A_1$, then $A_1, ..., A_n \Longrightarrow B_{n+1}$ and, finally, $B_2, ..., B_n \Longrightarrow A_n$. The first of these two implications is based on the local energy inequality (14.1), while the last of the two implications follows essentially from Sobolev's Embedding and like in Lemma (14.7), exactly like in the proof of Proposition 14.6, and estimate on the pressure, see Lemma 14.9 below, which essentially bounds $p$ in terms of $|u|^2$.

*Step 1: proof of $A_n$ for $n \le 4$.* We use, for $(p)_{B_r(a)}$ =average of $p$ in $B_r(a) = |B_r(a)|^{-1} \int_{B_r(a)} p$,

$$\begin{aligned}&\int_{Q_r(s,a)} |(p)_{B_r(a)}|^q \le \int_{Q_r(s,a)} (|p|)^q_{B_r(a)} = \int_{s-r^2}^s \frac{4\pi}{3} r^3 (|p|)^q_{B_r(a)} dr \\ &= \int_{s-r^2}^s \frac{4\pi}{3} r^3 \left( \frac{1}{\frac{4\pi}{3} r^3} \int_{B_r(a)} |p| \right)^q dr \le \int_{s-r^2}^s dr \frac{4\pi r^3}{3} \frac{1}{\frac{4\pi}{3} r^3} \int_{B_r(a)} |p|^q = \int_{Q_r(s,a)} |p|^q,\end{aligned}$$

where in the 1st inequality we used the obvious fact that $|(p)_{B_r(a)}| \le (|p|)_{B_r(a)}$ and the 2nd inequality follows by $q \ge 1$ and the Jensen inequality. Using $(\alpha+\beta)^q \le 2^{q-1}(\alpha^q + \beta^q)$ for $q \ge 1$ and for $\alpha, \beta \in \mathbb{R}_+$ (this by the convexity of $t \to t^q$), we obtain

$$\begin{aligned}&2^{2n} \int_{Q_{2^{-n}}(s,a)} |u|^3 + 2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(s,a)} |p - (p)_{B_{2^{-1}}(a)}|^{\frac{3}{2}} \\ &\le 2^{2n} \int_{Q_{2^{-1}}(s,a)} |u|^3 + 2^{\frac{3}{2}n + \frac{1}{2}} \int_{Q_{2^{-n}}(s,a)} \left( |p|^{\frac{3}{2}} + |(p)_{B_{2^{-1}}(a)}|^{\frac{3}{2}} \right) \\ &\le 2^{2n} \int_{Q_{2^{-n}}(s,a)} |u|^3 + 2^{\frac{3}{2}n + \frac{3}{2}} \int_{Q_{2^{-n}}(s,a)} |p|^{\frac{3}{2}} \le 2^{2n + \frac{3}{2}} \int_{Q_1(0,0)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) \le 2^{2n + \frac{3}{2}} \epsilon_0 < 2^{-3n} \epsilon_0^{\frac{2}{3}}\end{aligned}$$

for $2^{5n + \frac{3}{2}} \epsilon_0 < 2^{-15}$ for $n \le 4$.

*Step 2: proof of $A_1, ..., A_n \Longrightarrow B_{n+1}$.* We consider, for $t \le 0$,

$$\begin{aligned}&\int_{B_1} |u(t)|^2 \phi_n(t) + 2 \int_{-1}^t \int_{B_1} |\nabla u|^2 \phi_n \le \int_{-1}^t \int_{B_1} |u|^2 (\partial_t + \triangle) \phi_n \\ &+ \int_{-1}^t \int_{B_1} (|u|^2 + 2p)(u \cdot \nabla) \phi_n,\end{aligned}$$

and we conclude

$$\begin{aligned}&C_1^{-1} 2^n \sup_{s - 2^{-2n} < t < s} \int_{B_{2^{-n}}(a)} |u(t)|^2 + C_1^{-1} 2^n \int_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \\ &\le \frac{3}{2} C_1 2^{-2n} \int_{-1/9}^s \int_{B_{1/3}(a)} |u|^2 + 3 \int_{-1/9}^s \int_{B_{1/3}(a)} |u|^3 |\nabla \phi_n| + 2 \int_{-1/9}^s \int_{B_{1/3}(a)} p u \cdot \nabla \phi_n \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

We have already seen, in (14.28), that $I_1 \le C_1 2^{-2n} \epsilon_0^{2/3}$ and, in the inequality after (14.28), that $I_2 \le \frac{3}{2} C_1 2^{-2n} \epsilon_0^{\frac{2}{3}}$.

We now focus on $I_3$. We consider a sequence $\chi_k \in C_c^\infty((-\infty, s] \times \mathbb{R}^3, [0,1])$ for $k = 1, ...., n$, with

$$\chi_k = 1 \text{ in } Q_{\frac{7}{8}2^{-k}}(s,a), \ \mathrm{supp}\chi_k \cap \overline{Q_1(s,a)} \subset \overline{Q_{2^{-k}}(s,a)} \text{ and } |\nabla\chi_k| \leq 2^k 16. \qquad (14.31)$$

It is enough to pick $\chi_k(t,x) = T(2^{2k}(t-s))X(2^k(x-a))$ with $X(y) = 1$ for $|y| \leq \frac{7}{8}$ and $X(y) = 0$ for $|y| \geq 1$ and with $T(l) = 1$ for $|l| \leq \frac{7^2}{8^2}$ and $T(l) = 0$ $|l| \geq 1$.
Now we write

$$\begin{aligned}
I_3 &= 3\int_{-1/9}^{s}\int_{B_{1/3}(a)} pu\cdot\nabla\phi_n = 3\int_{-1/9}^{s}\int_{B_{1/3}(a)} pu\cdot\nabla\left[\phi_n\chi_1\right]\\
&= 3\sum_{k=1}^{n-1}\int_{Q_{1/3}(s,a)} pu\cdot\nabla\left[\phi_n\left(\chi_k-\chi_{k+1}\right)\right] + 3\int_{Q_{1/3}(s,a)} pu\cdot\nabla\left[\phi_n\chi_n\right]\\
&= 3\sum_{k=1}^{n-1}\int_{Q_{1/3}(s,a)}\left(p-(p)_{B_{2^{-k}}(a)}\right)u\cdot\nabla\left[\phi_n\left(\chi_k-\chi_{k+1}\right)\right] + 3\int_{Q_{1/3}(s,a)}\left(p-(p)_{B_{2^{-n}}(a)}\right)u\cdot\nabla\left[\phi_n\chi_n\right]\\
&= 3\sum_{k=1}^{n-1}\int_{Q_{2^{-k}}(s,a)}\left(p-(p)_{B_{2^{-k}}(a)}\right)u\cdot\nabla\left[\phi_n\left(\chi_k-\chi_{k+1}\right)\right] + 3\int_{Q_{2^{-n}}(s,a)}\left(p-(p)_{B_{2^{-n}}(a)}\right)u\cdot\nabla\left[\phi_n\chi_n\right],
\end{aligned}$$

where we used $\mathrm{supp}\chi_k \cap \overline{Q_1(s,a)} \subset \overline{Q_{2^{-k}}(s,a)}$. Then we have

$$|I_3| \leq 3\sum_{k=1}^{n-1}\int_{Q_{2^{-k}}(s,a)}\left|p-(p)_{B_{2^{-k}}(a)}\right||u|\left|\nabla\left[\phi_n\left(\chi_k-\chi_{k+1}\right)\right]\right| + 3\int_{Q_{2^{-n}}(s,a)}\left|p-(p)_{B_{2^{-n}}(a)}\right||u|\left|\nabla\left[\phi_n\chi_n\right]\right|.$$

Now we use the bounds

$$\begin{aligned}
&|(\chi_k-\chi_{k+1})\nabla\phi_n)| \leq (\chi_{Q_{2^{-k}}(s,a)\setminus Q_{2^{-k-1}}(s,a)} + \chi_{Q_{2^{-k-1}}(s,a)\setminus Q_{2^{-k-2}}(s,a)})\,|\nabla\phi_n|\\
&\leq C_1 2^{-2n}2^{4(k+1)} + C_1 2^{-2n}2^{4(k+2)}, \text{ from (ii) Lemma 14.8,}
\end{aligned}$$

where we used the fact that $\chi_k - \chi_{k+1} = 0$ in $Q_{\frac{7}{8}2^{-k-1}}(s,a)$ and outside $Q_{2^{-k}}(s,a)$ (in the region $\{t < s\}$),

$$\chi_n\,|\nabla\phi_n| \leq \chi_{Q_{2^{-n}}(s,a)}\,|\nabla\phi_n| \leq C_1 2^{2n}, \text{ from (i) Lemma 14.8,}$$

$$\begin{aligned}
&|\phi_n\left(\nabla\chi_k - \nabla\chi_{k+1}\right))| \leq \phi_n\left(16\ 2^k\chi_{Q_{2^{-k}}(s,a)\setminus Q_{\frac{7}{8}2^{-k}}(s,a)}) + 16\ 2^{k+1}\chi_{Q_{2^{-k-1}}(s,a)\setminus Q_{\frac{7}{8}2^{-k-1}}(s,a)})\right)\\
&\leq \phi_n\left(16\ 2^k\chi_{Q_{2^{-k}}(s,a)\setminus Q_{2^{-k-1}}(s,a)}) + 16\ 2^{k+1}\chi_{Q_{2^{-k-1}}(s,a)\setminus Q_{2^{-k-2}}(s,a)})\right)\\
&\leq 16\ 2^k C_1 2^{-2n}2^{3(k+1)} + 16\ 2^{k+1}C_1 2^{-2n}2^{3(k+2)} \text{ from (ii) Lemma 14.8 and (14.31)}
\end{aligned}$$

and, finally

$$|\phi_n\nabla\chi_n| \leq \phi_n 16\chi_{Q_{2^{-n}}(s,a)\setminus Q_{\frac{7}{8}2^{-n}}(s,a)}) \leq \phi_n 16\chi_{Q_{2^{-n}}(s,a)} \leq 16C_1 2^n \text{ from (i) Lemma 14.8 and (14.31).}$$

Then, for an appropriate $c_0'$, we have

$$|I_3| \le c_0' C_1 \sum_{k=1}^{n-1} 2^{-2n} 2^{4k} \int_{Q_{2^{-k}}(s,a)} \left| p - (p)_{B_{2^{-k}}(a)} \right| |u| + c_0' C_1 2^{2n} \int_{Q_{2^{-n}}(s,a)} \left| p - (p)_{B_{2^{-n}}(a)} \right| |u|$$

$$= c_0' C_1 2^{-2n} \sum_{k=1}^{n} 2^{4k} \int_{Q_{2^{-k}}(s,a)} \left| p - (p)_{B_{2^{-k}}(a)} \right| |u|$$

$$\le c_0' C_1 2^{-2n} \sum_{k=1}^{n} 2^{\frac{7}{3}k} 2^{\frac{2}{3}k} \|u\|_{L^3(Q_{2^{-k}}(s,a))} 2^k \|p - (p)_{B_{2^{-k}}(a)}\|_{L^{\frac{3}{2}}(Q_{2^{-k}}(s,a))}$$

$$\le c_0 C_1 2^{-2n} \sum_{k=1}^{n} 2^{\frac{7}{3}k} \left( 2^{2k} \int_{Q_{2^{-k}}(s,a)} |u|^3 + 2^{\frac{3}{2}k} \int_{Q_{2^{-k}}(s,a)} |p - (p)_{B_{2^{-n}}(a)}|^{\frac{3}{2}} \right)$$

$$\le c_0' C_1 2^{-2n} \sum_{k=1}^{n} 2^{-\frac{2}{3}k} \epsilon_0^{\frac{2}{3}} \le c_0' 2^{-2n} C_1 \epsilon_0^{\frac{2}{3}} \frac{1}{2^{\frac{2}{3}} - 1}.$$

So we have shown that $A_1, ..., A_n$ imply

$$I_1 + I_2 + I_3 \le C_1 (1 + 2 + c_0) 2^{-2n} \epsilon_0^{\frac{2}{3}} \text{ for } c_0 = \frac{c_0'}{2^{\frac{2}{3}} - 1}.$$

Then

$$2^n \sup_{s - 2^{-2n} < t < s} \int_{B_{2^{-n}}(a)} |u(t)|^2 + 2 \quad 2^n \int_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \le (1 + 2 + c_0) C_1^2 2^{-2n} \epsilon_0^{\frac{2}{3}}$$

and so also

$$2^{n+1} \sup_{s - 2^{-2n-2} < t < s} \int_{B_{2^{-n}}(a)} |u(t)|^2 + 2^{n+1} \int_{Q_{2^{-n-1}}(s,a)} |\nabla u|^2$$
$$\le 2 (1 + 2 + c_0) C_1^2 2^{-2(n+1)} \epsilon_0^{\frac{2}{3}}.$$

So, if we set $C_B = 2(1 + 2 + c_0) C_1^2$ we have $A_1, ..., A_n \Longrightarrow B_{n+1}$.

*Proof of* $B_2, ..., B_n \Longrightarrow A_n$.

Recall that we need the bound

$$2^{2n} \int_{Q_{2^{-n}}(s,a)} |u|^3 + 2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(s,a)} |p - (p)_{B_{2^{-n}}(a)}|^{\frac{3}{2}} < 2^{-3n} \epsilon_0^{\frac{2}{3}} \ .$$

The first term in this formula can be bounded using (14.11), that is, using $B_n$

$$2^{2n} \int_{Q_{2^{-n}}(s,a)} |u|^3 \le C_0 \left[ 2^n \sup_{s - 2^{-2n} < t < s} \int_{B_{2^{-n}}(a)} |u(t)|^2 + 2^n \int_{Q_{2^{-n}}(s,a)} |\nabla u|^2 \right]^{\frac{3}{2}}$$

$$\le C_0 \left[ C_B 2^{-2n} \epsilon_0^{\frac{2}{3}} \right]^{\frac{3}{2}} = \frac{1}{4} 4 C_0 C_B^{\frac{3}{2}} 2^{-3n} \epsilon_0 < \frac{1}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}} \text{ for } \epsilon_0 < 4^{-3} C_0^{-3} C_B^{-\frac{9}{2}}.$$

To finish, we will prove

$$2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(s,a)} |p - (p)_{B_{2^{-n}}(a)}|^{\frac{3}{2}} < \frac{3}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}}. \tag{14.32}$$

To this effect we will use the formula, valid for $0 < r \leq \rho/2$,

$$\begin{aligned} r^{-\frac{3}{2}} \int_{Q_r(s,a)} |p - (p)_{B_r(a)}|^{\frac{3}{2}} &< C_2 r^{-\frac{3}{2}} \int_{Q_{2r}(s,a)} |u|^3 \\ &+ C_2 r^5 \left[ \sup_{s-r^2<t<s} \int_{2r<|y-a|<\rho} \frac{|u(t)|^2}{|y-a|^4} \right]^{\frac{3}{2}} \\ &+ C_2 \frac{r^3}{\rho^{\frac{9}{2}}} \int_{Q_\rho(s,a)} \left( |u|^3 + |p|^{\frac{3}{2}} \right). \end{aligned} \tag{14.33}$$

We apply this formula for $r = 2^{-n}$ and $\rho = 1/2$, to get

$$\begin{aligned} 2^{\frac{3}{2}n} \int_{Q_{2^{-n}}(s,a)} |p - (p)_{B_{2^{-n}}(a)}|^{\frac{3}{2}} &< C_2 2^{\frac{3}{2}n} \int_{Q_{2^{-(n-1)}}(s,a)} |u|^3 \\ &+ C_2 2^{-5n} \left[ \sup_{s-2^{-2n}<t<s} \int_{2^{-(n-1)}<|y-a|<1/2} \frac{|u(t)|^2}{|y-a|^4} \right]^{\frac{3}{2}} \\ &+ C_2 2^{\frac{9}{2}} 2^{-3n} \int_{Q_{1/2}(s,a)} \left( |u|^3 + |p|^{\frac{3}{2}} \right). \end{aligned} \tag{14.34}$$

Then we estimate the three terms on the r.h.s. of (14.34).
For the first, we have, using inequality (14.11),

$$C_2 2^{\frac{3}{2}n} \int_{Q_{2^{-(n-1)}}(s,a)} |u|^3 = 4C_2 2^{-\frac{1}{2}n} 2^{2(n-1)} \int_{Q_{2^{-(n-1)}}(s,a)} |u|^3 \tag{14.35}$$

$$\leq 4C_2C_0 2^{-\frac{1}{2}n} \left[ 2^{n-1} \sup_{s-2^{-2(n-1)}<t<s} \int_{B_{2^{-(n-1)}}(a)} |u(t)|^2 dx + 2^{n-1} \int_{Q_{2^{-(n-1)}}(s,a)} |\nabla u|^2 \right]^{\frac{3}{2}}$$

$$\leq 4C_2C_0 2^{-\frac{1}{2}n} \left[ C_B 2^{-2(n-1)} \epsilon_0^{\frac{2}{3}} \right]^{\frac{3}{2}} = \frac{1}{4} 16 C_2 C_0 C_B^{\frac{3}{2}} 2^{-\frac{1}{2}n} 2^{-3(n-1)} \epsilon_0 \leq \left( 32 C_2 C_0 C_B^{\frac{3}{2}} \epsilon_0^{\frac{1}{3}} \right) \frac{1}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}} < \frac{1}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}}.$$

The last term in (14.34) is bounded using $\int_{Q_1(0,0)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) < \epsilon_0$, which yields

$$C_2 2^{\frac{9}{2}} 2^{-3n} \int_{Q_{1/2}(s,a)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) \leq \left( 4 C_2 2^{\frac{9}{2}} 2^{-3n} \epsilon_0^{\frac{1}{3}} \right) \frac{1}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}} < \frac{1}{4} 2^{-3n} \epsilon_0^{\frac{2}{3}}. \tag{14.36}$$

We consider now the middle term in (14.34). We have

$$\sup_{s-2^{-2n}<t<s}\int_{2^{-(n-1)}<|y-a|<1/2}\frac{|u(t)|^2}{|y-a|^4}=\sup_{s-2^{-2n}<t<s}\sum_{k=2}^{n-1}\int_{2^{-k}<|y-a|<2^{-(k-1)}}\frac{|u(t)|^2}{|y-a|^4}$$
$$\le 2^4\sum_{k=2}^{n-1}2^{4(k-1)}\sup_{s-2^{-2(k-1)}<t<s}\int_{B_{2^{-(k-1)}}(a)}|u(t)|^2=2^4\sum_{k=1}^{n-2}2^{4k}\sup_{s-2^{-2k}<t<s}\int_{B_{2^{-k}}(a)}|u(t)|^2$$
$$\le 2^4C_B\epsilon_0^{\frac23}\sum_{k=1}^{n-2}2^k\le 2^4C_B\epsilon_0^{\frac23}2^n.$$

Then

$$C_2 2^{-5n}\left[\sup_{s-2^{-2n}<t<s}\int_{2^{-(n-1)}<|y-a|<1/2}\frac{|u(t)|^2}{|y-a|^4}\right]^{\frac32}$$
$$\le C_2 2^{-5n}\left[2^4C_B\epsilon_0^{\frac23}2^n\right]^{\frac32}<\frac14\left(2^6C_2C_B^{\frac32}\epsilon_0^{\frac13}\right)2^{-2n}\epsilon_0^{\frac23}<\frac14 2^{-2n}\epsilon_0^{\frac23}. \tag{14.37}$$

So, summing up (14.35)–(14.37), we get (14.32), and this ends the proof of $B_2,...,B_n \implies A_n$. □

We will prove now formula (14.33).

**Lemma 14.9.** *There exists $C_2$ such that for $p\in L^{\frac32}(Q_\rho)$ and $u\in L^3(Q_\rho)\cap L^\infty((-\rho^2,0),L^2(B_\rho))$ and for $-\triangle p=\partial_i\partial_j(u_iu_j)$ in $Q_\rho$, then for any $0<r<\rho/2$ we have*

$$r^{-\frac32}\int_{Q_r}|p-(p)_{B_r}|^{\frac32}<C_2r^{-\frac32}\int_{Q_{2r}}|u|^3$$
$$+C_2r^5\left[\sup_{-r^2<t<0}\int_{2r<|y|<\rho}\frac{|u(t,y)|^2}{|y|^4}\right]^{\frac32} \tag{14.38}$$
$$+C_2\frac{r^3}{\rho^{\frac92}}\int_{Q_\rho}\left(|u|^3+|p|^{\frac32}\right).$$

*Proof.* We will start by assuming $u\in L^\infty((-\rho^2,0),C^N(B_\rho))$ with $N\gg 1$. This in turn implies that $p(t)\in L^\infty((-\rho^2,0)),C^k(B_{\rho'}))$ for a large $k<N$ and for $\rho'<\rho$: this is analogous to Lemma 11.5 valid for the pair $(u,\omega)$.

Let now $\phi\in\mathbb{C}_c^\infty(\mathbb{R}^3,[0,1])$ with

$$\phi(x)=\begin{cases}1 \text{ in } B_{3\rho/4}\\ 0 \text{ outside } B_\rho\end{cases}$$

with $|\nabla\phi| \le c\rho^{-1}$ and $|\partial_i\partial_j\phi| \le c\rho^{-2}$. Then, like in Lemma 11.2, we have

$$\phi p = (-\triangle)^{-1}(-\triangle)\phi p = \frac{1}{4\pi|x|} * ((-\triangle)\phi p) = \frac{1}{4\pi|x|} * (-\phi\triangle p - p\triangle\phi - 2\nabla\phi\cdot\nabla p)$$
$$= \frac{1}{4\pi|x|} * (\phi\partial_i\partial_j(u_iu_j) - p\triangle\phi - 2\nabla\phi\cdot\nabla p)$$
$$= \frac{1}{4\pi|x|} * (\partial_i\partial_j(\phi u_iu_j) - \partial_i\left(u_iu_j\partial_j\phi\right) - \partial_j(u_iu_j\partial_i\phi) + u_iu_j\partial_j\partial_i\phi - p\triangle\phi - 2\nabla\cdot(p\nabla\phi) + 2p\triangle\phi)$$
$$= \partial_i\frac{1}{4\pi|x|} * (\partial_j(\phi u_iu_j) - 2u_iu_j\partial_i\phi - 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (u_iu_j\partial_j\partial_i\phi + p\triangle\phi)$$
$$= -\frac{x_i}{4\pi|x|^3} * (\partial_j(\phi u_iu_j) - 2u_iu_j\partial_i\phi - 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (u_iu_j\partial_j\partial_i\phi + p\triangle\phi)$$

Now we can apply Lemma 11.6 and conclude that, for some constant $C$,

$$\phi p = -\frac{1}{4\pi}P.V.\left(\frac{\delta_{ij}}{|x|^3} - \frac{3x_ix_j}{|x|^5}\right) * (\phi u_iu_j) + C\phi|u|^2$$
$$+ \frac{x_i}{4\pi|x|^3} * (2u_iu_j\partial_i\phi + 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (u_iu_j\partial_j\partial_i\phi + p\triangle\phi)\,.$$

We have $p = \phi p$ in $Q_{3\rho/4}$. We write

$$p_{11} := -\frac{1}{4\pi}P.V.\left(\left(\frac{\delta_{ij}}{|x|^3} - \frac{3x_ix_j}{|x|^5}\right)\right) * (\chi_{B_{2r}}\phi u_iu_j) + C\phi|u|^2$$
$$p_{12} := -\frac{1}{4\pi}P.V.\left(\left(\frac{\delta_{ij}}{|x|^3} - \frac{3x_ix_j}{|x|^5}\right)\right) * ((1-\chi_{B_{2r}})\,\phi u_iu_j).$$

Then

$$\|p_{11}\|_{L^{\frac{3}{2}}(B_r)} \le C\sum_{i,j}\|\phi u_iu_j\|_{L^{\frac{3}{2}}(B_{2r})} \le C\sum_{i,j}\|u_iu_j\|_{L^{\frac{3}{2}}(B_{2r})} \le C'\|u\|^2_{L^3(B_{2r})}.$$

and so also

$$\|p_{11} - (p_{11})_{B_r}\|_{L^{\frac{3}{2}}(Q_r)} \le 2C'\|u\|^2_{L^3(Q_{2r})}$$

which is equivalent to

$$\int_{Q_r}|p_{11} - (p_{11})_{B_r}|^{\frac{3}{2}} \le (2C')^{\frac{3}{2}}\int_{Q_{2r}}|u|^3.$$

Next, we observe that by mean value there exists $x_0(t) \in B_r$ so that $(p_{12})_{B_r} = p_{21}(x_0(t))$

$$p_{12}(t,x) - (p_{12})_{B_r}(t) = \int_0^1 \nabla p_{12}(t, s(x - x_0(t)) + x_0(t))\cdot(x - x_0(t))ds$$

and so

$$\|p_{12}-(p_{12})_{B_r}\|_{L^{\frac{3}{2}}(Q_r)} \le 2r\int_0^1 ds\|\nabla p_{12}(s(x-x_0(t))+x_0(t))\|_{L^{\frac{3}{2}}(Q_r)} \le 2r|Q_r|^{\frac{2}{3}}\|\nabla p_{12}\|_{L^\infty(Q_r)}$$

$$= 2r\,(4\pi/3)^{\frac{2}{3}}\,r^{\frac{10}{3}}\frac{1}{4\pi}\|\nabla\left(\frac{\delta_{ij}}{|x|^3}-\frac{3x_ix_j}{|x|^5}\right)*((1-\chi_{B_{2r}})\,\phi u_iu_j)\|_{L^\infty(Q_r)}$$

$$\le Cr^{\frac{13}{3}}\left\|\int_{\rho>|y|\ge 2r}\frac{1}{|x-y|^4}|u(t,y)|^2dy\right\|_{L^\infty(Q_r)} \le 2^4Cr^{\frac{13}{3}}\left\|\int_{\rho>|y|\ge 2r}\frac{1}{|y|^4}|u(t,y)|^2dy\right\|_{L^\infty(Q_r)},$$

where we used $|x-y|=|y|\left(1-\dfrac{|x|}{|y|}\right)\ge|y|2^{-1}$.

So we conclude

$$\int_{Q_r}|p_{12}-(p_{12})_{B_r}|^{\frac{3}{2}} \le 2^6C^{\frac{3}{2}}r^{\frac{3}{2}}r^5\left(\sup_{-r^2<t<0}\int_{\rho>|y|\ge 2r}\frac{1}{|y|^4}|u(t,y)|^2dy\right)^{\frac{3}{2}}.$$

Now we set

$$p_2=p_{21}+p_{22}=\frac{x_i}{2\pi|x|^3}*u_iu_j\partial_i\phi+\frac{x_i}{4\pi|x|^3}*p\partial_j\phi.$$

Then, also from $|\nabla\phi|\le c\rho^{-1}$, $\mathrm{supp}|\nabla\phi|\subseteq B_\rho\setminus B_{3\rho/4}$ and, for $x\in B_r$,

$$|x-y|=|y|\left(1-\frac{|x|}{|y|}\right)\ge|y|\left(1-\frac{r}{\frac{3}{4}\rho}\right)\ge|y|\left(1-\frac{\frac{1}{2}\rho}{\frac{3}{4}\rho}\right)=|y|\left(1-\frac{2}{3}\right)=\frac{|y|}{3},$$

we obtain

$$\|p_{21}-(p_{21})_{B_r}\|_{L^{\frac{3}{2}}(B_r)} \le 2r|B_r|^{\frac{2}{3}}\|\nabla p_{21}\|_{L^\infty(B_r)} \le Cr^3\rho^{-1}\int_{\frac{3}{4}\rho\le|y|\le\rho}\frac{|u(y)|^2}{|x-y|^3}dy$$

$$\le 3^3Cr^3\rho^{-1}\int_{\frac{3}{4}\rho\le|y|\le\rho}\frac{|u(y)|^2}{|y|^3}dy \le 4^3Cr^3\rho^{-4}\int_{\frac{3}{4}\rho\le|y|\le\rho}|u(y)|^2dy$$

$$\le 4^3Cr^3\rho^{-4}|\{\frac{3}{4}\rho\le|y|\le\rho\}|^{\frac{1}{3}}\left(\int_{\frac{3}{4}\rho\le|y|\le\rho}|u(y)|^3dy\right)^{\frac{2}{3}} \le C'Cr^3\rho^{-3}\left(\int_{\frac{3}{4}\rho\le|y|\le\rho}|u(y)|^3dy\right)^{\frac{2}{3}}.$$

Then

$$\int_{Q_r}|p_{21}-(p_{21})_{B_r}|^{\frac{3}{2}} \le (C')^{\frac{3}{2}}r^{\frac{9}{2}}\rho^{-\frac{9}{2}}\int_{Q_\rho}|u|^3dy.$$

By the exact same argument,

$$\int_{Q_r}|p_{22}-(p_{22})_{B_r}|^{\frac{3}{2}} \le (C')^{\frac{3}{2}}r^{\frac{9}{2}}\rho^{-\frac{9}{2}}\int_{Q_\rho}|p|^{\frac{3}{2}}dy.$$

Finally we set

$$p_3 = p_{31} + p_{32} = \frac{1}{4\pi|x|} * (u_i u_j \partial_j \partial_i \phi) + \frac{1}{4\pi|x|} * (p\triangle\phi).$$

They can be treated like $p_{21}$ and $p_{22}$, due to $|\partial_i\partial_j\phi| \le c\rho^{-2}$. Indeed, for example

$$\|p_{31} - (p_{31})_{B_r}\|_{L^{\frac{3}{2}}(B_r)} \le 2r|B_r|^{\frac{2}{3}} \|\nabla p_{31}\|_{L^\infty(B_r)} \le Cr^3\rho^{-2} \int_{\frac{3}{4}\rho\le|y|\le\rho} \frac{|u(y)|^2}{|x-y|^2} dy$$
$$\le 3^2 Cr^3\rho^{-2} \int_{\frac{3}{4}\rho\le|y|\le\rho} \frac{|u(y)|^2}{|y|^2} dy \le 4^2 Cr^3\rho^{-4} \int_{\frac{3}{4}\rho\le|y|\le\rho} |u(y)|^2 dy$$
$$\le C'Cr^3\rho^{-3} \left( \int_{\frac{3}{4}\rho\le|y|\le\rho} |u(y)|^3 dy \right)^{\frac{2}{3}} \text{ etc.}$$

All the above estimates have been obtained by assuming $u \in L_t^\infty C_x^N(\overline{Q_\rho})$. In general, we consider a sequence $L^\infty((-\rho^2,0), C_x^N(\overline{B_\rho})) \ni u_n \xrightarrow{n\to\infty} u$, with the convergence occurring in $L^3(Q_\rho) \cap L^\infty((-\rho^2,0), L^2(B_\rho))$.

□

## 15 A second result of Caffarelli, Kohn and Nirenberg

In this section we will the following theorem.

**Theorem 15.1.** *There exists absolute constants $\epsilon_1 > 0$ s.t. if $(u,p)$ is a suitable weak solution of the NS in $Q_R(t_0,x_0)$ for some $R > 0$ and we have either*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r(t_0,x_0)} |\nabla u|^2 < \epsilon_1 \text{ or} \tag{15.1}$$

$$\limsup_{r\to 0} \frac{1}{r} \sup_{t_0-r^2<t<t_0} \int_{B_r(x_0)} |u|^2 < \epsilon_1, \tag{15.2}$$

*then $u \in L^\infty(Q_\rho(t_0,x_0)))$ for some $\rho \in (0,R)$.*

Specifically, we will show that

$$(2\rho)^{-2} \int_{Q_{2\rho}(t_0,x_0)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dtdx < \epsilon_0^*, \tag{15.3}$$

with $\epsilon_0^*$ the small positive constant in Theorem 14.2. Then, the conclusion follows from Theorem 14.2.

Like in the previous section, we will at first prove a simplified version of this theorem, where there is no pressure.

## 15.1 A simplified result, without pressure

We oversimplify and we assume that there is no pressure in the local energy inequality (13.5), so that, as before in (14.5) we have

$$\int_{B_R(x_0)} |u(s)|^2\phi(s)dx + 2\int_{t_0-R^2}^{s}\int_{B_R(x_0)} |\nabla u|^2\phi \le \int_{t_0-R^2}^{s}\int_{B_R(x_0)} |u|^2(\partial_t + \triangle)\phi \tag{15.4}$$
$$+ \int_{t_0-R^2}^{s}\int_{B_R(x_0)} |u|^2(u\cdot\nabla)\phi \text{ for all } \phi \in C_c^\infty(Q_R(t_0,x_0),[0,\infty)).$$

Then using (15.4) it is possible to prove rigorously the following.

**Proposition 15.2.** *There exists and absolute constant $\epsilon_1 > 0$ such that if for some $R > 0$*

$$u \in L^\infty((t_0-R^2,t_0),L^2(B_R(x_0),\mathbb{R}^3)) \text{ and } \nabla u \in L^2(Q_R(t_0,x_0)) \tag{15.5}$$

*and $u$ satisfies* (15.4) *then, if $u$ satisfies either* (15.1) *or* (15.2)*, there exists $\rho \in (0,R)$ s.t.*

$$\rho^{-2}\int_{Q_\rho(t_0,x_0)} |u|^3 < \epsilon_0^*. \tag{15.6}$$

Before proving Proposition 15.2 we give a sketch. First of all, we can assume $(t_0,x_0) = (0,0)$. Next, suppose that (15.1) is true and define

$$\widetilde{E}(r) = \frac{1}{r}\sup_{-r^2<t<0}\int_{B_r}|u|^2.$$

Then it will be shown that there exists a fixed $\theta \in (0,1)$ s.t. $\widetilde{E}(\theta r) \le 2^{-1}\epsilon_1 + 2^{-1}\widetilde{E}(r)$ for all $r \in (0,r_0]$ for $r_0 > 0$ small enough. Then

$$\widetilde{E}(\theta^n r) \le 2^{-1}\epsilon_1 + 2^{-1}\widetilde{E}(\theta^{n-1}r) \le \left(2^{-1}+2^{-2}\right)\epsilon_1 + 2^{-2}\widetilde{E}(\theta^{n-2}r) \le \sum_{j=1}^{n} 2^{-j}\epsilon_1 + 2^{-n}\widetilde{E}(r)$$

so that, assuming that $\widetilde{E}(r)$ is uniformly bounded in $(0,r_0]$, then picking $n$ sufficiently large, we find that there exists an $r_1 > 0$ s.t. $\widetilde{E}(r) < 2\epsilon_1$ for all $r \in (0,r_1]$. Then, by (14.11)

$$r^{-2}\int_{Q_r}|u|^3dx \le C_0\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2dx + r^{-1}\int_{Q_r}|\nabla u|^2\right]^{\frac{3}{2}} \le C_0\left[3\epsilon_1\right]^{\frac{3}{2}}$$
$$= C_0 3^{\frac{3}{2}}\epsilon_1^{\frac{3}{2}} < \epsilon_0^* \text{ for } \epsilon_1 \in \left(0, C_0^{-\frac{2}{3}}3^{-1}\left(\epsilon_0^*\right)^{\frac{2}{3}}\right)$$

and this, in turn, gives (15.6). So the key point of Proposition 15.2 is that if (15.1) is true, then $\widetilde{E}(r) < 2\epsilon_1$ and also a similar case with (15.1) and (15.2) interchanged.

The proof of Proposition 15.2 exploits the following lemma, about cutoffs.

**Lemma 15.3.** *There exists a constant $C_3 \geq 1$ such that, for any fixed $r > 0$ and $\theta \in (0, 1/2]$, there exists $\phi \in C_c^\infty(\mathbb{R}^4, [0, \infty))$ such that $supp\phi \cap \overline{Q_1} \subseteq \overline{Q_r}$,*

$$\phi \geq C_3^{-1}(\theta r)^{-1} \text{ in } Q_{\theta r} \text{ and} \tag{15.7}$$

$$0 \leq \phi \leq C_3(\theta r)^{-1}, \quad |\nabla \phi| \leq C_3(\theta r)^{-2} \text{ and } |(\partial_t + \triangle)\phi| \leq C_3 \theta^2 r^{-3} \text{ in } Q_r. \tag{15.8}$$

*Proof.* We write

$$\phi(t, x) = (\theta r)^2 \vartheta(t, x)\psi(t, x). \tag{15.9}$$

Here we choose $\psi$ such that

$$(\partial_t + \triangle)\psi(t, x) = 0 \text{ for } t < (\theta r)^2 \text{ and with initial value } \psi((\theta r)^2, x) = \delta(x). \tag{15.10}$$

Then we know that

$$\psi(t, x) = K_{(\theta r)^2 - t}(x) = (4\pi((\theta r)^2 - t))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4((\theta r)^2 - t)}}. \tag{15.11}$$

Then we have

$$\begin{aligned} \psi(t, x) &= (4\pi((\theta r)^2 - t))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4((\theta r)^2 - t)}} \leq (4\pi((\theta r)^2 - t))^{-\frac{3}{2}} \\ &\leq (4\pi(\theta r)^2)^{-\frac{3}{2}} = (4\pi)^{-\frac{3}{2}} (\theta r)^{-3} \text{ in } Q_r = (-r^2, 0) \times B_r \end{aligned} \tag{15.12}$$

$$\begin{aligned} \psi(t, x) &= (4\pi((\theta r)^2 + |t|))^{-\frac{3}{2}} e^{-\frac{|x|^2}{4((\theta r)^2 + |t|)}} \\ &\geq (4\pi 2(\theta r)^2)^{-\frac{3}{2}} e^{-\frac{\theta^2 r^2}{4(\theta r)^2}} = (8\pi)^{-\frac{3}{2}} e^{-\frac{1}{4}} (\theta r)^{-3} \text{ in } Q_{\theta r}. \end{aligned} \tag{15.13}$$

Next,

$$\begin{aligned} \nabla \psi(t, x) &= -2^{-1}\pi^{-\frac{3}{2}} (4((\theta r)^2 - t))^{-\frac{5}{2}} e^{-\frac{|x|^2}{4((\theta r)^2 - t)}} \; x \\ &= -2^{-1}\pi^{-\frac{3}{2}} (4((\theta r)^2 - t))^{-2} e^{-\frac{|x|^2}{4((\theta r)^2 - t)}} \; \frac{x}{\sqrt{4((\theta r)^2 - t)}} \end{aligned}$$

so in $Q_r$

$$|\nabla \psi(t, x)| \leq 2^{-6}\pi^{-\frac{3}{2}} (\theta r)^{-4} e^{-\frac{|x|^2}{4((\theta r)^2 - t)}} \; \frac{|x|}{\sqrt{4((\theta r)^2 - t)}} \leq 2^{-6}\pi^{-\frac{3}{2}} (\theta r)^{-4} \sup_{\alpha \geq 0} \alpha e^{-\alpha^2}. \tag{15.14}$$

We define, for $X \in C_c^\infty(\mathbb{R}^3, [0, 1])$ and $T, \eta \in C_c^\infty(\mathbb{R}, [0, 1])$,

$$\begin{aligned} &\vartheta(t, x) = X(x/r) T(t/r^2) \eta(t) \text{ where} \\ &X(x) = \begin{cases} 1 \text{ in } B_{1/2} \\ 0 \text{ outside } B_1 \end{cases} \text{ and } T(t) = \begin{cases} 1 \text{ for } t \in (-1/4, 1/4) \\ 0 \text{ for } |t| \geq 1, \end{cases} \end{aligned} \tag{15.15}$$

and

$$\eta(t) = \begin{cases} 1 \text{ for } t \leq r^2/4 \\ 0 \text{ for } t \geq r^2/2 \end{cases}$$

Now we check if (15.9) satisfies the desired results. First of all, in $Q_1$ we have $\phi(t,x) \neq 0$ only if $X(x/r) \neq 0$, that is only if $|x| > 1$, and $T(t/r^2) \neq 0$, that is only if $-r^2/4 < t < 0$. Hence it is true that $\mathrm{supp}\phi \cap Q_1 \subseteq Q_r$.
Now, let us look at the estimates. In $Q_{\theta r}$ we have

$$\phi(t,x) = (\theta r)^2 X(x/r)T(t/r^2)\eta(t)\psi(t,x) = (\theta r)^2\psi(t,x) \geq (8\pi)^{-\frac{3}{2}} e^{-\frac{1}{8}}(\theta r)^{-1},$$

yielding (15.7) and in $Q_r$ we have

$$\phi(t,x) = (\theta r)^2 X(x/r)T(t/r^2)\eta(t)\psi(t,x) \leq (\theta r)^2\psi(t,x) \leq (4\pi)^{-\frac{3}{2}}(\theta r)^{-1},$$

so yielding the first estimate in (15.8).
Turning to the gradient, we have

$$\nabla\phi(t,x) = (\theta r)^2\psi(t,x)T(t/r^2)\eta(t)r^{-1}\nabla X(x/r) + (\theta r)^2 T(t/r^2)\eta(t)r^{-1}X(x/r)\nabla\psi(t,x).$$

In $Q_r$ we have

$$(\theta r)^2\psi(t,x)T(t/r^2)\eta(t)r^{-1}|\nabla X(x/r)| \leq \|\nabla X\|_{L^\infty}(\theta r)^2\psi(t,x) \leq \|\nabla X\|_{L^\infty}(4\pi)^{-\frac{3}{2}}(\theta r)^{-1},$$

and

$$(\theta r)^2 T(t/r^2)\eta(t)r^{-1}X(x/r)|\nabla\psi(t,x)| \leq (\theta r)^2|\nabla\psi(t,x)| \leq C\theta^{-2}r^{-2}.$$

Finally, we have

$$\begin{aligned}(\partial_t + \triangle)\phi(t,x) = (\theta r)^2\psi(t,x)(\partial_t + \triangle)\left(X(x/r)T(t/r^2)\eta(t)\right)\\ + 2(\theta r)^2 r^{-1}T(t/r^2)\eta(t)\nabla X(x/r)\cdot\nabla\psi(t,x).\end{aligned}$$

In $Q_r$, using $|(\nabla X)(x/r)| \neq 0 \Longrightarrow 1/2 \leq |x/r| \leq 1$, we have

$$\begin{aligned}&2r^{-1}(\theta r)^2 T(t/r^2)\eta(t)\,|(\nabla X)(x/r)\cdot\nabla\psi(t,x)| \leq \theta^2 r\,|(\nabla X)(x/r)|\,2^{-6}\pi^{-\frac{3}{2}}\theta^{-5}r^{-4}e^{-\frac{1}{2^4\theta^2}}\frac{|x|}{r}\\ &\leq 2^{-6}\theta^2\pi^{-\frac{3}{2}}r^{-3}\|\nabla X\|_{L^\infty}\sup_{\theta>0}\theta^{-5}e^{-\frac{1}{2^4\theta^2}} \leq C\theta^2 r^{-3}.\end{aligned}$$

Finally, in $Q_r$, using also $T'(t/r^2) \neq 0 \Longrightarrow 1/4 \leq |t/r^2| \leq 1$

$$\begin{aligned}&(\theta r)^2\psi(t,x)|(\partial_t + \triangle)\left(X(x/r)T(t/r^2)\eta(t)\right)| = (\theta r)^2\psi(t,x)|(\partial_t + \triangle)\left(X(x/r)T(t/r^2)\right)|\\ &\leq (\theta r)^2(4\pi((\theta r)^2 + |t|))^{-\frac{3}{2}}e^{-\frac{|x|^2}{4((\theta r)^2+|t|)}}r^{-2}\left(X(x/r)\left|T'(t/r^2)\right| + T(t/r^2)\,|(\triangle X)(x/r)|\right)\\ &\leq \theta^2(4\pi)^{-\frac{3}{2}}(1/4)^{-\frac{3}{2}}\|T'\|_{L^\infty(\mathbb{R})} + \theta^2(4\pi)^{-\frac{3}{2}}(\theta r)^{-3}e^{-\frac{1}{2^4\theta^2}}\|\triangle X\|_{L^\infty} \leq C\theta^2 r^{-3}.\end{aligned}$$

□

*Proof of Proposition 15.2.* In the proof it is enough to consider $(t_0, x_0) = (0, 0)$. The first important step is the following.

**Lemma 15.4.** *For $C_0$ the constant in Lemma 14.7, $C_3$ the constant in Lemma 15.3 and $C_4$ the constant of the Poincaré inequality* (15.17) *reminded in the proof, we have*

$$\begin{aligned}
&\max\left(\sup_{-(\theta r)^2<t<0}\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2, \frac{1}{\theta r}\int_{-(\theta r)^2}^t\int_{B_{\theta r}}|\nabla u|^2\right)\\
&\leq 3C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+r^{-1}\int_{Q_r}|\nabla u|^2\right]\\
&+4C_4^2C_3^2\theta^{-6}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)\left(r^{-1}\int_{Q_r}|\nabla u|^2\right).
\end{aligned}\tag{15.16}$$

*Proof.* Applying (15.4) for the $\phi$ of Lemma 15.3, we get

$$\begin{aligned}
&\int_{B_1}|u(t)|^2\phi(t)+2\int_{-1}^t\int_{B_1}|\nabla u|^2\phi\leq\int_{-1}^t\int_{B_1}|u|^2(\partial_t+\triangle)\phi\\
&+\int_{-1}^t\int_{B_1}\left(|u|^2-\left(|u|^2\right)_{B_r}\right)(u\cdot\nabla)\phi.
\end{aligned}$$

Using the estimates in Lemma 15.3, we obtain

$$\begin{aligned}
&\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2+\frac{1}{\theta r}\int_{-(\theta r)^2}^t\int_{B_{\theta r}}|\nabla u|^2\leq C_3^2\theta^2r^{-3}\int_{Q_r}|u|^2\\
&+C_3^2\theta^{-2}r^{-2}\int_{Q_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u|\\
&\leq C_3^2\theta^2r^{-3}|Q_r|^{\frac{1}{3}}\|u\|^2_{L^3(Q_r)}+C_3^2\theta^{-2}r^{-2}\int_{Q_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u|\\
&=C_3^2\theta^2r^{-3+\frac{5}{3}}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\|u\|^2_{L^3(Q_r)}+C_3^2\theta^{-2}r^{-2}\int_{Q_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u|\\
&\leq 2C_3^2\theta^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{2}{3}}+C_3^2\theta^{-2}r^{-2}\int_{Q_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u|.
\end{aligned}$$

Now we have

$$\begin{aligned}
\int_{B_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u|&\leq\left\||u|^2-\left(|u|^2\right)_{B_r}\right\|_{L^{\frac{3}{2}}(B_r)}\|u\|_{L^3(B_r)}\leq C_4\|\nabla|u|^2\|_{L^1(B_r)}\|u\|_{L^3(B_r)}\\
&\leq 2C_4\|u\|_{L^2(B_r)}\|\nabla u\|_{L^2(B_r)}\|u\|_{L^3(B_r)},
\end{aligned}$$

where we used the Poincaré inequality

$$\left\||u|^2-\left(|u|^2\right)_{B_r}\right\|_{L^{\frac{3}{2}}(B_r)}\leq C_4\|\nabla|u|^2\|_{L^1(B_r)},\tag{15.17}$$

see [10] Theorem 8.11, where, by scale invariance, the constant $C_4$ does not depend on $r$.

Then, from Hölder with $\frac{1}{6}+\frac{1}{2}+\frac{1}{3}=1$

$$\int_{Q_r}\left||u|^2-\left(|u|^2\right)_{B_r}\right|\ |u| \leq 2C_4\|\nabla u\|_{L^2(Q_r)}\|u\|_{L^3(Q_r)}\|\|u\|_{L^2(B_r)}\|_{L^6(-r^2,0)}$$

$$\leq 2r^{\frac{1}{3}}C_4\|\nabla u\|_{L^2(Q_r)}\|u\|_{L^3(Q_r)}\sup_{-r^2<t<0}\|u\|_{L^2(B_r)}$$

$$=2C_4r^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)^{\frac{1}{2}}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)^{\frac{1}{2}}.$$

Then we conclude

$$\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2+\frac{1}{\theta r}\int_{-(\theta r)^2}^{t}\int_{B_{\theta r}}|\nabla u|^2 \leq 2C_3^2\theta^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{2}{3}}$$

$$+2C_4C_3^2\theta^{-2}\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)^{\frac{1}{2}}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)^{\frac{1}{2}}$$

$$\leq 2C_3^2\theta^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{2}{3}}$$

$$+C_3^2\theta^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{2}{3}}+4C_4^2C_3^2\theta^{-6}\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right).$$

Then, using the inequality

$$r^{-2}\int_{Q_r}|u|^3 \leq C_0\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+r^{-1}\int_{Q_r}|\nabla u|^2\right]^{\frac{3}{2}}, \tag{15.18}$$

we obtain the following, which is (15.16),

$$\max\left(\sup_{-(\theta r)^2<t<0}\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2, \frac{1}{\theta r}\int_{-(\theta r)^2}^{t}\int_{B_{\theta r}}|\nabla u|^2\right)$$

$$\leq 3C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+r^{-1}\int_{Q_r}|\nabla u|^2\right]$$

$$+4C_4^2C_3^2\theta^{-6}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)\left(r^{-1}\int_{Q_r}|\nabla u|^2\right).$$

□

Having obtained inequality (15.16) we move to the conclusion of the proof of Proposition 15.2.

We assume either (15.1) or (15.2). For definiteness we assume (15.1) , that is

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}|\nabla u|^2 < \epsilon_1$$

but, assuming instead (15.2), that is

$$\limsup_{r\to 0}\frac{1}{r}\sup_{t_0-r^2<t<t_0}\int_{B_r}|u|^2<\epsilon_1,$$

the argument is the same, due to the symmetry with respect to the above quantities of inequality (15.16). Then for $r$ sufficiently small, we have

$$\frac{1}{r}\int_{Q_r}|\nabla u|^2<2\epsilon_1.$$

Then, by (15.16), we have

$$\sup_{-(\theta r)^2<t<0}\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2\le 3C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+2\epsilon_1\right]$$
$$+8C_4^2C_3^2\theta^{-6}\epsilon_1\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right).$$

Setting now

$$\widetilde{E}(r):=\frac{1}{r}\sup_{-r^2<t<0}\int_{B_r}|u|^2.$$

we have

$$\widetilde{E}(\theta r)\le 6C_3^2C_0^{\frac{2}{3}}\theta^2\epsilon_1+\left(3C_3^2C_0^{\frac{2}{3}}\theta^2+8C_4^2C_3^2\theta^{-6}\epsilon_1\right)\widetilde{E}(r).$$

Now if we use $\theta\le 1/2$ so small that $6C_3^2C_0^{\frac{2}{3}}\theta^2<1/2$ and $\epsilon_1>0$ so small that $8C_4^2C_3^2\theta^{-6}\epsilon_1<1/4$, we obtain

$$\widetilde{E}(\theta r)\le\frac{1}{2}\epsilon_1+\frac{1}{2}\widetilde{E}(r)\text{ for all }r\in(0,r_0]\text{ for }r_0>0\text{ small enough.}\tag{15.19}$$

This implies

$$\widetilde{E}(\theta^n r)\le\epsilon_1+2^{-n}\widetilde{E}(r).\tag{15.20}$$

We assume now

$$\limsup_{r\to 0}\widetilde{E}(r)<\infty,\tag{15.21}$$

which implies $\widetilde{E}(r)\le C_5<\infty$ for $r\in(0,r_1]$. Then (15.20) implies $\widetilde{E}(\theta^n r)\le\epsilon_1+2^{-n}C_5\le 2\epsilon_1$ for $n>\log_2\left(C_5\epsilon_1^{-1}\right)$ and $0<r<\min\{r_0,r_1\}$. This implies $\widetilde{E}(r)\le 2\epsilon_1$ for $0<r<r_2$,

with $r_2 = \theta^{n_0} \min\{r_0, r_1\}$, for a fixed $n_0 > \log_2\left(C_5 \epsilon_1^{-1}\right)$. Inserting $\widetilde{E}(r) \le 2\epsilon_1$ and (15.1) in (15.18) we obtain

$$r^{-2} \int_{Q_r} |u|^3 \le C_0 3^{3/2} \epsilon_1^{\frac{3}{2}} < \epsilon_0^* \text{ for } \epsilon_1 < 3\left(C_0^{-1}\epsilon_0^*\right)^{\frac{2}{3}}, \tag{15.22}$$

yielding (15.6). To complete the proof of Proposition 15.2, we need to prove (15.21). If (15.21) is false, there exists $R_n \searrow 0$ with $\widetilde{E}(R_n) \nearrow \infty$. Using (15.19),

$$\widetilde{E}(R_n) \le \frac{1}{2}\epsilon_1 + \frac{1}{2}\widetilde{E}(\theta^{-1}R_n) \le \epsilon_1 + 2^{-m_n}\widetilde{E}(\theta^{-m_n}R_n)$$

for $m_n := \left[\left|\frac{\log(R_n/r_0)}{\log\theta}\right|\right]$ which is the largest $m_n$ s.t. $\theta^{-m_n}R_n \le r_0$, so that we have $\theta r_0 \le \theta^{-m_n}R_n \le r_0$. This implies

$$\widetilde{E}(R_n) \le \frac{1}{2}\epsilon_1 + \frac{1}{2}\widetilde{E}(\theta^{-1}R_n) \le \epsilon_1 + 2^{-1} \sup_{\theta r_0 \le r \le r_0} \widetilde{E}(r)$$

which, from $\widetilde{E}(R_n) \xrightarrow{n\to\infty} \infty$, implies $\sup_{\theta r_0 \le r \le r_0} \sup_{-r^2<t<0} \int_{B_r} |u|^2 = \infty$. But then, this would imply $u \notin L^\infty((-R^2,0), L^2(B_R,\mathbb{R}^3))$, contradicting the hypothesis (15.5).

□

## 15.2 Proof of Theorem 15.1

We can focus on the case $(t_0, x_0) = (0,0)$. Then using (14.1) like in Sect. 15.1, we have

$$\begin{aligned}
&\int_{B_1} |u(t)|^2\phi(t) + 2\int_{-1}^t \int_{B_1} |\nabla u|^2\phi \le \int_{-1}^t \int_{B_1} |u|^2(\partial_t + \triangle)\phi \\
&+ \int_{-1}^t \int_{B_1} \left(|u|^2 - \left(|u|^2\right)_{B_r}\right)(u\cdot\nabla)\phi + \int_{-1}^t \int_{B_1} p(u\cdot\nabla)\phi,
\end{aligned}$$

with the test function from Lemma 15.3. Then, by Lemma 15.3 we get

$$\begin{aligned}
&\frac{1}{\theta r}\int_{B_{\theta r}} |u(t)|^2 + \frac{1}{\theta r}\int_{-(\theta r)^2}^t \int_{B_{\theta r}} |\nabla u|^2 \le C_3^2\theta^2 r^{-3}\int_{Q_r} |u|^2 \\
&+ C_3^2\theta^{-2}r^{-2}\int_{Q_r} \left||u|^2 - \left(|u|^2\right)_{B_r}\right| \ |u| + C_3^2\theta^{-2}r^{-2}\int_{Q_r} |p| \ |u|.
\end{aligned}$$

Now, by the discussion in Sect. 15.1, see (15.16), we have

$$\begin{aligned}
&\frac{1}{\theta r}\int_{B_{\theta r}} |u(t)|^2 + \frac{1}{\theta r}\int_{-(\theta r)^2}^t \int_{B_{\theta r}} |\nabla u|^2 \le 3C_3^2 C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2 + r^{-1}\int_{Q_r}|\nabla u|^2\right] \\
&+ 4C_4^2C_3^2\theta^{-6}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)\left(r^{-1}\int_{Q_r}|\nabla u|^2\right) + C_3^2\theta^{-2}r^{-1}\int_{Q_r}|p| \ |u|.
\end{aligned}$$

We focus now on the additional term

$$C_3^2\theta^{-2}r^{-2}\int_{Q_r}|p|\ \ |u|\le C_3^2\theta^{-2}r^{-2}\|u\|_{L^3(Q_r)}\|p\|_{L^{\frac{3}{2}}(Q_r)}=C_3^2\theta\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{1}{3}}\theta^{-3}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{2}{3}}$$
$$\le C_3^2\theta^2\left(r^{-2}\int_{Q_r}|u|^3\right)^{\frac{2}{3}}+C_3^2\theta^{-6}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}.$$

So if we apply (15.18), we obtain

$$\max\left(\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2,\frac{1}{\theta r}\int_{-(\theta r)^2}^{t}\int_{B_{\theta r}}|\nabla u|^2\right)\le 4C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+r^{-1}\int_{Q_r}|\nabla u|^2\right]$$
$$+4C_4^2C_3^2\theta^{-6}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)+C_3^2\theta^{-6}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}.$$

Now introduce the estimate

$$\left((\theta r)^{-2}\int_{Q_{\theta r}}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}\le 2C_5^{\frac{4}{3}}\theta^{-2}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)$$
$$+2C_5^{\frac{4}{3}}\theta^{\frac{4}{3}}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}. \tag{15.23}$$

We need now to exploit one of (15.1)–(15.2). We choose

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r}|\nabla u|^2<\epsilon_1. \tag{15.24}$$

Then, for $r_0>0$ small enough, we have

$$\frac{1}{r}\int_{Q_r}|\nabla u|^2<2\epsilon_1 \text{ for } r\in(0,r_0]. \tag{15.25}$$

Then we have

$$\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2\le 4C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+2\epsilon_1\right]$$
$$+8C_4^2C_3^2\theta^{-6}\epsilon_1\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)+C_3^2\theta^{-6}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}$$

and

$$\left((\theta r)^{-2}\int_{Q_{\theta r}}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}\le 4C_5^{\frac{4}{3}}\epsilon_1\theta^{-2}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)+2C_5^{\frac{4}{3}}\theta^{\frac{4}{3}}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}.$$

Then we obtain

$$\begin{aligned}
&\frac{1}{\theta r}\int_{B_{\theta r}}|u(t)|^2+\theta^{-7}\left((\theta r)^{-2}\int_{Q_{\theta r}}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}\\
&\le 4C_3^2C_0^{\frac{2}{3}}\theta^2\left[r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2+2\epsilon_1\right]\\
&+8C_4^2C_3^2\theta^{-6}\epsilon_1\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)+C_3^2\theta\theta^{-7}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}\\
&+4C_5^{\frac{4}{3}}\epsilon_1\theta^{-9}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)+2C_5^{\frac{4}{3}}\theta^{\frac{4}{3}}\theta^{-7}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}}.
\end{aligned}$$

Setting

$$E(r):=\frac{1}{r}\int_{B_r}|u(t)|^2+\theta^{-7}\left(r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\right)^{\frac{4}{3}},$$

we obtain

$$E(\theta r)\le 8C_3^2C_0^{\frac{2}{3}}\theta^2\epsilon_1+\left(4C_3^2C_0^{\frac{2}{3}}\theta^2+8C_4^2C_3^2\theta^{-6}\epsilon_1+C_3^2\theta+4C_5^{\frac{4}{3}}\epsilon_1\theta^{-9}+2C_5^{\frac{4}{3}}\theta^{\frac{4}{3}}\right)E(r).$$

Choosing $\theta$ small enough, we can assume $8C_3^2C_0^{\frac{2}{3}}\theta^2<1/2$ and $C_3^2\theta+2C_5^{\frac{4}{3}}\theta^{\frac{4}{3}}<1/5$, so that

$$E(\theta r)\le 5^{-1}\epsilon_1+\left(\frac{3}{10}+8C_4^2C_3^2\theta^{-6}\epsilon_1+4C_5^{\frac{4}{3}}\epsilon_1\theta^{-9}\right)E(r).$$

We choose $\epsilon_1$ so that $8C_4^2C_3^2\theta^{-6}\epsilon_1+4C_5^{\frac{4}{3}}\epsilon_1\theta^{-9}<1/5$. Then we obtain

$$E(\theta r)\le 2^{-1}\epsilon_1+2^{-1}E(r).$$

Then, proceeding as in Sect. 15.1, if we know that $\limsup_{r\to 0}E(r)<\infty$, we conclude $E(r)\le 2\epsilon_1$ for $0<r<r_2$ for some appropriately small $r_2$. Then we get

$$r^{-2}\int_{Q_r}|u|^3\le C_0 3^{3/2}\epsilon_1^{\frac{3}{2}}<2^{-1}\epsilon_0^*\text{ for }\epsilon_1<3\left(2^{-1}C_0^{-1}\epsilon_0^*\right)^{\frac{2}{3}},\tag{15.26}$$

Similarly

$$r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}\le\theta^{\frac{21}{4}}E^{\frac{3}{4}}(r)\le\theta^{\frac{21}{4}}2^{\frac{3}{4}}\epsilon_1^{\frac{3}{4}}<2^{-1}\epsilon_0^*\text{ for }\epsilon_1<2^{-7}2^{-\frac{28}{3}}(\epsilon_0^*)^{\frac{4}{3}}.$$

Then we get (15.3). To complete the proof, up to (15.23), we need to show $\limsup_{r\to 0}E(r)<\infty$. By the argument in (15.1), having $\limsup_{r\to 0}E(r)=\infty$ would imply

$$\sup_{\theta r_0\le r\le r_0}\left[\sup_{-r^2<t<0}\int_{B_r}|u|^2+\int_{Q_r}|p|^{\frac{3}{2}}\right]=\infty.$$

But this would imply either $u \notin L^\infty((-R^2,0), L^2(B_R,\mathbb{R}^3))$ or $p \notin L^{3/2}(Q_R,\mathbb{R})$, contradicting the hypotheses.

□

Finally, we state the lemma needed for (15.23).

**Lemma 15.5.** *There exists $C_5$ such that for $p \in L^{\frac{3}{2}}(Q_r)$ and $u \in L^3(Q_r) \cap L^\infty((-r^2,0), L^2(B_r))$ and for $-\triangle p = \partial_i\partial_j(u_iu_j)$ in $Q_r$, then for any $0<\theta<1/2$ we have*

$$(\theta r)^{-2}\int_{Q_{\theta r}} |p|^{\frac{3}{2}} \le C_5\theta^{-3/2}\left(r^{-1}\sup_{-r^2<t<0}\int_{B_r}|u(t)|^2\right)^{\frac{3}{4}}\left(r^{-1}\int_{Q_r}|\nabla u|^2\right)^{\frac{3}{4}} + C_5\theta r^{-2}\int_{Q_r}|p|^{\frac{3}{2}}. \tag{15.27}$$

*Proof.* By scaling, it suffices to consider case $r=1$. We will start by assuming $u \in C^\infty(\overline{Q_1})$. This in turn implies that $p(t) \in L^{\frac{3}{2}}((-1,0)), C^k(B_1))$ for all $k$. Let now $\phi \in \mathbb{C}_c^\infty(\mathbb{R}^3,[0,1])$ with

$$\phi(x) = \begin{cases} 1 \text{ in } B_{3/4} \\ 0 \text{ outside } B_{4/5}. \end{cases}$$

Let $U_{ij} = u_i(u_j-(u_j)_1)$ where $(u_j)_r = (u_j)_{B_r} = v|B_rv|^{-1}\int_{B_r}u_j$. Notice that $-\triangle p = \partial_i\partial_jU_{ij}$. Then, by Lemma 11.1, we have

$$\begin{aligned}
\phi p &= (-\triangle)^{-1}(-\triangle)\phi p = \frac{1}{4\pi|x|} * ((-\triangle)\phi p) = \frac{1}{4\pi|x|} * (-\phi\triangle p - p\triangle\phi - 2\nabla\phi\cdot\nabla p) \\
&= \frac{1}{4\pi|x|} * (\phi\partial_i\partial_jU_{ij} - p\triangle\phi - 2\nabla\phi\cdot\nabla p) \\
&= \frac{1}{4\pi|x|} * (\partial_i\partial_j(\phi U_{ij}) - \partial_i(U_{ij}\partial_j\phi) - \partial_j(U_{ij}\partial_i\phi) + U_{ij}\partial_j\partial_i\phi - p\triangle\phi - 2\nabla\cdot(\nabla\phi p) + 2p\triangle\phi) \\
&= \partial_i\frac{1}{4\pi|x|} * (\partial_j(\phi U_{ij}) - 2U_{ij}\partial_i\phi - 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (U_{ij}\partial_j\partial_i\phi + p\triangle\phi) \\
&= -\frac{x_i}{4\pi|x|^3} * (\partial_j(\phi U_{ij}) - 2U_{ij}\partial_i\phi - 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (U_{ij}\partial_j\partial_i\phi + p\triangle\phi)
\end{aligned}$$

Like in Lemma 14.9 we conclude

$$\phi p = -\frac{1}{4\pi}P.V.\left(\frac{\delta_{ij}}{|x|^3} - \frac{3x_ix_j}{|x|^5}\right) * (\phi U_{ij}) + \frac{x_i}{4\pi|x|^3} * (2U_{ij}\partial_i\phi + 2p\partial_j\phi) + \frac{1}{4\pi|x|} * (U_{ij}\partial_j\partial_i\phi + p\triangle\phi).$$

We have $p = \phi p$ in $Q_\theta$. We write

$$p_1 = -\frac{1}{4\pi}P.V.\left(\left(\frac{\delta_{ij}}{|x|^3} - \frac{3x_ix_j}{|x|^5}\right)\right) * (\phi U_{ij}).$$

Then

$$\|p_1\|_{L^{\frac{3}{2}}(B_\theta)} \le C\sum_{i,j}\|\phi U_{ij}\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le C\sum_{i,j}\|u_i(u_j-(u_j)_1)\|_{L^{\frac{3}{2}}(B_{4/5})} \le C'\|u\|_{L^2(B_1)}\|u_j-(u_j)_1\|_{L^6(B_1)}$$
$$\le C\|u\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}.$$

Next, we write

$$p_2 = \frac{x_i}{4\pi|x|^3} * U_{ij}\partial_i\phi = \frac{1}{2\pi}\int_{3/4\le|y|\le 4/5}\frac{x_i-y_i}{4\pi|x-y|^3}U_{ij}\partial_i(y)\phi(y).$$

Then, using Yang's inequality,

$$\|p_2\|_{L^{\frac{3}{2}}(B_\theta)} \lesssim \|\frac{1}{|x|^2}\|_{L^1(1/4\le|x|\le 2)}\|u_i(u_j-(u_j)_1)\|_{L^{\frac{3}{2}}(B_{4/5})} \le C\|u\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}.$$

Similarly, for

$$p_3 = \frac{1}{4\pi|x|} * U_{ij}\partial_j\partial_i\phi$$

we have

$$\|p_3\|_{L^{\frac{3}{2}}(B_\theta)} \lesssim \|\frac{1}{|x|}\|_{L^1(1/4\le|x|\le 2)}\|u_i(u_j-(u_j)_1)\|_{L^{\frac{3}{2}}(B_{4/5})} \le C\|u\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}.$$

For

$$p_4 = \frac{1}{4\pi|x|} * p\triangle\phi$$

we have

$$\|p_4\|_{L^{\frac{3}{2}}(B_\theta)} \le c\theta^2\|p_4\|_{L^\infty(B_\theta)} \lesssim \theta^2\|p\|_{L^1(B_1)} \le C\theta^2\|p\|_{L^{\frac{3}{2}}(B_1)}$$

and, similarly, for

$$p_5 = \frac{x_i}{4\pi|x|^3} * 2p\partial_j\phi$$

we have

$$\|p_5\|_{L^{\frac{3}{2}}(B_\theta)} \le C\theta^2\|p\|_{L^{\frac{3}{2}}(B_1)}.$$

Thus, we have

$$\|p\|_{L^{\frac{3}{2}}(B_\theta)} \le C\|u\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)} + C\theta^2\|p\|_{L^{\frac{3}{2}}(B_1)}$$

and so

$$
\begin{aligned}
\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_\theta)} &\le C\int_{-\theta^2}^{0} \|u\|^{\frac{3}{2}}_{L^2(B_1)}\|\nabla u\|^{\frac{3}{2}}_{L^2(B_1)}dt + C\theta^3\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} \\
&\le C\|\|u\|^{\frac{3}{2}}_{L^2(B_1)}\|_{L^4(-\theta^2,0)}\|\|\nabla u\|^{\frac{3}{2}}_{L^2(B_1)}\|_{L^{4/3}(-\theta^2,0)} + C\theta^3\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} \\
&\le C\|\|u\|_{L^2(B_1)}\|^{\frac{3}{2}}_{L^6(-\theta^2,0)}\|\nabla u\|^{\frac{3}{2}}_{L^2(Q_1)} + C\theta^3\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)} \\
&\le C\theta^{2\frac{1}{6}\cdot\frac{3}{2}}\|u\|^{\frac{3}{2}}_{L^\infty((-\theta^2,0),L^6(B_1))}\|\nabla u\|^{\frac{3}{2}}_{L^2(Q_1)} + C\theta^3\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(Q_1)},
\end{aligned}
$$

that is

$$
\begin{aligned}
\int_{Q_\theta}|p|^{\frac{3}{2}} \le C\theta^{1/2}&\left(\sup_{-1<t<0}\int_{B_1}|u(t)|^2\right)^{\frac{3}{4}}\left(\int_{Q_1}|\nabla u|^2\right)^{\frac{3}{4}} \\
&+ C\theta^3\int_{Q_r}|p|^{\frac{3}{2}},
\end{aligned}
$$

which is (15.27) for $r = 1$.

□

# A Appendix. On the Bochner integral

For this part see [3]. Let $X$ be a Banach space.

**Definition A.1** (Strong measurability)**.** Let $I$ be an interval. A function $f : I \to X$ is strongly measurable if there exists a set $E$ of measure 0 and a sequence $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I\backslash E$.

*Remark* A.2. Notice that when $\dim X < \infty$ a function is measurable (in the sense that $f^{-1}(B)$ is measurable for any Borel set $B$) if an only if it is strongly measurable in the above sense. Indeed if $f$ is strongly measurable in the above sense then as a point wise limit of measurable functions $f$ is measurable, see Theorem 1.14 p. 14 Rudin [15]. Viceversa if $f$ is measurable, then $f$ is strongly measurable in the above sense, see the Corollary to Lusin's Theorem in Rudin [15] p. 54.

*Example* A.3. Consider $\{x_j\}_{j=1}^n$ in $X$ and $\{A_j\}_{j=1}^n$ measurable sets in $I$ with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim that the *simple* function

$$
f(t) := \sum_{j=1}^{n} x_j\chi_{A_j}(t) : I \to X \tag{A.1}
$$

is measurable. Indeed, see Rudin [15] p. 54, there are sequences $\{\varphi_{j,k}\}_{k\in\mathbb{N}}$ in $C_c^0(I, \mathbb{R})$ with $\varphi_{j,k}(t) \stackrel{k\to\infty}{\to} \chi_{A_j}(t)$ a.e. and hence

$$
C_c^0(I, \mathbb{R}) \ni f_k(t) := \sum_{j=1}^{n} x_j\varphi_{j,k}(t) \stackrel{k\to\infty}{\to} f(t) \text{ a.e. in } I.
$$

**Proposition A.4.** *If $(f_n)$ is a sequence of strongly measurable functions from $I$ to $X$ convergent a.e. to a $f : I \to X$, then $f$ is strongly measurable.*

*Proof.* There is an $E$ with $|E| = 0$ s.t. $f_n(t) \overset{n\to\infty}{\to} f(t)$ for any $t \in I\backslash E$. Consider for any $n$ a sequence $C_c(I, X) \ni f_{n,k} \overset{k\to\infty}{\to} f_n$ a.e. We will suppose first that $|I| < \infty$. By applying Egorov Theorem to $\{\|f_{n,k} - f_n\|\}_{k\in\mathbb{N}}$ there is $E_n \subset I$ with $|E_n| \le 2^{-n}$ s.t. $\|f_{n,k} - f_n\| \overset{k\to\infty}{\to} 0$ uniformly in $I\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I\backslash E_n$ and set $g_n = f_{n,k(n)}$. Set $F := E \bigcup (\bigcap_m \bigcup_{n>m} E_n)$. Then $|F| = 0$. Indeed for any $m$

$$|F| \le |E| + \sum_{n=m}^{\infty} |E_n| \le |E| + \sum_{n=m}^{\infty} 2^{-n} \overset{m\to\infty}{\to} 0.$$

Let $t \in I\backslash F$. Since $t \notin E$ we have $f_n(t) \overset{n\to\infty}{\to} f(t)$. Furthermore, for $n$ large enough we have $t \in I\backslash E_n$. Indeed

$$t \notin \bigcap_m \bigcup_{n>m} E_n \Rightarrow \exists\ m \text{ s.t. } t \notin \bigcup_{n>m} E_n \Rightarrow\ t \notin E_n\ \forall\ n > m.$$

Then $\|g_n(t) - f_n(t)\| < 1/n$ and $g_n(t) \overset{n\to\infty}{\to} f(t)$. So $f(t)$ is measurable in the case $|I| < \infty$. Now we consider the case $|I| = \infty$. We express $I = \cup_n I_n$ for an increasing sequence of intervals with $|I_n| < \infty$. Consider for any $n$ a sequence $C_c(I_n, X) \ni f_{n,k} \overset{k\to\infty}{\to} f$ a.e. in $I_n$. Then by Egorov Theorem to $\|f_{n,k} - f_n\|$ there is $E_n \subset I_n$ with $|E_n| \le 2^{-n}$ s.t. $f_{n,k} \overset{k\to\infty}{\to} f_n$ uniformly in $I_n\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I_n\backslash E_n$ and set $g_n = f_{n,k(n)}$. Then defining $F$ like above, the remainder of the proof works exactly like for the case $|I| < \infty$.

□

*Example* A.5. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $X$ and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in $I$ with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.2}$$

is measurable. Indeed if we set $f_n(t) := \sum_{j=1}^{n} x_j \chi_{A_j}(t)$, then we have $\lim_{n\to\infty} f_n(t) = f(t)$ for any $t$, since if $t \notin \cup_{j=1}^{\infty} A_j$ both sides are 0, and if $t \in A_{n_0}$ then for $n \ge n_0$ we have $f_n(t) = x_{n_0} = f(t)$. Hence by Proposition A.4 the function $f$ is measurable.
When the sum in (A.2) is finite then the function $f$ is called *simple*.

*Example* A.6. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in $X$ and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in $I$ where again $A_j \cap A_k = \emptyset$ for $j \ne k$ but we allow $|A_j| = \infty$. Then

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.3}$$

is measurable. To see this consider $f_n(t) = \chi_{[-n,n]}(t) f(t)$. Then

$$f_n(t) = \sum_{j=1}^{\infty} x_j \chi_{A_j \cap [-n,n]}(t)$$

and by Example A.5 we know that each $f_n(t)$ is strongly measurable. Since $f_n(t) \to f(t)$ for any $t \in I$ we conclude by Proposition A.4 that $f$ is strongly measurable.

Another natural definition of measurability is the following one.

**Definition A.7** (Weak measurability)**.** Let $I$ be an interval. A function $f : I \to X$ is weakly measurable if for any $x' \in X'$ the function $t \to \langle x', f(t) \rangle_{X'X}$ is a measurable function $I \to \mathbb{R}$.

Obviously, strongly measurable implies weakly measurable. Let us explore the viceversa.

**Definition A.8.** Let $I$ be an interval. A function $f : I \to X$ is *almost separably valuable* if there exists a 0 measure set $N \subset I$ s.t. $f(I \backslash N)$ is separable.

The following lemma shows that strongly measurable functions are almost separably valuable.

**Lemma A.9.** *If $f : I \to X$ is strongly measurable with $(f_n(t))$ a sequence in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$ for a 0 measure set $E \subset I$ then $f(I \backslash E)$ is separable and there exists a separable Banach subspace $Y \subseteq X$ with $f(I \backslash E) \subseteq Y$.*

*Proof.* First of all $f_n(I \cap \mathbb{Q})$ is a countable dense set in $f_n(I)$. So $f_n(I)$ is separable. In a separable metric space any subspace is separable. So $f_n(I \backslash E)$ is separable. The closed vector space $Y$ generated by $\cup_n f_n(I \backslash E)$ is separable. Indeed let $C \subseteq \cup_n f_n(I \backslash E)$ be a countable set dense in $\cup_n f_n(I \backslash E)$. Let $\mathrm{Span}_{\mathbb{Q}}(C)$ be the vector space on $\mathbb{Q}$ generated by $C$. Then $\mathrm{Span}_{\mathbb{Q}}(C)$ is dense in $Y$. For $C = \{x_1, x_2, ...\}$ we have $\mathrm{Span}_{\mathbb{Q}}(C) = \cup_{n=1}^{\infty} \mathrm{Span}_{\mathbb{Q}}(\{x_1, ..., x_n\})$. This proves that $\mathrm{Span}_{\mathbb{Q}}(C)$ is countable and that $Y$ is separable. □

*Example* A.10. Let $X$ be a Hilbert space with an orthonormal basis $\{e_t\}_{t \in \mathbb{R}}$. Then the map $f : \mathbb{R} \to X$ given by $f(t) = e_t$ is not strongly measurable. This follows from the fact that it is not almost separably valuable.
On the other hand if $x \in X$ then $t \to \langle f(t), x \rangle$ is different from 0 only on a countable subset of $\mathbb{R}$, and as such it is measurable. Hence $f$ is weakly measurable.
Notice however that if $C \subset [0, 1]$ is the standard Cantor set (which has 0 measure and has same cardinality of $\mathbb{R}$) and if $\{\widetilde{e}_t\}_{t \in C}$ is another basis of $X$, then the map

$$g(t) = \begin{cases} \widetilde{e}_t \text{ for } t \in C \text{ and} \\ 0 \text{ otherwise} \end{cases}$$

is weakly measurable (like $f$ and for the same reasons) and is almost separably valuable. Pettis Theorem, which we prove below, implies that $g : \mathbb{R} \to X$ is strongly measurable.

The following lemma will be used for Pettis Theorem.

**Lemma A.11.** *Let $X$ be a separable Banach space and let $S'$ be the unit ball of the dual $X'$. Then $X'$ is separable for the weak topology $\sigma(X', X)$, see Brezis [2] p.62, that is there exists a sequence $\{x'_n\}$ in $S'$ s.t. for any $x' \in S'$ there exists a subsequence $\{x'_{n_k}\}$ s.t. for any $x \in X$ we have $\lim_{k\to\infty} \langle x'_{n_k}, x\rangle_{X'X} = \langle x', x\rangle_{X'X}$.*

*Proof.* Let $\{x_n\}$ be dense in $X$. For any $n$ consider

$$F_n : S' \to \mathbb{R}^n \text{ defined by } F_n(x') := (\langle x', x_1\rangle_{X'X}, ..., \langle x', x_n\rangle_{X'X}).$$

Since $\mathbb{R}^n$ is separable, and so is $F_n(S')$, there exists a sequence $\{x'_{n,k}\}_k$ s.t. $\{F_n(x'_{n,k})\}_k$ is dense in $F_n(S')$. Obviously $\{x'_{n,k}\}_{n,k}$ can be put into a sequence. For any $x' \in S'$ for any $n$ there is a $k_n$ s.t. $|\langle x' - x'_{n,k_n}, x_i\rangle_{X'X}| < 1/n$ for all $i \le n$. This implies that for any fixed $i$ we have $\lim_{n\to\infty} \langle x'_{n,k_n}, x_i\rangle_{X'X} = \langle x', x_i\rangle_{X'X}$. By density this holds for any $x \in X$. $\square$

**Proposition A.12** (Pettis's Theorem)**.** *Consider $f : I \to X$. Then $f$ is strongly measurable if and only if it is weakly measurable and almost separable valuable.*

*Proof.* The necessity has been already proved, so we focus on the sufficiency only. By modifying $f$ we can assume that $f(I)$ is separable. By replacing $X$ by a smaller space, we can assume that $X$ is separable.
Fix now $x \in X$. Then we claim that $t \to \|f(t) - x\|$ is measurable. Indeed for any $a > 0$

$$\{t \in I : \|f(t) - x\| \le a\} = \cap_{x'\in S'}\{t \in I : |\langle x', f(t) - x\rangle_{X'X}| \le a\}.\}$$

Using the fact that $S'$ is separable in the weak topology $\sigma(X', X)$ and the notation in Lemma A.11, we have

$$\{t \in I : \|f(t) - x\| \le a\} = \cap_{n\in\mathbb{N}}\{t \in I : |\langle x'_n, f(t) - x\rangle_{X'X}| \le a\}.$$

Since the set in the r.h.s. is measurable, we conclude that $t \to \|f(t)-x\|$ is measurable and so our claim is correct.
Consider now $n \ge 1$. Since $f(I)$ is separable there is a sequence of balls $\{B(x_j, \frac{1}{n})\}_{j\ge0}$ whose union contains $f(I)$. Set now

$$\begin{cases} \omega_0^{(n)} := \{t : f(t) \in B(x_0, \frac{1}{n})\}\,, \\ \omega_j^{(n)} := \{t : f(t) \in B(x_j, \frac{1}{n})\} \setminus \cup_{k<j}\, \omega_k^{(n)} \end{cases}$$

and

$$f_n(t) := \sum_{j=0}^{\infty} x_j \chi_{\omega_j^{(n)}}(t).$$

Notice that $\cup_{j\ge0}\omega_j^{(n)} = I$ and they are pairwise disjoint and measurable. By Example A.6 we know that $f_n : I \to X$ is strongly measurable. Furthermore, for any $t \in I$ there is a $j$ s.t. $t \in \omega_j^{(n)}$ and this implies

$$\frac{1}{n} > \|f(t) - x_j\| = \|f(t) - f_n(t)\|.$$

In other words, $\|f(t) - f_n(t)\| \leq 1/n$ for any $t \in I$. Then $f_n(t) \to f(t)$ for any $t$, and so by Proposition A.4 the function $f : I \to X$ is strongly measurable. □

*Example* A.13. Consider the map $f : (0,1) \to L^\infty(0,1)$ defined by $t \overset{f}{\to} \chi_{(0,t)}$. This map is not almost separable valued. Indeed $t \neq s$ implies $\|f(t) - f(s)\|_\infty = 1$. If $f$ was almost separable valued then there would exist a 0 measure subset $E$ in $(0,1)$ and a countable set $\mathcal{N} = \{t_n\}_n$ in $(0,1)\backslash E$ such that for any $t \in (0,1)\backslash(E \cup \mathcal{N})$ there would exist a subsequence $n_k$ with $f(t_{n_k}) \overset{k\to\infty}{\to} f(t)$ in $L^\infty(0,1)$. But this is impossible since $\|f(t) - f(t_{n_k})\|_\infty = 1$.
On the other hand $f : (0,1) \to L^2(0,1)$ defined in the same way, is strongly measurable. First of, since $L^2(0,1)$ is separable, it is almost separable valued. Next for any given any $w \in L^2(0,1)$ we have

$$\langle f(t), w\rangle_{L^2(0,1)} = \int_0^t w(x)dx$$

which is a continuous, and hence measurable, function. So $f$ is also weakly measurable and hence it is strongly measurable by Pettis Theorem.

Recall that in Remark A.2 we mentioned another possible notion of measurability, that is that $f : I \to X$ could be defined as measurable if $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$. We have the following fact.

**Proposition A.14.** *Consider $f : I \to X$. Then $f$ is strongly measurable $\Leftrightarrow$ it almost separably valuable and $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$.*

*Proof.* The "$\Leftarrow$" follows from the fact that for any $\mathfrak{a}$ open subset of $\mathbb{R}$ and for any $x' \in X$ the set $A = \{x \in X : \langle x, x'\rangle_{X,X'} \in \mathfrak{a}\}$ is open and for $g(t) := \langle f(t), x'\rangle_{X,X'}$ we have $f^{-1}(A) = g^{-1}(\mathfrak{a})$. So the latter being measurable it follows that $g$ is measurable and hence $f$ is weakly measurable. Hence by Pettis Theorem we conclude that $f$ is strongly measurable.
We now assume that $f$ is strongly measurable. We know from Lemma A.9 that $f$ is almost separably valuable. Let $U$ be an open subset of $X$. Let $(f_n)_n$ be a sequence in $C^0_c(I, X)$ with $f_n(t) \overset{n\to\infty}{\to} f(t)$ a.e. outside a 0 measure set $E \subset I$. Let $U_r = \{x \in X : \text{dist}(x, U^c) > r\}$. Then

$$f^{-1}(U)\backslash E = (\cup_{m\geq 1} \cup_{n\geq 1} \cap_{k\geq n} f_k^{-1}(U_{\frac{1}{m}}))\backslash E. \tag{A.4}$$

To check this, notice that if $t$ belongs to the left hand side , then $f(t) \in U_{\frac{1}{m_0}}$ for some $m_0$ and, since $f_n(t) \overset{n\to\infty}{\to} f(t)$, for $n$ large we have $f_k(t) \in U_{\frac{1}{m_1}}$ if $k \geq n$ for $m_1 > m_0$ preassigned. Viceversa if $t$ belongs to the right hand side, then there exist $n$ and $m$ s.t. $f_k(t) \in U_{\frac{1}{m}}$ for all $k \geq n$. Then by $f_k(t) \overset{k\to\infty}{\to} f(t)$ it follows that $f(t) \in \overline{U_{\frac{1}{m}}}$ with the latter a subset of $U$. This proves (A.4). Since the r.h.s. is a measurable set, this completes the proof. □

**Definition A.15** (Bochner integrability)**.** A strongly measurable function $f : I \to X$ is Bochner–integrable if there exists a sequence $(f_n(t))$ in $C_c(I, X)$ s.t.

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X dt = 0. \tag{A.5}$$

Notice that $\|f_n(t) - f(t)\|_X$ is measurable.

*Example* A.16. Consider the situation of Example A.13 of a Hilbert space $X$ with an orthonormal basis $\{e_t\}_{t\in\mathbb{R}}$ and the map $f : \mathbb{R} \to X$, which we saw is not strongly measurable and hence is not Bochner–integrable. Notice that $f$ is Riemann integrable in any compact interval $[a, b]$ with $\int_a^b f(t)dt = 0$.
To see this recall that the Riemann integral is, if it exists, the limit

$$\int_a^b f(t)dt = \lim_{|\triangle|\to 0} \sum_{I_j\in\triangle} f(t_j)|I_j| \text{ with } t_j \in I_j \text{ arbitrary}$$

where $\triangle$ varies among all possible decompositions of $[a, b]$ and $|\triangle| = \max_{I\in\triangle} |I|$. We have

$$\| \sum_{I_j\in\triangle} e_{t_j} |I_j| \|^2 = \sum_{j,k} \langle e_{t_j}, e_{t_k}\rangle |I_j||I_k| \le 2 \sum_j |I_j||\triangle| = 2|\triangle|(b - a) \overset{|\triangle|\to 0}{\to} 0.$$

**Proposition A.17.** *Let $f : I \to X$ be Bochner–integrable. Then there exists an $x \in X$ s.t. if $(f_n(t))$ is a sequence in $C_c(I, X)$ satisfying* (A.5) *then we have*

$$\lim_{n\to\infty} x_n = x \text{ where } x_n := \int_I f_n(t)dt. \tag{A.6}$$

*Proof.* First of all we check that $x_n$ is Cauchy. This follows immediately from (A.5) and from

$$\|x_n - x_m\|_X = \| \int_I (f_n(t) - f_m(t))dt\|_X \le \int_I \|f_n(t) - f_m(t))\|_X dt$$
$$\le \int_I \|f_n(t) - f(t))\|_X dt + \int_I \|f(t) - f_m(t))\|_X dt.$$

Let us set $x = \lim x_n$. Let $(g_n(t))$ be another sequence in $C_c(I, X)$ satisfying (A.5). Then $\lim \int_I g_n = x$ by

$$\| \int_I g_n(t)dt - x\|_X = \| \int_I (g_n(t) - f_n(t))dt + \int_I f_n(t)dt - x\|_X$$
$$\le \int_I \|g_n(t) - f_n(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt$$
$$\le \int_I \|g_n(t) - f(t)\|_X dt + \int_I \|f_n(t) - f(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt.$$

□

**Definition A.18.** Let $f : I \to X$ be Bochner–integrable and let $x \in X$ be the corresponding element obtained from Proposition A.17. The we set $\int_I f(t)dt = x$.

**Theorem A.19** (Bochner's Theorem)**.** *Let $f : I \to X$ be strongly measurable. Then $f$ is Bochner–integrable if and only if $\|f\|$ is Lebesgue integrable. Furthermore, we have*

$$\| \int_I f(t)dt\| \leq \int_I \|f(t)\|dt. \tag{A.7}$$

*Proof.* Let $f$ be Bochner–integrable. Then there is a sequence $(f_n(t))$ in $C_c(I, X)$ satisfying (A.5). We have $\|f\| \leq \|f_n\| + \|f - f_n\|$. Since both functions in the r.h.s. are Lebesgue integrable and $\|f\|$ is measurable it follows that$\|f\|$ is Lebesgue integrable.

Conversely let $\|f\|$ be Lebesgue integrable. Then there exist a sequence $(g_n(t))$ in $C_c(I, \mathbb{R})$ and $g \in L^1(I)$ s.t. $\int_I |g_n(t) - \|f(t)\||dt \to 0$ and $|g_n(t)| \leq g(t)$. In fact it is possible to choose such a sequence so that $\|g_n - g_m\|_{L^1(I)} < 2^{-n}$ for any $n$ and any $m \geq n$ (just by extracting an appropriate subsequence from a starting $g_n$ [3]). Then if we set

$$S_N(t) := \sum_{n=1}^{N} |g_n(t) - g_{n+1}(t)| \tag{A.8}$$

we have $\|S_N\|_{L^1(I)} \leq 1$. Since $\{S_N(t)\}_{N\in\mathbb{N}}$ is increasing, the limit $S(t) := \lim_{n\to+\infty} S_n(t)$ remains defined, is finite a.e. and $\|S\|_{L^1(I)} \leq 1$. Then $|g_n(t)| \leq |g_1(t)| + S(t) =: g(t)$ everywhere, where $g \in L^1(I)$. Notice that $\lim\limits_{n\to\infty} g_n(t)$ is convergent almost everywhere (it convergent in all points where $\lim_{n\to+\infty} S_n(t)$ is convergent). By dominated convergence it follows that this limit holds also in $L^1(I)$ and hence it is equal to $\|f\|$.
Let $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ a.e. (this sequence exists by the strong measurability of $f(t)$). Set

$$u_n(t) := \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t).$$

Notice that $(u_n(t))$ is in $C_c(I, X)$. We have

$$\|u_n(t)\| \leq \frac{|g_n(t)| \; \|f_n(t)\|}{\|f_n(t)\| + \frac{1}{n}} \leq |g_n(t)| \leq g(t).$$

We have (where the 2nd equality holds because because $\lim\limits_{n\to\infty} g_n(t) = \|f(t)\|$ and $\lim\limits_{n\to\infty} \|f_n(t)\| = \|f(t)\|$ a.e.)

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t) = \lim_{n\to\infty} f_n(t) = f(t) \text{ a.e.}.$$

[3] Suppose we start with a given $\{g_n\}$. Then for any $2^{-n}$ there exists $N_n$ s.t. $n_1, n_2 > N_n$ implies $\|g_{n_1} - g_{n_2}\|_{L^1(I)} < 2^{-n}$. Let now $\{\varphi(n)\}$ be a strictly increasing sequence in $\mathbb{N}$ s.t. $\varphi(n) > N_n$ for any $n$. Then $\|g_{\varphi(n)} - g_{\varphi(m)}\|_{L^1(I)} < 2^{-n}$ for any pair $m > n$. Rename $g_{\varphi(n)}$ as $g_n$.

Then we have

$$\lim_{n\to\infty} \|u_n(t) - f(t)\| = 0 \text{ a.e. with } \|u_n(t) - f(t)\| \le g(t) + \|f(t)\| \in L^1(I).$$

By dominated convergence we conclude

$$\lim_{n\to\infty} \int_I \|u_n(t) - f(t)\| dt = 0.$$

This implies that $f$ is Bochner–integrable. Finally, we have

$$\| \int_I f(t)dt\| = \lim_{n\to\infty} \| \int_I u_n(t)dt\| \le \lim_{n\to\infty} \int_I \|u_n(t)\|dt = \int_I \|f(t)\|dt.$$

□

**Corollary A.20** (Dominated Convergence)**.** *Consider a sequence $(f_n(t))$ of Bochner–integrable functions $I \to X$, $g : I \to \mathbb{R}$ Lebesgue integrable and let $f : I \to X$. Suppose that*

$$\|f_n(t)\| \le g(t) \text{ for all } n$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t.$$

*Then $f$ is Bochner–integrable with $\int_I f(t) = \lim_n \int_I f_n(t)$.*

*Proof.* By Dominated Convergence in $L^1(I, \mathbb{R})$ we have $\int_I \|f(t)\| = \lim_n \int_I \|f_n(t)\|$. By Proposition A.4, as a pointwise limit a.e. of a sequence of strongly measurable functions, $f$ is strongly measurable. By Bochner's Theorem $f$ is Bochner–integrable. By the triangular inequality

$$\limsup_n \| \int_I (f(t) - f_n(t))\| \le \lim_n \int_I \|f(t) - f_n(t)\| = 0$$

where the last inequality follows from $\|f(t) - f_n(t)\| \le \|f(t)\| + g(t)$ and the standard Dominated Convergence. □

**Definition A.21.** Let $p \in [1, \infty]$. We denote by $L^p(I, X)$ the set of equivalence classes of strongly measurable functions $f : I \to X$ s.t. $\|f(t)\| \in L^p(I, \mathbb{R})$. We set $\|f\|_{L^p(I,X)} := \|\|f\|\|_{L^p(I,\mathbb{R})}$.

**Proposition A.22.** *$(L^p(I, X), \| \|_{L^p})$ is a Banach space.*

*Proof.* The proof is similar to the case $X = \mathbb{R}$ , see [2].
(**Case** $p = \infty$). Let $(f_n)$ be Cauchy sequence in $L^\infty(I, X)$. For any $k \ge 1$ there is a $N_k$ s.t.

$$\|f_n - f_m\|_{L^\infty(I,X)} \le \frac{1}{k} \text{ for all } n, m \ge N_k.$$

So there exists an $E_k \subset I$ with $|E_k| = 0$ s.t.

$$\|f_n(t) - f_m(t)\|_X \le \frac{1}{k} \text{ for all } n, m \ge N_k \text{ and for all for } t \in I \backslash E_k.$$

Set $E := \cup_k E_k$. Then for any $t \in I\backslash E$ the sequence $(f_n(t))$ is convergent. So a function $f(t)$ remains defined with

$$\|f_n(t) - f(t)\|_X \leq \frac{1}{k} \text{ for all } n \geq N_k \text{ and for all for } t \in I\backslash E. \tag{A.9}$$

By Proposition A.4 the function $f$ is strongly measurable. By (A.9) we have $f \in L^\infty(I, X)$ and

$$\|f_n - f\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n \geq N_k$$

and so $f_n \to f$ in $L^\infty(I, X)$.

(**Case** $p < \infty$). Let $(f_n)$ be Cauchy sequence in $L^p(I, X)$ and let $(f_{n_k})$ be a subsequence with

$$\|f_{n_k} - f_{n_{k+1}}\|_{L^p(I,X)} \leq 2^{-k}.$$

Set now

$$g_l(t) = \sum_{k=1}^{l} \|f_{n_k}(t) - f_{n_{k+1}}(t)\|_X$$

Then

$$\|g_l\|_{L^p(I,\mathbb{R})} \leq 1.$$

By monotone convergence we have that $(g_l(t))_l$ converges a.e. to a $g \in L^p(I, \mathbb{R})$. Furthermore, for $2 \leq k < l$

$$\|f_{n_k}(t) - f_{n_l}(t)\|_X = \sum_{j=k}^{l-1} \|f_{n_j}(t) - f_{n_{j+1}}(t)\|_X \leq g(t) - g_{k-1}(t).$$

Then a.e. the sequence $(f_{n_k}(t))$ is Cauchy in $X$ for a.e. $t$ and so it converges for a.e. $t$ to some $f(t)$. By Proposition A.4 the function $f$ is strongly measurable. Furthermore,

$$\|f(t) - f_{n_k}(t)\|_X \leq g(t).$$

It follows that $f - f_{n_k} \in L^p(I, X)$, and so also $f \in L^p(I, X)$. Finally we claim $\|f - f_{n_k}\|_{L^p(I,X)} \to 0$. First of all we have $\|f(t) - f_{n_k}(t)\|_X \to 0$ for a.e. $t$ and

$$\|f(t) - f_{n_k}(t)\|_X^p \leq g^p(x)$$

by dominated convergence we obtain that $\|f - f_{n_k}\|_X \to 0$ in $L^p(I, \mathbb{R})$. Hence $f_{n_k} \to f$ in $L^p(I, X)$. □

**Proposition A.23.** *$C_c^\infty(I, X)$ is a dense subspace of $L^p(I, X)$ for $p < \infty$.*

*Proof.* We split the proof in two parts. We first show that $C_c^0(I,X)$ is a dense subspace of $L^p(I,X)$ for $p<\infty$. For $p=1$ this follows from the definition of integrable functions in Definition A.15. For $1<p<\infty$ going through the proof of Bochner's Theorem A.19, the functions $u_n$ considered in that proof can be taken to belong to $C_c^0(I,X)$ and converge to $f$ in $L^p(I,X)$.

The second part of the proof consists in showing that $C_c^\infty(I,X)$ is a dense subspace of $C_c^0(I,X)$ inside $L^p(I,X)$ for $p<\infty$. Let $f\in C_c^0(I,X)$. We consider $\rho\in C_c^\infty(\mathbb{R},[0,1])$ s.t. $\int\rho(x)dx=1$. Set $\rho_\epsilon(x):=\epsilon^{-1}\rho(x/\epsilon)$. Then for $\epsilon>0$ small enough $\rho_\epsilon * f\in C_c^\infty(I,X)$. We extend both $f$ and $\rho_\epsilon * f$ on $\mathbb{R}$ setting them 0 in $\mathbb{R}\backslash I$. In this way $\rho_\epsilon * f\in C_c^\infty(\mathbb{R},X)$ and $f\in C_c^0(\mathbb{R},X)$ and it is enough to show that $\rho_\epsilon * f\overset{\epsilon\to 0^+}{\to} f$ in $L^p(\mathbb{R},X)$..
We have

$$\rho_\epsilon * f(t)-f(t)=\int_{\mathbb{R}}(f(t-\epsilon s)-f(s))\rho(s)dy$$

so that, by Minkowski inequality and for $\Delta(s):=\|f(\cdot-s)-f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(t)-f(t)\|_{L^p}\le\int|\rho(s)|\Delta(\epsilon\ s)ds.$$

Now we have $\lim_{s\to 0}\Delta(s)=0$ and $\Delta(s)\le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0}\|\rho_\epsilon * f-f\|_{L^p}=\lim_{\epsilon\searrow 0}\int|\rho(s)|\Delta(\epsilon\ s)ds=0.$$

So

$$\lim_{\epsilon\searrow 0}\rho_\epsilon * f=f \text{ in } L^p(\mathbb{R},X). \tag{A.10}$$

□

Proceeding as in the previous proof, we can prove the following.

**Proposition A.24.** *Let $p\in[1,\infty)$ and $f\in L^p(\mathbb{R},X)$. Set*

$$T_hf(t)=h^{-1}\int_t^{t+h}f(s)ds \text{ for } t\in\mathbb{R} \text{ and } h\neq 0.$$

*Then $T_hf\in L^p(\mathbb{R},X)\cap L^\infty(\mathbb{R},X)\cap C^0(\mathbb{R},X)$ and $T_hf\overset{h\to 0}{\to} f$ in $L^p(\mathbb{R},X)$ and for almost every $t$.*

□

**Definition A.25.** We denote by $\mathcal{D}'(I,X)$ the space $\mathcal{L}(\mathcal{D}(I,\mathbb{R}),X)$.

**Corollary A.26.** *Let $f\in L^1_{loc}(I,X)$ be such that $f=0$ in $\mathcal{D}'(I,X)$. Then $f=0$ a.e.*

*Proof.* First of all we have $\int_J fdt=0$ for any $J\subset I$ compact. Indeed, let $(\varphi_n)\in\mathcal{D}(I)$ with $0\le\varphi_n\le 1$ and $\varphi_n\to\chi_J$ a.e. Then

$$\int_J fdt=\lim_{n\to+\infty}\int_J\varphi_n fdt=0$$

where we applied Dominated Convergence for the last equality.
Set now $\overline{f}(t) = f(t)$ in $J$ and $\overline{f}(t) = 0$ outside $J$. Then $T_h\overline{f} = 0$ for all $h > 0$. Then $\overline{f}(t) = 0$ for a.e. $t$. So $f(t) = 0$ for a.e. $t \in J$. This implies $f(t) = 0$ for a.e. $t \in \mathbb{R}$. $\square$

**Corollary A.27.** *Let $g \in L^1_{loc}(I, X)$, $t_0 \in I$, and $f \in C(I, X)$ given by $f(t) = \int_{t_0}^t g(s)ds$. Then:*

*(1) $f' = g$ in $\mathcal{D}'(I, X)$;*

*(2) $f$ is differentiable a.e. with $f' = g$ a.e.*

*Proof.* It is not restrictive to consider the case $I = \mathbb{R}$ and $g \in L^1(\mathbb{R}, X)$. We have

$$T_h g(t) = h^{-1} \int_t^{t+h} g(s)ds = \frac{f(t+h) - f(t)}{h}.$$

By Proposition A.24 $T_h g \overset{h\to 0}{\to} g$ for almost every $t$. This yields (2).

For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle f', \varphi \rangle = -\int_{\mathbb{R}} f(t)\varphi'(t)dt.$$

Furthermore

$$\lim_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \varphi'(t) \text{ in } L^\infty(\mathbb{R}).$$

So

$$\begin{aligned}\langle f', \varphi \rangle &= -\lim_{h\to 0} \int_{\mathbb{R}} f(t)\frac{\varphi(t+h) - \varphi(t)}{h} dt = -\lim_{h\to 0} \int_{\mathbb{R}} \varphi(t)\frac{f(t-h) - f(t)}{h} dt \\ &= -\lim_{h\to 0} \int_{\mathbb{R}} \varphi(t) T_{-h} g(t)dt = \langle g, \varphi \rangle.\end{aligned}$$

$\square$

**Definition A.28.** Let $p \in [1, \infty]$. We denote by $W^{1,p}(I, X)$ the space formed by the $f \in L^p(I, X)$ s.t. $f' \in \mathcal{D}(I, X)$ is also $f' \in L^p(I, X)$ and we set $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f'\|_{L^p}$.

**Lemma A.29.** *Let $u, g \in L^1(I, X)$ be such that*

$$\langle u(t_2), f \rangle_{XX^*} - \langle u(t_1), f \rangle_{XX^*} = \int_{t_1}^{t_2} \langle g(s), f \rangle_{XX^*} \, ds \text{ for any } f \in X^*.$$

*Then $\partial_t u = g$ in $\mathcal{D}'(I, X)$.*

*Proof.* We immediately obtain $\langle u(t), f \rangle_{XX^*} \in AC(I)$ with derivative $\partial_t \langle u(t), f \rangle_{XX^*} = \langle g(t), f \rangle_{XX^*}$. For any $\varphi \in \mathcal{D}(I)$ and any $f \in X^*$

$$\left\langle -\int_I u(t)\varphi'(t)dt, f \right\rangle_{XX^*} = -\int_I \langle u(t), f \rangle_{XX^*} \varphi'(t)dt = \int_I \langle g(t), f \rangle_{XX^*} \varphi(t)dt = \left\langle \int_I g(t)\varphi(t)dt, f \right\rangle_{XX^*}$$

which yields

$$-\int_I u(t)\varphi'(t)dt = \int_I g(t)\varphi(t) \text{ for all } \varphi \in \mathcal{D}(I)$$

and so $\partial_t u = g$ in $\mathcal{D}'(I, X)$.

□

# References


[1] Bahouri, Chemin, Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011.

[2] H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[3] T.Cazenave, Hareaux, *Semilinear Equations*, Oxford Univ.Press.

[4] Chemin, Desjardins, Gallagher, Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier-Stokes equations.* Oxford Lecture Series in Mathematics and its Applications, 32, The Clarendon Press, Oxford University Press, Oxford, 2006

[5] L. Escauriaza, G.A. Seregin and V. Shverak, *$L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness*, Uspekhi Mat. Nauk 58 (2003), no. 2(350), 3–44; translation in Russian Math. Surveys 58 (2003), no. 2, 211–250

[6] L. Evans and R.Gariepy, *Measure theory and fine properties of functions*, Taylor and Francis, Revised Edition (2015).

[7] M. Keel and T, Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

[8] L.Hörmander *The Analysis of Linear Partial Differential Operators I*, Second Edition, Springer (1990).

[9] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63 (1934), 193–248.

[10] E. Lieb and M. Loss *Analysis*, Second Edition, AMS (2001).

[11] W.S.Ozanski, B.C. Pooley, *Leray's fundamental work on the Navier-Stokes equations: a modern review of "Sur le mouvement d'un liquide visqueux emplissant l'espace"*, Partial differential equations in fluid mechanics, 113–203, London Math. Soc. Lecture Note Ser., 452, Cambridge Univ. Press, Cambridge, 2018.

[12] P.G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem,* Chapman & Hall/CRC Research Notes in Mathematics Series (2002).

[13] P.G. Lemarié-Rieusset, *The Navier-Stokes problem in the 21st century,* CRC Press, Boca Raton, FL, 2016CRC Press, Boca Raton, FL, (2016).

[14] J. C. Robinson, J. L. Rodrigo, W Sadowski, *The Three–Dimensional Navier–Stokes Equations,* Cambridge University Press, 2016.

[15] W.Rudin, *Real and Complex Analysis*, McGraw–Hill (1970).

[16] L. Schwartz, *Théorie des distributions. Tome I*, Actualités Sci. Ind., no. 1091, Publ. Inst. Math. Univ. Strasbourg 9, Hermann & Cie., Paris (1950).

[17] S. Sobolev (1936), *Méthode nouvelle a résoudre le probleme de cauchy pour les equations lineaires hyperboliques normales*, Rec. Math. [Mat. Sbornik] N.S. (43)(1936), 39–72.

[18] E.M.Stein, *Singular Integrals and differentiability properties of functions*, Princeton Un. Press (1970).

[19] E.M.Stein, *Harmonic Analysis*, Princeton Un. Press (1993).

[20] S. Takahashi, *On Interior Regularity Solutions for Weak Solutions of the Navier–Stokes Equations*, Manuscripta Math. 69 (1990), 237–254.