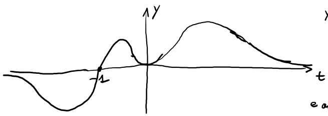
23 dicembre

$$f(x) = \int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt$$

studiarla

$$\begin{cases} e^{-\frac{1}{t^2}} & t \neq 0 \\ 0 & t = 0 \end{cases} \quad \text{è in } C^\infty(\mathbb{R})$$



$y = e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}$

$\Uparrow$

$L[0,x]$ $\forall x > 0$

e ad $L[x,0]$ $\forall x < 0$

Dominio $f = \mathbb{R}$, $f(x) = \int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt$ è ben definita $\forall\ x \in \mathbb{R}$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$x >> 1$

$$f(x) = \int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt = \int_0^1 e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt + \int_1^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt$$

$$\int_1^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt = \int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt + \underbrace{\int_1^x e^{-\frac{1}{t^2}} \frac{1}{1+t^2}\, dt}_{L_1(1+o(1))}$$

Per confronto, $0 < e^{-\frac{1}{t^2}} \frac{1}{1+t^2} < \frac{1}{1+t^2} \in L[1,+\infty)$

$\Rightarrow e^{-\frac{1}{t^2}} \frac{1}{1+t^2} \in L[1,+\infty)$, cioè $\exists$

$$\lim_{x \to +\infty} \int_1^x e^{-\frac{1}{t^2}} \frac{1}{1+t^2}\, dt = L_1 \in \mathbb{R}_+$$

$\int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt \qquad 0 < e^{-\frac{1}{t^2}} \frac{t}{1+t^2}$

$$\lim_{t \to +\infty} \frac{e^{-\frac{1}{t^2}} \frac{t}{1+t^2}}{\frac{1}{t}} = \lim_{t \to +\infty} \frac{t^2}{\cancel{1}+t^2} = 1 \in \mathbb{R}_+$$

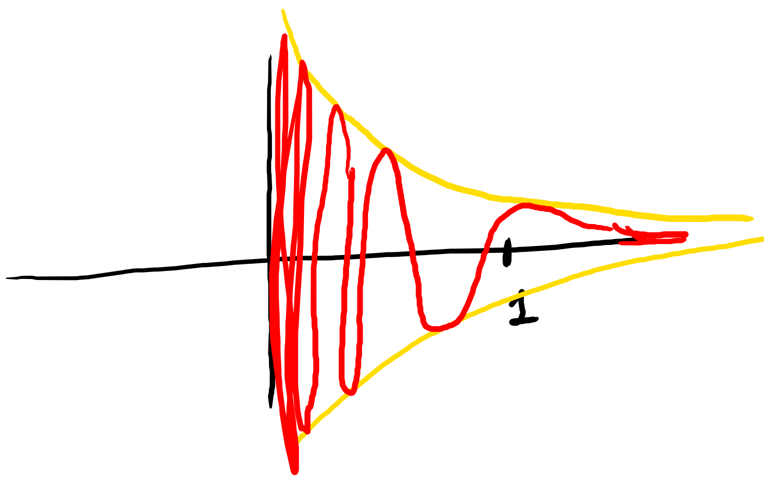
$\frac{1}{t} \notin L[1,+\infty) \Leftrightarrow e^{-\frac{1}{t^2}} \frac{t}{1+t^2} \notin L[1,+\infty)$

Anzi $\Rightarrow$ $\lim_{x \to +\infty} \int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt = +\infty$

$$f(x) = \int_0^1 e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt + L_1(1+o(1)) + \int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt \xrightarrow{x \to +\infty} +\infty$$

1) $\frac{1}{x}\sin\left(\frac{1}{x}\right)$ È integrabile in $[0,1]$ nel senso di Riemann (o Darboux)?

Non è integrabile per Riemann, perché non è una funzione limitata in $(0,1]$



$y = \frac{1}{x}\sin\left(\frac{1}{x}\right)$

$f(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right)$ in $L(0,1]$?

$$\int_1^{+\infty} \frac{\sin x}{x^p}\,dx$$

$p > 0$

esiste finito

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x} \sin\left(\frac{1}{x}\right) dx$$

$y = \frac{1}{x}$ $\quad x = \frac{1}{y}$

$dy = -\frac{1}{x^2}\,dx$

$dx = -x^2\,dy = -\frac{1}{y^2}\,dy$

$$\int_\varepsilon^1 \frac{1}{x} \sin\left(\frac{1}{x}\right) dx = -\int_{\frac{1}{\varepsilon}}^1 y \sin(y) \frac{1}{y^2}\,dy =$$

$$= \int_1^{\frac{1}{\varepsilon}} \frac{\sin(y)}{y}\,dy$$

$R = \frac{1}{\varepsilon}$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x} \sin\left(\frac{1}{x}\right) dx = \lim_{\varepsilon \to 0^+} \int_1^{\frac{1}{\varepsilon}} \frac{\sin(y)}{y}\,dy =$$

$$= \lim_{R \to +\infty} \int_1^R \frac{\sin(y)}{y}\,dy \in \int_1^{+\infty} \frac{\sin y}{y}\,dy \in \mathbb{R}$$

$\frac{1}{x} \sin\left(\frac{1}{x}\right) \in L(0,1]$

$\int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt$

$e^{-\frac{1}{t^2}} = 1-\frac{1}{t^2}+o(\frac{1}{t^2})$ $\frac{1}{1+y} = 1-\frac{y}{1+y}$

$e^x = 1+x+o(x)$

$\frac{t}{1+t^2} = \frac{t}{t^2} \quad \frac{1}{1+t^{-2}} = \frac{1}{t} \quad \frac{1}{1+t^{-2}} = \frac{1}{t}\left(1-\frac{t^{-2}}{1+t^{-2}}\right)$

$\int_1^x e^{-\frac{1}{t^2}} \frac{t}{1+t^2}\, dt = \int_1^x \frac{1}{t}\Big(1\underbrace{-\frac{1}{t^2}+o(\frac{1}{t^2})}_{o(\frac{1}{t})}\Big)\Big(1\underbrace{-\frac{t^{-2}}{1+t^{-2}}}_{o(\frac{1}{t})}\Big)\, dt$

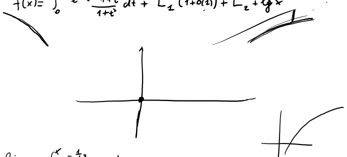
$= \int_1^x \frac{1}{t}\,(1+o(\frac{1}{t}))\,(1+o(\frac{1}{t}))\, dt$

$= \int_1^x \frac{1}{t}\,(1+o(\frac{1}{t}))\, dt = \underbrace{\int_1^x \frac{1}{t}\, dt}_{\lg x} + \underbrace{\int_1^x \frac{1}{t}\, o(\frac{1}{t})\, dt}_{L_2+o(1)} =$

$\frac{1}{t}\, o(\frac{1}{t}) \in L\,[1,+\infty)$

$\Rightarrow \exists\, L_2 \in \mathbb{R}$ t.c. $\lim_{x\to+\infty} \int_1^x \frac{1}{t}\, o(\frac{1}{t})\, dt = L_2$

$f(x) = \int_0^1 e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt + L_1\,(1+o(1)) + L_2 + \lg x$



$\lim_{x\to-\infty} \int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2} = +\infty$

$\int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt = \int_0^{-1} e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt + L_3 + o(1) + \lg|x|$

$f'(x) = e^{-\frac{1}{x^2}} \frac{1+x}{1+x^2}$ $\forall\, x \in \mathbb{R}$

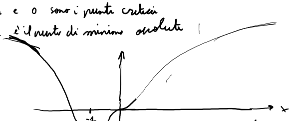
$f'(0) = 0$

$f'(x) = 0 \iff x = -1$,

$f'(x) > 0$ per $x > -1$, $f'(x) < 0$ per $x < -1$

-1 e 0 sono i punti critici

-1 è il punto di minimo assoluto



per $0 > x > -1$ e per $-1 \le x \le t \le 0$ ho $e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2} \ge 0$

$f(x) = \int_0^x e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt = -\int_x^0 e^{-\frac{1}{t^2}} \frac{1+t}{1+t^2}\, dt < 0$

$f'(x) = e^{-\frac{1}{x^2}} \frac{1+x}{1+x^2}$ $x \neq 0$

$f''(x) = e^{-\frac{1}{x^2}}\left[\frac{2}{x^3}\,\frac{1+x}{1+x^2} + \frac{1+x^2-(1+x)2x}{(1+x^2)^2}\right] =$

$= \frac{e^{-\frac{1}{x^2}}}{1+x^2}\left[\frac{2}{x^3}(1+x) + \frac{1-2x-x^2}{1+x^2}\right]$

$= \frac{e^{-\frac{1}{x^2}}}{(1+x^2)^2}\,\frac{1}{x^3}\left[(2+2x)(1+x^2) + x^3(1-2x-x^2)\right]$

$= \frac{e^{-\frac{1}{x^2}}}{(1+x^2)^2}\,\frac{1}{x^3}\left[-x^5-2x^4+3x^3+2x^2+2x+2\right]$