Esercizi 23 dicembre 2021

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Esercizi su ortonormalizzazione:

$\mathcal{B} = \{v_1, v_2, v_3\}$ base

g prodotto scalare

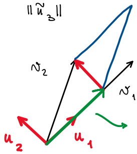
$$u_1 := \frac{v_1}{\|v_1\|}$$

$$\tilde{u}_2 := \underline{v_2 - \overbrace{g(v_2, u_1)}^{=\alpha} \cdot u_1} \quad , \quad g(\tilde{u}_2, u_1) = 0$$

$$u_2 := \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \underline{\frac{1}{\|\tilde{u}_2\|} \cdot \tilde{u}_2} \quad , \quad g\left(\frac{1}{\|\tilde{u}_2\|} \cdot \tilde{u}_2, u_1\right) = \frac{1}{\|\tilde{u}_2\|} g(\tilde{u}_2, u_1) = 0$$

$$\underline{\tilde{u}_3 := v_3 - g(v_3, u_1) \cdot u_1 - g(v_3, u_2) \cdot u_2}$$

$$u_3 := \frac{\tilde{u}_3}{\|\tilde{u}_3\|}$$



proiez. ortog. di v_2 su u_1

cerco $\alpha \in \mathbb{R}$ t.c.

il vettore $v_2 - \alpha \cdot u_1$ sia ortogonale a u_1, cioè:

$$g(v_2 - \alpha u_1, u_1) = 0$$

$$\Big(= g(v_2, u_1) - \alpha \underbrace{g(u_1, u_1)}_{= \|u_1\|^2 = 1} = 0$$

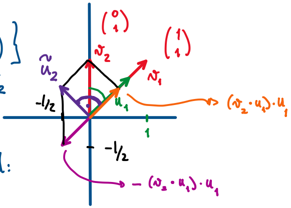
$$\Rightarrow \alpha = g(v_2, u_1)$$

OSS: v_1 e v_2 lin. INDIP.

$$\tilde{u}_2 := v_2 - g(v_2, u_1) u_1 \notin \mathrm{Span}(u_1)$$

Esempi numerici: in $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \underset{=v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{=v_2}{\begin{pmatrix}0\\1\end{pmatrix}} \right\}$$


$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}1\\1\end{pmatrix}$
v_2
v_1
$\tilde{u}_2$
$-1/2$
u_1
1
$-1/2$
$(v_2\cdot u_1)\cdot u_1$
$-(v_2\cdot u_1)\cdot u_1$

con prodotto standard:

$$\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = 1\cdot 0 + 1\cdot 1 = 1 \quad \neq 0$$

$$u_1 = \frac{v_1}{\|v_1\|}, \quad \|v_1\| = \left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$u_1 = \frac{v_1}{\sqrt{2}} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

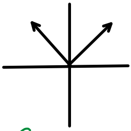
$$\tilde{u}_2 = v_2 - (v_2\cdot u_1)\,u_1 = \begin{pmatrix}0\\1\end{pmatrix} - 1/\sqrt{2}\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$v_2\cdot u_1 = \begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} = 0\cdot 1/\sqrt{2} + 1\cdot 1/\sqrt{2} = 1/\sqrt{2} \quad \neq 0$$

$$= \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\end{pmatrix}$$

$$\|\tilde{u}_2\| = \left\|\begin{pmatrix}-1/2\\1/2\end{pmatrix}\right\| = \sqrt{(-1/2)^2 + (1/2)^2} = \sqrt{\tfrac{1}{4}+1/4} = \sqrt{1/2} = \frac{1}{\sqrt{2}} < 1$$

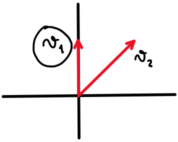
$$u_2 := \frac{1}{\|\tilde{u}_2\|}\cdot\tilde{u}_2 = \frac{1}{1/\sqrt{2}}\cdot\begin{pmatrix}-1/2\\1/2\end{pmatrix} = \sqrt{2}\cdot\begin{pmatrix}-1/2\\1/2\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$



OSS.: Se invece cons.

$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ e applico GS,

trovo un'altra base ortonormale:

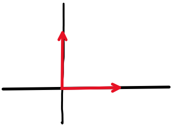


$$\| v_1 \| = \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \sqrt{0^2+1^2} = \sqrt{1} = 1$$

$u_1 := v_1$, $v_2 \cdot u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot 0 + 1 \cdot 1 = 1$

$$\tilde{u}_2 = v_2 - (v_2 \cdot u_1) \cdot u_1$$

$$\overset{!}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Base ortonorm. è

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$\neq \mathcal{E}$$

$\| \tilde{u}_2 \| = 1 \implies u_2 = \tilde{u}_2$

Esempio in $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \underset{= v_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{= v_2}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}, \underset{= v_3}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}} \right\} \text{ (es. LIN. INDIP)}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 2 = 3 \neq 0$$

$u_1 := \dfrac{v_1}{\| v_1 \|}$, $\| v_1 \| = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$$u_1 = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\tilde{u}_2 := \vartheta_2 - (\vartheta_2 \cdot u_1)\, u_1$$

$$\vartheta_2 \cdot u_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = 1 \cdot 1/\sqrt{3} + 0 \cdot 1/\sqrt{3} + 2 \cdot 1/\sqrt{3}$$

$$= 3/\sqrt{3} = \sqrt{3}$$

$$\tilde{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \sqrt{3} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\|\tilde{u}_2\| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$$

$$u_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{u}_3 := \vartheta_3 - (\vartheta_3 \cdot u_1) u_1 - (\vartheta_3 \cdot u_2) u_2$$

$$\vartheta_3 \cdot u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = 2/\sqrt{3} + 1/\sqrt{3} = \sqrt{3}$$

$$\vartheta_3 \cdot u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = -1/\sqrt{2}$$

$$\Rightarrow \tilde{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{3} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \left(-1/\sqrt{2}\right) \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$$

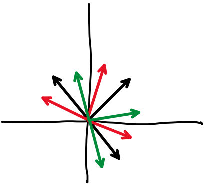
$$= \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix}$$

$$\|\tilde{u}_3\| = \sqrt{1 + 1/4 + 1/4} = \sqrt{3/2} = \sqrt{3}/\sqrt{2}$$

$$u_3 := \frac{1}{\sqrt{3}/\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$$

Basi ortonormali di $\mathbb{R}^2$: sono infinite



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Secondo punto es. 4 della simulazione:

$$r: \begin{cases} x = 2 + t \\ y = -1 - t \\ z = t \end{cases} \qquad s: \begin{cases} x = 4 + 3\tau \\ y = -3 + 2\tau \\ z = 2 \end{cases}$$

Sono complanari?

Sono parallele? $\quad W_r = \text{Span}\left(\left(\begin{matrix}1\\-1\\1\end{matrix}\right)\right)$

NO $\quad \nparallel$

$W_s = \text{Span}\left(\left(\begin{matrix}3\\2\\0\end{matrix}\right)\right)$

$2+2 = 4+3\cdot 0$ ✓

Sono incidenti? sì

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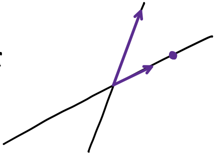
$$\textcircled{*}\begin{cases} 2+t = 4+3z \xRightarrow{} \\ \quad | \\ -1-t = -3+2z \Rightarrow 2z = -t+2 = -2+2 = 0 \\ t = 2 \Rightarrow t = 2 \end{cases} \qquad \Rightarrow z = 0$$

⇓

$\textcircled{*}$ è compatibile

⇒ sono complanari



$H \supseteq r, \quad H \supseteq s$

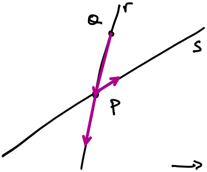
$$H: \begin{cases} x = 2 + t + 3\cdot s \\ y = -1 + (-t) + 2\cdot s \\ z = 0 + t + 0\cdot s \end{cases}$$

$$\begin{cases} x = 2+t+3s \\ y = -1-t+2s \\ z = t \end{cases}$$

ALTRO MODO: r e s sono complanari

$\Leftrightarrow$

$$\det\begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & -2 \\ 1 & 0 & 2 \end{pmatrix} = 0$$



$Q = (2, -1, 0)$

$P = (4, -3, 2)$

$$\overrightarrow{QP} = \begin{pmatrix} 4-2 \\ -3-(-1) \\ 2-0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

VERIFICA PASSANDO ATTRAVERSO
EQ. CARTESIANE

Eq. cartesiane per s:

$$\operatorname{rg}\begin{pmatrix} 3 & x-4 \\ 2 & y+3 \\ 0 & z-2 \end{pmatrix} = 1 \xrightarrow{II - \frac{2}{3} I} \begin{pmatrix} 3 & x-4 \\ 0 & y+3-\frac{2}{3}x+\frac{8}{3} \\ 0 & z-2 \end{pmatrix}$$

eq. cart.: $-\frac{2}{3}x + y + \frac{17}{3} = 0 \,/\cdot(-3)$

$$S: \begin{cases} 2x - 3y = 17 \\ z = 2 \end{cases}$$

eq. cart. per r: $\operatorname{rg}\begin{pmatrix} 1 & x-2 \\ -1 & y+1 \\ 1 & z \end{pmatrix} = 1$

$$\xrightarrow[\underline{III} - I]{\underline{II} + I} \begin{pmatrix} 1 & x-2 \\ 0 & y+1+x-2 \\ 0 & z-x+2 \end{pmatrix}$$

eq. cart.: $\begin{cases} x + y = 1 \\ -x + z = -2 \end{cases}$

$$r \quad \begin{cases} x + y = 1 \\ x - z = 2 \end{cases}$$

$$\begin{cases} x + y = 1 \\ x - z = 2 \\ 2x - 3y = 17 \\ z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 1 & 0 & -1 & 2 \\ 2 & -3 & 0 & 17 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{array}{c} \underline{\mathrm{II}} - \mathrm{I} \\ \rightsquigarrow \\ \underline{\mathrm{III}} - 2 \cdot \mathrm{I} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & \boxed{-5} & 0 & 15 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{array}{c} \rightsquigarrow \\ \underline{\mathrm{III}} - 5 \cdot \underline{\mathrm{II}} \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\rightsquigarrow \underline{\mathrm{IV}} - \frac{1}{5} \underline{\mathrm{III}}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{-1} & -1 & 1 \\ 0 & 0 & \boxed{5} & 10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

le rette non parallele

$\Uparrow$

$\mathrm{rg}\, A = 3 = \mathrm{rg}\,(A|b)$

$\xRightarrow{\text{RC}}$ sono incidenti

Auguro un sereno Natale e
Anno Nuovo a tutti
Valentina Beorchia