Ex 4,5

mercoledì 12 gennaio 2022 16:55

④ ⓐ $A = \begin{pmatrix} -3 & 0 & 0 \\ 2a & b & a \\ 10 & 0 & 2 \end{pmatrix}$, $a, b \in \mathbb{R}$.

QUANDO È DIAGONALIZZABILE?

STEP 1: POLINOMIO CARATTERISTICO E AUTOVALORI.

$$P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & 0 \\ 2a & b-\lambda & a \\ 10 & 0 & 2-\lambda \end{vmatrix} \underset{\substack{\uparrow \\ \text{SVILUPPO} \\ 1^a \text{ RIGA}}}{=}$$

$$= (-3-\lambda) \cdot \begin{vmatrix} b-\lambda & a \\ 0 & 2-\lambda \end{vmatrix} + 0 =$$

$$= (-3-\lambda)(b-\lambda) \cdot (2-\lambda).$$

-> AUTOVALORI $\lambda_1 = -3$, $\lambda_2 = b$, $\lambda_3 = 2$.

3 CASI

- i) $b = -3$
- ii) $b = 2$
- iii) $b \neq 2, -3$

i) $\boxed{b = -3} \Rightarrow m_A(2) = 1$, $m_A(-3) = 2$.

-> $1 \leq m_G(2) \leq m_A(2) = 1$ ✓.

$\underline{m_G(-3) = 2?}$ -> A DIAGONALIZZABILE

$m_G(-3) = 3 - rg(A-(-3)I).$

$$A+3I = \begin{pmatrix} 0 & 0 & 0 \\ 2a & 0 & a \\ 10 & 0 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - \frac{2a}{5}R_3]{} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10 & 0 & 5 \end{pmatrix}$$

$\Rightarrow rg(A+3I) = 1 \Rightarrow m_G(-3) = 2$ ✓.

ii) $\boxed{b=2} \to m_A(2)=2, \; m_A(-3)=1$

$\to 1 \leq m_G(-3) \leq m_A(-3) = 1 \to m_G(-3) = m_A(-3)$ ✓.

$\underline{m_G(2)=2?}$ $\to$ A DIAGONALIZZABILE.

$$A-2I = \begin{pmatrix} -5 & 0 & 0 \\ 2a & 0 & a \\ 10 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

$\nearrow a = 0 \Rightarrow rg(A-2I) = 1 \to m_G(2) = 3-1 = 2$

$\searrow a \neq 0 \to rg(A-2I) = 2 \to m_G(2) = 3-2 = 1$

$\to$ A DIAGONALIZZABILE $\Leftrightarrow a = 0.$

iii) $\boxed{b \neq -3, 2}$

$\rightsquigarrow$ PER OGNI AUTOVALORE $\lambda = -3, b, 2$ VALE

$1 \leq m_G(\lambda) \leq m_A(\lambda) = 1 \to m_G(\lambda) = m_A(\lambda)$

$\Rightarrow$ A DIAGONALIZZABILE ✓✓✓.

CONCLUSIONE:

A É DIAGONALIZZABILE SE E SOLO SE

$b \neq 2$ o $a = 0$.

ⓑ $C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ c & 0 & 0 & 0 \end{pmatrix}, \quad c \in \mathbb{R}.$

QUANDO É DIAGONALIZZABILE?

STEP 1: $P_C(\lambda)$ E AUTOVALORI

$$|C - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & c-\lambda & 0 \\ c & 0 & 0 & -\lambda \end{vmatrix} \underset{\substack{\uparrow \\ \text{SVILUPPO} \\ 2^a \text{ RIGA}}}{=} (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & c-\lambda & 0 \\ c & 0 & -\lambda \end{vmatrix} =$$

$$= (1-\lambda)(c-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ c & -\lambda \end{vmatrix} = (1-\lambda)(c-\lambda)(\lambda^2 - \lambda - c).$$

$\lambda^2 - \lambda - c = 0 \iff \lambda = \dfrac{1 \pm \sqrt{1+4c}}{2}$

$\rightsquigarrow$ HA RADICI REALI $\iff c \geq 1/4$.

$$\Rightarrow P_C(\lambda) = (1-\lambda)(c-\lambda)\left(\frac{1+\sqrt{1+4c}}{2} - \lambda\right)\left(\frac{1-\sqrt{1+4c}}{2} - \lambda\right)$$

$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = c, \ \lambda_3 = \frac{1+\sqrt{1+4c}}{2}, \ \lambda_4 = \frac{1-\sqrt{1+4c}}{2}.$$

- SE SONO TUTTI DISTINTI (E REALI $c \geq -1/4$) (VEDI ⓐ iii)) É DIAGONALIZZABILE!

ALTRIMENTI:

i) $\lambda_1 = \lambda_2$ $(c = 1)$ $\Rightarrow \lambda_3, \lambda_4 = \dfrac{1 \pm \sqrt{5}}{2}$

$\Rightarrow$ DA CONTROLLARE $m_G(1)=2$.

ii) $\lambda_1 = \lambda_3 ? \to 1 = \dfrac{1+\sqrt{1+4c}}{2} \to c=0$

$c=0 \to \lambda_4 = \dfrac{1-\sqrt{1}}{2} = 0$

$\rightsquigarrow \lambda_1=\lambda_3=1, \quad \lambda_2=\lambda_4=0$

$\rightsquigarrow$ DA CONTROLLARE $m_G(1)=2, \quad m_G(0)=2$.

iii) $\lambda_1=\lambda_4?$ IMPOSSIBILE, $\begin{cases} \lambda_1=1 \\ \lambda_4 = \dfrac{1}{2} - \dfrac{\sqrt{1+4c}}{2} < \dfrac{1}{2} \end{cases}$.

iv) $\lambda_2=\lambda_3?$

$c = \dfrac{1+\sqrt{1+4c}}{2} \to 2c-1 = \sqrt{1+4c}$

$$\Rightarrow \begin{cases} 2c-1>0 \\ (2c-1)^2 = 1+4c \end{cases} \Rightarrow \begin{cases} 4c^2-4c+1 = 1+4c \\ c > \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} 4c^2-8c=0 \\ c>\frac{1}{2} \end{cases} \Rightarrow \begin{cases} 4c(c-2)=0 \\ c>\frac{1}{2} \end{cases} \Rightarrow c=2.$$

$c=2 \Rightarrow \lambda_1=1, \quad \lambda_2=c=2, \quad \lambda_3=2, \quad \lambda_4=\dfrac{1-3}{2}=-1$

$\rightsquigarrow$ DA CONTROLLARE $m_G(2)=2$.

v) $\lambda_2=\lambda_4?$

$c = \underline{1+\sqrt{1+4c}} \to 2c-1 = -\sqrt{1+4c}$

2

$$\Rightarrow \begin{cases} 2c-1<0 \\ (2c-1)^2 = 1+4c \end{cases} \Rightarrow \begin{cases} 4c^2-4c+1 = 1+4c \\ c < \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} 4c^2-8c = 0 \\ c < \frac{1}{2} \end{cases} \Rightarrow \begin{cases} 4c(c-2) = 0 \\ c < \frac{1}{2} \end{cases} \Rightarrow c = 0.$$

→ COINCIDE COL CASO ii)

vi) $\underline{\lambda_3 = \lambda_4 ?}$

$$\frac{1+\sqrt{1+4c}}{2} = \frac{1-\sqrt{1+4c}}{2} \Longleftrightarrow 1+4c=0 \Longleftrightarrow c=-\frac{1}{4}.$$

$\rightarrow \lambda_1 = 1, \quad \lambda_2 = c = -\frac{1}{4}, \quad \lambda_3 = \lambda_4 = \frac{1}{2}$

→ DA CONTROLLARE $m_G(\frac{1}{2}) = 2.$

SOLUZIONI

i) $c=1 \rightarrow m_G(1) = 2?$

$$C - I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \Rightarrow rg(C-I) = 2$$

$\Rightarrow m_G(1) = 4-2 = 2 \checkmark. \rightarrow$ C DIAGONALIZZABILE

ii) $c=0 \rightarrow m_G(0) = 2, \quad m_G(1) = 2?$

$$\bullet \underset{\overset{\uparrow}{= C-0I}}{C} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow rg(C) = 2 \Rightarrow m_G(0) = 4-2 = 2 \checkmark$$

- $C - I = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{2 COLONNE} \\ \text{NON NULLE} \\ \text{L.I.}}}{\Rightarrow} rg(C - I) = 2 \Rightarrow m_G(1) = 2 \checkmark$

$\rightarrow$ C DIAG $\checkmark$.

iv) $c = 2 \rightarrow m_G(2) = 2$?

$$C - 2I = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} \rightarrow rg(C - 2I) = 3$$

$\Rightarrow m_G(2) = 4 - 3 = 1 < 2 = m_A(2)$

$\Rightarrow$ C NON DIAGONALIZZABILE

vi) $c = -\frac{1}{4} \rightarrow m_G(\frac{1}{2}) = 2$?

$$C - \frac{1}{2}I = \begin{pmatrix} 1/2 & 0 & 1 & 1 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -3/4 & 0 \\ -1/4 & 0 & 0 & -1/2 \end{pmatrix} \rightarrow rg(C - \frac{1}{2}I) = 3$$

$\Rightarrow m_G(\frac{1}{2}) = 4 - 3 = 1 < m_A(\frac{1}{2}) = 2$

$\Rightarrow$ C NON DIAGONALIZZABILE.

CONCLUSIONE

C È DIAGONALIZZABILE $\Longleftrightarrow \begin{cases} c > -1/4 \\ c \neq 2 \end{cases}$.

⑤ $A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. DIAGONALIZZABILE IN $\mathbb{R}, \mathbb{C}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$?

STEP 1 (COMUNE): $P_A(\lambda)$?

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}.$$

SARRUS: $\begin{bmatrix} 1-\lambda & 1 & -1 & 1-\lambda & 1 \\ -1 & -\lambda & 1 & -1 & -\lambda \\ 1 & 1 & -\lambda & 1 & 1 \end{bmatrix}$

$$\rightarrow P_A(\lambda) = \lambda^2(1-\lambda) + 1 + 1 - (1-\lambda) - \lambda - \lambda$$
$$= \lambda^2(1-\lambda) + 2(1-\lambda) - (1-\lambda)$$
$$= (1-\lambda)(\lambda^2+1).$$

STEP 2: RADICI DI $P_A(\lambda)$ E SOLUZIONE

- $\boxed{\text{IN } \mathbb{C}}$

$$(1-\lambda)(\lambda^2+1) = (1-\lambda)(-i-\lambda)(i-\lambda).$$

→ TUTTE RADICI DISTINTE → A DIAGONALIZZABILE ✓.

- $\boxed{\text{IN } \mathbb{R}}$

λ^2+1 NON HA SOLUZIONI ⇒ A NON DIAG.BILE ✗.

- $\boxed{\text{IN } \mathbb{Z}_2}$

$\lambda^2+1 \underset{\text{IN } \mathbb{Z}_2}{\overset{\uparrow}{=}} (\lambda+1)^2 \underset{\text{IN } \mathbb{Z}_2}{\overset{\uparrow}{=}} (\lambda-1)^2 \Rightarrow 1$ AUTOVALORE, $m_A(1)=2$.

$\underline{m_G(1)=2?}$

$$A-I = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \overset{\text{IN } \mathbb{Z}_2}{\overset{\downarrow}{=}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\Rightarrow rg(A-I)=2 \Rightarrow m_G(1)=3-2=1<2$

$\Rightarrow$ A NON DIAG.BILE X.

- $\boxed{\text{IN } \mathbb{Z}_3}$

$q(\lambda)=\lambda^2+1 \to q(0)=1,\ q(1)=1+1=2,$

$(P_A(\lambda)=(1-\lambda)\cdot q(\lambda))$ $\quad q(2) \underset{\text{IN } \mathbb{Z}_3}{\overset{\uparrow}{=}} q(-1) = 1+1=2$

$\Rightarrow q(\lambda)$ NON HA RADICI IN $\mathbb{Z}_3$

$\Rightarrow P_A(x)$ NON HA 3 RADICI IN $\mathbb{Z}_3$

$\Rightarrow$ A NON DIAG.BILE X

- $\boxed{\text{IN } \mathbb{Z}_5}$

$q(\lambda)=\lambda^2+1 \to q(0)=1,\ q(1)=2,\ q(2)=5=0$

$q(3)=q(-2)=5=0 \qquad q(4)=q(-1)=2$

$\to$ RADICI $\lambda_1=2,\ \lambda_2=3.$

$\to P_A(\lambda) \underset{\text{IN } \mathbb{Z}_5[\lambda]}{\overset{\uparrow}{=}} (1-\lambda)(2-\lambda)(3-\lambda)$

HA 3 RADICI DISTINTE $\Rightarrow$ A É DIAG.BILE ✓✓✓.