ANOMALIES & PATH INTEGRAL

Consideriamo $\quad \mathcal{L} = \overline{\Psi}(i\not{D} - m)\Psi \qquad \not{D} = \gamma^\mu(\partial_\mu + iA_\mu)$

fermioni
di Dirac in
rap. R

abeliano o
non-ab. $(A_\mu = A_\mu^a t_R^a)$

In dimensioni d ← PARI

Passiamo all'Euclidea (P.I. è meglio definito)

$$x^0 = -i x^4 \qquad \partial_0 = i\partial_d \qquad \gamma_M^0 = -i\gamma_E^d \qquad \gamma_M^i = \gamma_E^i$$

$$A_0 = iA_d \qquad\qquad\qquad\qquad i = 1, \dots, d-1$$

$$\{\gamma_M^\mu, \gamma_M^\nu\} = 2\eta^{\mu\nu} \underset{+---}{} \rightarrow \{\gamma_E^\mu, \gamma_E^\nu\} = -2\delta^{\mu\nu}$$

$$\not{D}_M = \gamma_M^\mu \partial_\mu = \gamma_M^0 \partial_0 + \gamma_M^i \partial_i = \gamma_E^d \partial_d + \gamma_E^i \partial_i = \not{D}_E$$

$$(\gamma_M^5)^2 = \mathbb{1} \qquad \{\gamma_M^5, \gamma_M^\mu\} = 0 \quad \rightarrow \quad \gamma_M^5 = -(i)^{\frac{d}{2}+1}\gamma^0\gamma^1\cdots\gamma^{d-1}$$

Controlliamo $(\gamma_M^5)^2 = (-1)^{\frac{d}{2}+1}\gamma^0\overbrace{\gamma^1\cdots\gamma^{d-1}}^{d-1}\underbrace{\gamma^0}_{(\gamma^0)^2=1}\gamma^1\cdots\gamma^{d-1} =$

$$= (-1)^{\frac{d}{2}+1}(-1)^{d-1}\gamma^1\cdots\gamma^{d-1}\gamma^1\cdots\gamma^{d-1} =$$

$$= (-1)^{\frac{d}{2}+1}(-1)^{d-1}(-1)^{d-1}(-1)^{\sum_{k=1}^{d-2}k} \overbrace{}^{d-1}\underbrace{(\frac{d-2+1})d-2}{2} $$

$$= (-1)^{\frac{d}{2}+1+\frac{d^2-3d}{2}+1} \underset{(\gamma_E^i)^2}{} = (-1)^{\frac{d^2}{2}-d} \underset{\text{pari}}{\uparrow} = (-1)^{\frac{d^2}{2}} = 1$$

d pari $\Rightarrow d^2 \in 4\mathbb{Z}$

$$\gamma_E^5 = -(i)^{\frac{d}{2}+1}(-i)\gamma_E^d\gamma_E^1\cdots\gamma_E^{d-1} = -i^{d/2}(-1)^{d-1}\gamma_E^1\cdots\gamma_E^d =$$

$$= (-i)^{3d/2}\gamma_E^1\cdots\gamma_E^d \qquad\qquad \underset{1/i \cdot 2d-2}{\uparrow}$$

Consistentemente con $(\gamma_E^\mu)^2 = -1$ possiamo prendere γ_E^μ ANTI-HERM.

$$\Rightarrow \not{D}^\dagger = (\gamma^\mu)^\dagger(-\partial_\mu - iA_\mu^\dagger) = \gamma^\mu(\partial_\mu + iA_\mu) = \not{D}$$

$$\Rightarrow \not{D} \text{ è HERMITIANO! } \leftarrow \text{q.to vale in Euclideo (non in Mink.)}$$

$$iS_M = i \int d^d x \, \bar{\Psi}(i\not{D}_M - m)\psi = i(-i) \int d^d x_E \, \bar{\Psi}(i\not{D}_E - m)\psi =$$

$$= - \int d^d x_E \, \bar{\Psi}(-i\not{D}_E + m)\psi = -S_E$$

Consideriamo il funzionale generatore (def in Eucl.)

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\Psi} \, e^{\int dx \, \bar{\Psi}(i\not{D} - m)\psi}$$

$$\stackrel{!}{=} \det(i\not{D} - m)$$

$\mathcal{L}$ è invariante sotto trasf. assiali (globali) se $m = 0$

Troviamo le WI relative alla simm. (globale) assiale :

consideriamo trasf. assiali locali

$$\psi \mapsto \psi' = e^{i\beta(x)\gamma_5}\psi \qquad \bar{\Psi} \mapsto \bar{\Psi}' = \bar{\Psi} e^{i\beta(x)\gamma_5} \quad \text{(\textasteriskcentered)} \quad \beta = \beta^a \hat{t}_R$$

$$\bar{\Psi}(i\not{D} - m)\psi \mapsto \bar{\Psi}'(i\not{D} - m)\psi' = \bar{\Psi}(i\not{D} - m)\psi - 2im\beta\,\bar{\Psi}\gamma^5\psi$$

$$\underbrace{}_{\text{trasf. infinitesima}} \qquad\qquad - (\partial_\mu \beta)\underbrace{\bar{\Psi}\gamma^\mu\gamma^5\psi}_{j_A^\mu}$$

$$S' = S - \int dx \, \beta(x)\left[\partial_\mu j_{A(x)}^\mu - 2im\, P(x) \right]$$

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\Psi} \, e^{-S[\psi,\bar{\Psi},A]}$$

cambiamo nome delle variab. integr. →
$$= \int \mathcal{D}\psi' \mathcal{D}\bar{\Psi}' \, e^{-S[\psi',\bar{\Psi}',A]}$$

cambiamo coordinate usando (\textasteriskcentered)
$$\to \stackrel{!}{=} \int \mathcal{D}\psi \mathcal{D}\bar{\Psi} \, J[\beta,A] \, e^{-S[\psi,\bar{\Psi},A]} \, e^{\int dx\,\beta(x)\left(\partial_\mu j_A^\mu - 2imP\right)}$$

tipicamente lo Jacobiano è assunto essere $= 1$ (misura invariante)

e se pto è uno ottenimbus WI compatibili

con ciò che uno si aspetta classicamente.

Ma in questo caso (trasf. assiale) $J \neq 1$ e questo

porta a una VIOLAZIONE delle WI, cioè a un'ANOMALIA.

Riscriviamo lo Jacobiano come (vedi dopo)

$$J[\beta, A] = e^{-\int dx \, \beta(x) \, \mathcal{A}[A](x)}$$

Prendendo $\beta \ll 1$:

$$Z[A] = \int D\psi \, D\bar{\psi} \, e^{-S} \left(1 + \int dx \, \beta(x) \left[\partial_\mu j_A^\mu - 2im P - \mathcal{A}[A] \right] \right)$$

$$\Big\downarrow$$

$$= Z[A] + \int dx \, \beta(x) \left\langle \partial_\mu j_A^\mu - 2im P - \mathcal{A}[A] \right\rangle_A$$

$$\Rightarrow \quad \partial_\mu \left\langle j_A^\mu \right\rangle_A = 2im \left\langle P \right\rangle + \mathcal{A}[A] \qquad \begin{array}{c} \text{ANOMALOUS} \\ \text{A WI} \end{array}$$

Metodo di Fujikawa per calcolare lo Jacobiano:

$\not{D}$ è HERM. $\rightarrow$ autovalori REALI

$$\not{D} \, \varphi_n = \lambda_n \, \varphi_n \qquad \lambda_n \in \mathbb{R} \qquad\qquad \varphi_n(x) = \langle x | n \rangle$$

Il set di autofunz. è ortonormale e completo:

$$\int dx \, \varphi_m^\dagger(x) \, \varphi_n(x) = \int dx \, \langle m | x \rangle \langle x | n \rangle = \langle m | n \rangle = \delta_{mn}$$

$$\sum_n \varphi_n(x) \, \varphi_n^\dagger(y) = \delta(x-y)$$

Espandiamo ψ e $\bar{\psi}$ in autofunz. di $\not{D}$

$$\psi(x) = \sum_n a_n \, \varphi_n(x) = \sum_n a_n \, \langle x | n \rangle$$

matrici di cambio
di base

$$\bar{\psi}(x) = \sum_m \varphi_m^\dagger(x) \, \bar{b}_m = \sum_m \langle m | x \rangle \, \bar{b}_m$$

$a_n, \bar{b}_m$ sono variabili di Grassmann

φ_n sono spinori di Dirac in rap. R del gruppo G

Prendiamo la misura:

$$\mathcal{D}\psi\,\mathcal{D}\bar\psi = \prod_n da_n \prod_m d\bar b_m \left(\underbrace{\det \langle m|x\rangle \cdot \det \langle x|n\rangle}_{\det\langle m|n\rangle\,=\,1}\right)^{-1}$$

numeri
d'Grassmann

$$= \prod_n da_n \prod_m d\bar b_m$$

Azione:

$$\int dx\,\bar\psi\,(i\slashed{D}-m)\psi = \sum_n (i\lambda_n - m)\,\bar b_n a_n$$

$$\longrightarrow \det(i\slashed{D}-m) = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\; e^{\int dx\,\bar\psi(i\slashed{D}-m)\psi} = \int \prod_n da_n d\bar b_n\; e^{\sum_n (i\lambda_n - m)\bar b_n a_n}$$

$$= \prod_n (i\lambda_n - m)$$

Vediamo come trasformano a_n e $\bar b_m$ sotto trasf. assiale.

$$\psi'(x) = \sum_n a_n'\,\varphi_n(x)$$

$$\bar\psi'(x) = \dots$$
$$= \dots$$

$$= (1 + i\beta(x)\gamma_5)\sum_m a_m\,\varphi_m(x)$$

$\Downarrow$ usiamo o.n. della base $\{\varphi_n\}$

$$a_n' = \sum_n C_{nm}\,a_m$$

$$\bar b_m' = \sum_k C_{km}\,\bar b_k$$

con $\quad C_{nm} = \delta_{nm} + i\int dx\,\beta(x)\,\varphi_n^\dagger\,\gamma_5\,\varphi_m$

(Variabili di Grassmann)

Nel caso della QED,
b trasforma con $(\bar C^{-1})^t$
$[\;\rightarrow \mathcal{D}\psi'\,\mathcal{D}\bar\psi' = \mathcal{D}\psi\,\mathcal{D}\bar\psi\;]$

$$\prod_n da_n' = (\det C)^{-1}\prod_n da_n$$

$$\prod_m d\bar b_m' = (\det C)^{-1}\prod_m d\bar b_m$$

$$\Rightarrow \quad \mathcal{D}\psi'\mathcal{D}\bar{\psi}' = (\det C)^{-2}\,\mathcal{D}\psi\mathcal{D}\bar{\psi} \quad\Rightarrow\quad J[\beta,A] = (\det C)^{-2}$$

Ora $\quad \det C = e^{Tr \ln C} \quad$ e $\quad \ln(1+\xi) = \xi + \mathcal{O}(\xi^2)$

$$\downarrow$$

$$J[\beta,A] = e^{-2Tr \ln C} = e^{-2Tr \ln\left(\delta_{nm} + i\int dx\,\beta(x)\,\varphi_n^\dagger(x)\,\gamma_5\,\varphi_m(x)\right)} \simeq$$

$$\simeq e^{-2\,Tr\, i\int dx\,\beta(x)\,\varphi_n^\dagger(x)\,\gamma_5\,\varphi_m(x)}$$

$$= e^{-2i\int dx\,\beta(x)\,\sum_n \varphi_n^\dagger(x)\,\gamma_5\,\varphi_n(x)}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{Mal definita :}}$$

$$\underbrace{\sum_n \varphi_n^\dagger\,\gamma_5\,\varphi_n}_{\substack{\text{spin.}\\\text{ind.}}} = \mathrm{tr}\left(\gamma_5 \underbrace{\sum_n \varphi_n(x)\varphi_n^\dagger(x)}_{\delta(x-x)}\right) \sim \mathrm{tr}\,\gamma_5 \cdot \delta(0)$$

$$\text{``}\,0\,\cdot\,\infty\,\text{''}$$

$$\longrightarrow \text{necessità di REGOLARIZZARE}$$

Idea di Fujikawa: usare un CUTOFF GAUSSIANO

$$\sum_n \varphi_n^\dagger(x)\,\gamma_5\,\varphi_n(x) = \lim_{M\to\infty}\; \sum_n \varphi_n^\dagger(x)\,\gamma_5\,e^{-\lambda_n^2/M^2}\,\varphi_n(x)$$

$$= \lim_{M\to\infty}\; \sum_n \varphi_n^\dagger(x)\,\gamma_5\,e^{-\slashed{D}^2/M^2}\,\varphi_n(x)$$

Si può generalizzare la regolarizzazione:

$$\sum_n \varphi_n^\dagger\,\gamma_5\,\varphi_n \equiv \lim_{M\to\infty}\; \sum_n \varphi_n^\dagger\,\gamma_5\,f\left(\slashed{D}^2/M^2\right)\,\varphi_n$$

con $f(y)$ t.c. $f^{(n)}(y) \xrightarrow[y\to\infty]{} 0 \qquad \forall n = 0,1,2,\ldots$

e $f(0) = 1$

Calcoliamo la somma regolarizzata nello sp. di Fourier :

$$\varphi_n(x) = \int \frac{dk}{(2\pi)^d} e^{ikx} \tilde{\varphi}_n(k)$$

$v^T A w = \text{tr } v^T A w =$
$= \text{tr } A w v^T$

$$"\sum_n \varphi_n^+ \gamma_5 \varphi_n" = \lim_{M \to \infty} \int \frac{dk}{(2\pi)^d} \frac{dq}{(2\pi)^d} \sum_n \tilde{\varphi}_n^+(q) e^{-iqx} \gamma_5 f\left(\frac{\not{D}^2}{M^2}\right) e^{ikx} \tilde{\varphi}_n(k)$$

$$= \lim_{M \to \infty} \int \frac{dk}{(2\pi)^d} \frac{dq}{(2\pi)^d} e^{-iqx} \text{tr } \gamma_5 f\left(\frac{\not{D}^2}{M^2}\right) \underbrace{\sum_n \tilde{\varphi}_n(k) \tilde{\varphi}_n^+(q)}_{(2\pi)^d \delta(k-q)} e^{ikx}$$

traccia su indici
spinoriali e di rap. R

$$= \lim_{M \to \infty} \int \frac{dk}{(2\pi)^d} e^{-ikx} \text{tr}\left[\gamma_5 f\left(\frac{\not{D}^2}{M^2}\right)\right] e^{ikx}$$

$$\not{D}^2 = \gamma^\mu \gamma^\nu D_\mu D_\nu = \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) D_\mu D_\nu =$$

$$= -D^2 + \frac{i}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}$$

Inoltre $\quad e^{-ikx} F(\partial_\mu) e^{ikx} = F(\partial_\mu + ik_\mu)$

$$"\sum_n \varphi_n^+ \gamma_5 \varphi_n" = \lim_{M \to \infty} \int \frac{dk}{(2\pi)^d} \text{tr}\left[\gamma_5 f\left(-\frac{(D_\mu + ik_\mu)(D_\mu + ik_\mu)}{M^2} + i\frac{\gamma^\mu \gamma^\nu F_{\mu\nu}}{2M^2}\right)\right]$$

$$= \lim_{M \to \infty} \int \frac{dk}{(2\pi)^d} \text{tr}\left[\gamma_5 f\left(\frac{k^2}{M^2} - \frac{2ik_\mu D_\mu + D^2}{M^2} + i\frac{\gamma^\mu \gamma^\nu F_{\mu\nu}}{2M^2}\right)\right]$$

- $\text{tr } \gamma_5 = 0$, $\quad \text{tr } \gamma_5 \gamma^{\mu_1} \cdots \gamma^{\mu_\ell} = 0 \quad$ se $\ell \neq d$,

  $\text{tr } \gamma_5 \gamma^{\mu_1} \cdots \gamma^{\mu_d} = i^{\frac{3d}{2}} 2^{d/2} \epsilon^{\mu_1 \cdots \mu_d} \quad (\epsilon^{12\cdots d} = 1)$

- espandiamo f in Taylor attorno a k^2/M

$$= \lim_{\eta\to\infty} \int \frac{dk}{(2\pi)^d}\, \frac{i^{-\frac{3d}{2}}\, 2^{d/2}}{\left(\frac{d}{2}\right)!\,(2\eta^2)^{d/2}}\, f^{(d/2)}\!\left(k^2/\eta^2\right)\, \epsilon^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{d/2}\nu_{d/2}} \cdot tr\; iF_{\mu_1\nu_1}\cdots iF_{\mu_{\frac{d}{2}}\nu_{\frac{d}{2}}}$$

$k \to \eta k$

$$= \lim_{\eta\to\infty} \int \frac{dk}{(2\pi)^d}\, \frac{i^{-\frac{3d}{2}}\, i^{d/2}}{\left(\frac{d}{2}\right)!}\, f^{(d/2)}\!\left(k^2\right)\, \epsilon^{\mu_1\cdots\nu_{d/2}}\, tr\; F_{\mu_1\nu_1}\cdots F_{\mu_{d/2}\nu_{d/2}} + O(1/\eta)$$

$$= \epsilon^{\mu_1\cdots\nu_{d/2}}\, tr\; F\cdots F\; \frac{1}{(d/2)!} \underbrace{\int \frac{d^dk}{(2\pi)^d}\, f^{(d/2)}(k^2)}$$

coor. polari

$r=k^2$
$dr=2k\,dk$

$$\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{k^{d-1}\,dk}{(2\pi)^d}\, f^{(d/2)}(k^2)$$

$$\frac{\pi^{d/2}}{\left(\frac{d}{2}-1\right)!} \int_0^\infty \frac{r^{d/2-1}\,dr}{(2\pi)^d}\, f^{(d/2)}(r) = \frac{(-1)^{d/2}}{2^d\,\pi^{d/2}} \int$$

$$\frac{1}{(n-1)!} \int_0^\infty r^{n-1} f^{(n)}(r)\,dr = \frac{1}{(n-1)!} \int_0^\infty \left[\underbrace{\frac{d}{dr}\left(r^{n-1} f^{(n-1)}\right)}_{=0} - (n-1) r^{n-2} f^{(n-1)}(r) \right] dr$$

$$= -\frac{(n-1)}{(n-1)!} \int_0^\infty r^{n-2} f^{(n-1)}\,dr = \cdots = (-1)^{n-1}\frac{(n-1)!}{(n-1)!} \int_0^\infty f'(r)\,dr =$$

$$= (-1)^{n-1} f\Big|_0^\infty = (-1)^n$$

$$"\sum_n \Phi_n^\dagger \gamma_5 \Phi_n" = \epsilon^{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}}\, tr\; F_{\mu_1\nu_1}\cdots F_{\mu_{d/2}\nu_{d/2}}\; \frac{(-1)^{d/2}}{2^d\left(\frac{d}{2}\right)!\,\pi^{d/2}}$$

$$J[\beta,A] = e^{-2i\int dx \beta(x) \sum_n \varphi_n^+ \gamma_5 \varphi_n} \qquad \left[J[\beta,A] = e^{-\int dx\, \beta(x)\, A[A](x)} \right]$$

$$\Rightarrow A[A] = 2i\; \epsilon^{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}}\; tr\, F_{\mu_1\nu_1}\cdots F_{\mu_{d/2}\nu_{d/2}}\; \frac{(-1)^{d/2}}{2^d\left(\frac{d}{2}\right)!\,\pi^{d/2}}$$

$$\begin{cases} d=4 \quad \dfrac{1}{16\cdot 2\cdot\pi^2} = \dfrac{1}{32\pi^2} \\[2mm] d=2 \quad \dfrac{-1}{4\cdot 1\cdot\pi} = \dfrac{-1}{4\pi} \end{cases}$$

Euclideo
$\downarrow$

Minkowski
$\downarrow$

Per $d=4$

$$A[A] = \frac{i}{16\pi^2}\; \epsilon^{\mu\nu\alpha\beta}\; tr\, F_{\mu\nu} F_{\alpha\beta} \longrightarrow -\frac{1}{16\pi^2}\; \epsilon^{\mu\nu\alpha\beta}\; tr\, F_{\mu\nu} F_{\alpha\beta}$$

(lezione 22; singlet anom.)

risultati trovati col calcolo dei correlatori

(lezione 20)

Per $d=2$

$$A[A] = -\frac{i}{2\pi}\; \epsilon^{\mu\nu}\, F_{\mu\nu} \longrightarrow \frac{1}{2\pi}\; \epsilon^{\mu\nu}\, F_{\mu\nu} \qquad (\epsilon^{01}=+1)$$

caso abeliano

$$\partial_0 = i\partial_d$$
$$A_0 = iA_d$$

Continuazione a Minkowski

$d=4$:
$$\underset{\substack{\text{EUCL} \\ \epsilon_E^{1234}=+1}}{\epsilon^{\mu\nu\alpha\beta}}\, tr\, F_{\mu\nu} F_{\alpha\beta} = 4!\,(-)\, tr\, F_{41} F_{23} = 4!\, i\, tr\, F_{01} F_{23} =$$
$$= i\, \underset{\text{MINK.}}{\epsilon^{\mu\nu\alpha\beta}}\, tr\, F_{\mu\nu} F_{\alpha\beta}$$

$d=2$:
$$\underset{\substack{\text{EUCL.} \\ \epsilon_E^{12}=+1}}{\epsilon^{\mu\nu}\, F_{\mu\nu}} = 2F_{12} = 2(\partial_1 A_2 - \partial_2 A_1) = -2i(\partial_1 A_0 - \partial_0 A_1) =$$
$$= 2i(\partial_0 A_1 - \partial_1 A_0) = 2i\, F_{01} = +i\, \underset{\text{MINK.}}{\epsilon^{\mu\nu}\, F_{\mu\nu}}$$

convenzione
$\epsilon^{01} = +1$