1) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ diagonalizzazione

$$p_A = \begin{vmatrix} 1-x & 2 \\ 0 & 1-x \end{vmatrix} = (1-x)^2 \rightsquigarrow \lambda = 1 \qquad m_a(\lambda) = 2$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad m_g(\lambda) = 1 \qquad A \text{ non diagonalizzabile su } \mathbb{C}$$

<u>SVD</u> $\quad B = {}^t\!A A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

$$p_B = \begin{vmatrix} 1-x & 2 \\ 2 & 5-x \end{vmatrix} = (1-x)(5-x) - 4 = x^2 - 6x + 1$$

$$\lambda_{1,2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

$$\lambda_1 = 3 + 2\sqrt{2} \qquad \begin{pmatrix} -2-2\sqrt{2} & 2 \\ 2 & 2-2\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$x - (\sqrt{2}-1)\, y = 0 \qquad v_1 = \begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix} \qquad \|v_1\| = \sqrt{4-2\sqrt{2}}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 - 2\sqrt{2} \qquad \begin{pmatrix} -2+2\sqrt{2} & 2 \\ 2 & 2+2\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$x + (1+\sqrt{2})\, y = 0 \qquad v_2 = \begin{pmatrix} \sqrt{2}+1 \\ -1 \end{pmatrix} \qquad \|v_2\| = \sqrt{4+2\sqrt{2}}$$

$$u_2 = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} \sqrt{2}+1 \\ -1 \end{pmatrix}$$

$$U = (u_1 \;\; u_2) \in O(2)$$

$$\mu_1 = \sqrt{\lambda_1} = \sqrt{3+2\sqrt{2}}, \quad \mu_2 = \sqrt{\lambda_2} = \sqrt{3-2\sqrt{2}}$$

$$A v_1 = \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix} \qquad w_1 = \frac{A v_1}{\|A v_1\|} = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix}$$

$$A v_2 = \begin{pmatrix} \sqrt{2}-1 \\ -1 \end{pmatrix} \qquad w_2 = \frac{A v_2}{\|A v_2\|} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} \sqrt{2}-1 \\ -1 \end{pmatrix}$$

$$W = (w_1 \ \ w_2) \in O(2)$$

$$A = W D \, {}^t U \quad , \quad D = \begin{pmatrix} \sqrt{3+2\sqrt{2}} & 0 \\ 0 & \sqrt{3-2\sqrt{2}} \end{pmatrix}$$

2) $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \quad , \ \theta \in \mathbb{R}$

Trovare $U \in O(2)$, $D \in M_2(\mathbb{R})$ diagonale t.c. $D = U^{-1} S_\theta U$

$\sin\theta = 0 \quad (\theta = k\pi, \ k \in \mathbb{Z}) \implies S_{k\pi} = (-1)^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Supponiamo $\theta \neq k\pi \ \forall k \in \mathbb{Z} \iff \sin\theta \neq 0$

$$p_{S_\theta} = \begin{vmatrix} \cos\theta - x & \sin\theta \\ \sin\theta & -\cos\theta - x \end{vmatrix} = x^2 - \cos^2\theta - \sin^2\theta = x^2 - 1$$

$\lambda_1 = 1, \quad \lambda_2 = -1$

$\lambda_1 = 1: \quad \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$(\cos\theta - 1)x + \sin\theta \, y = 0 \qquad v_1 = \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix}$$

$$\|v_1\| = \sqrt{\sin^2\theta + (1-\cos\theta)^2} = \sqrt{2(1-\cos\theta)}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2(1-\cos\theta)}} \begin{pmatrix} \sin\theta \\ 1 - \cos\theta \end{pmatrix}$$

$\lambda_2 = -1: \quad \begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$(\cos\theta+1)\,x+\sin\theta\,y=0 \qquad v_2=\begin{pmatrix}-\sin\theta\\ 1+\cos\theta\end{pmatrix}$$

$$\|v_2\|=\sqrt{\sin^2\theta+(1+\cos\theta)^2}=\sqrt{2(1+\cos\theta)}$$

$$u_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{2(1+\cos\theta)}}\begin{pmatrix}-\sin\theta\\ 1+\cos\theta\end{pmatrix}$$

$U=(u_1\ \ u_2)\in O(2)$ matrice con colonne u_1, u_2.

$$D=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

3) Trovare equazioni cartesiane di $U=\text{Span}(u,v)\subset\mathbb{R}^4$

$$u=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix},\quad v=\begin{pmatrix}0\\3\\1\\0\end{pmatrix}.$$

$X=su+tv\,,\quad s,t\in\mathbb{R}$

$$\begin{cases}x_1=s\\ x_2=3t\\ x_3=-2s+t\\ x_4=s\end{cases}\qquad\begin{cases}s=x_1\\ t=\frac{1}{3}x_2\\ x_3=-2x_1+\frac{1}{3}x_2\\ x_4=x_1\end{cases}$$

$$\begin{cases}2x_1-x_2+3x_3=0\\ x_1-x_4=0\end{cases}$$ equazioni cartesiane di U.