Ex 3,4, 5

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EX3

$W \subseteq \mathbb{R}^4$, $W = \{ \underline{x} \in \mathbb{R}^4 \mid x_1 + x_3 - 3x_4 = x_1 + x_2 - x_3 = 0 \}$

$\langle \cdot, \cdot \rangle$ PRODOTTO SCALARE STANDARD.

BASE ON DI W?

STEP 1: BASE DI W

$$\begin{cases} x_1 + x_3 - 3x_4 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases} \rightsquigarrow \begin{cases} x_3 = x_1 + x_2 \\ x_4 = \frac{1}{3}(x_1 + x_3) = \frac{1}{3}(2x_1 + x_2) \end{cases}$$

$\rightarrow W = \{ \underline{x} \in \mathbb{R}^4 \mid \underline{x} = (x_1, x_2, x_1 + x_2, \frac{2}{3}x_1 + \frac{1}{3}x_2) \}$

$\rightarrow$ BASE DI $W = \left((1, 0, 1, \frac{2}{3}), (0, 1, 1, \frac{1}{3}) \right)$.

STEP 2: RICAVIAMO BASE ON. (GRAM-SCHMIDT)

$v_1 = (1, 0, 1, \frac{2}{3})$, $v_2 = (0, 1, 1, \frac{1}{3})$

$\rightarrow$ COSTRUIAMO UNA BASE O.N. DI W (u_1, u_2)

$u_1 = \frac{v_1}{\|v_1\|}$.

$\|v_1\| = \sqrt{1^2 + 0^2 + 1^2 + \left(\frac{2}{3}\right)^2} = \sqrt{2 + \frac{4}{9}} = \sqrt{\frac{22}{9}} =$

$= \frac{\sqrt{22}}{3}$.

$\rightarrow u_1 = \frac{3}{\sqrt{22}} v_1 = \left(\frac{3}{\sqrt{22}}, 0, \frac{3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right)$.

$v_2' = v_2 - \underbrace{\langle v_2, u_1 \rangle u_1}$ PROIEZIONE ORTOGONALE DI v_2 SU $\langle u_1 \rangle$

$\rightarrow \langle v_2', u_1 \rangle = 0 \rightarrow$ ORTOGONALE

$\rightsquigarrow u_2 = \frac{v_2'}{\|v_2'\|}$ ✓.

$\rightsquigarrow \langle v_2, u_1 \rangle = 0 \cdot \frac{3}{\sqrt{22}} + 1 \cdot 0 + 1 \cdot \frac{3}{\sqrt{22}} + \frac{1}{3} \cdot \frac{2}{\sqrt{22}}$

$= \frac{3}{\sqrt{22}} + \frac{2}{3} \cdot \frac{1}{\sqrt{22}} = \frac{3 + \frac{2}{3}}{\sqrt{22}} = \frac{11}{3\sqrt{22}}$

$\rightarrow v_2' = (0, 1, 1, \frac{1}{3}) - \frac{11}{3\sqrt{22}} \left(\frac{3}{\sqrt{22}}, 0, \frac{3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right)$

$= (0, 1, 1, \frac{1}{3}) - \left(\frac{11}{22}, 0, \frac{11}{22}, \frac{2}{3} \frac{11}{22} \right) =$

$= (0, 1, 1, \frac{1}{3}) - \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3} \right) = \left(-\frac{1}{2}, 1, \frac{1}{2}, 0 \right)$

$\rightarrow \|v_2'\| = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{3}{2}}$

$\rightarrow u_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{2}}{\sqrt{3}} \left(-\frac{1}{2}, 1, \frac{1}{2}, 0 \right) = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0 \right)$.

($1 = \frac{\sqrt{2}}{\sqrt{2}}$)

• COMPLETO A UNA BASE O.N. DI $\mathbb{R}^4$?

STEP 1: COMPLETO A UNA BASE DI $\mathbb{R}^4$.

$u_1 = \left(\frac{3}{\sqrt{22}}, 0, \frac{3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right)$, $u_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0 \right)$

$\rightarrow u_1, u_2, e_2 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$ BASE DI $\mathbb{R}^4$

(INFATTI LA MATRICE $\begin{pmatrix} 3/\sqrt{22} & 0 & 3/\sqrt{22} & 2/\sqrt{22} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ HA DETERMINANTE $\neq 0$ $\rightarrow$ RANGHE L.I. ✓)

• SCELTI e_2, e_4 PER FAR APPARIRE + ZERI POSSIBILI.

STEP 2: ORTONORMALIZZIAMO

$\{u_1\}$ O.N. ✓ $\{u_1, u_2\}$ O.N. ✓

$\{u_1, u_2, e_2\}$ O.N.?

$\langle u_2, e_2 \rangle = \frac{2}{\sqrt{6}} \neq 0 \rightarrow$ NO. DOBBIAMO ORTONORMALIZZARE

$\rightsquigarrow e_2' = e_2 - \big(\underbrace{\langle e_2, u_1 \rangle u_1 + \langle e_2, u_2 \rangle u_2}_{\text{PROJ. ORTOGONALE SU } \langle u_1, u_2 \rangle} \big)$

$\Rightarrow \langle e_2', u_1 \rangle = \langle e_3', u_2 \rangle = 0$

$\Rightarrow u_3 = \frac{e_2'}{\|e_2'\|}$.

$\langle e_2, u_1 \rangle = 0$, $\langle e_2, u_2 \rangle = \frac{2}{\sqrt{6}}$

$\rightarrow e_2' = (0, 1, 0, 0) - \frac{2}{\sqrt{6}} \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0 \right)$

$= \left(\frac{2}{6}, 1 - \frac{4}{6}, -\frac{2}{6}, 0 \right) = \frac{1}{3}(1, 1, -1, 0)$

$\rightarrow \|e_2'\| = \frac{1}{3} \sqrt{1^2 + 1^2 + (-1)^2 + 0^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

$\rightarrow u_3 = \frac{e_2'}{\|e_2'\|} = \frac{\sqrt{3}}{3}(1, 1, -1, 0) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{3}, 0 \right)$

$\{u_1, u_2, u_3, e_4\}$ O.N.?

$\langle e_4, u_1 \rangle = \frac{2}{\sqrt{22}} \neq 0 \rightarrow$ DA ORTONORMALIZZARE.

$e_4' = e_4 - \underbrace{\langle e_4, u_1 \rangle}_{= 2/\sqrt{22}} u_1 - \underbrace{\langle e_4, u_2 \rangle}_{=0} u_2 - \underbrace{\langle e_4, u_3 \rangle}_{=0} u_3$

$= (0, 0, 0, 1) - \frac{2}{\sqrt{22}} \left(\frac{3}{\sqrt{22}}, 0, \frac{3}{\sqrt{22}}, \frac{2}{\sqrt{22}} \right) = \left(-\frac{6}{22}, 0, -\frac{6}{22}, 1 - \frac{4}{22} \right)$

$= \left(-\frac{3}{11}, 0, -\frac{3}{11}, \frac{9}{11} \right) = \frac{3}{11}(-1, 0, -1, 3)$

$\rightarrow \|e_4'\| = \frac{3}{11} \sqrt{(-1)^2 + 0^2 + (-1)^2 + 3^2} = \frac{3}{11}\sqrt{11} = \frac{3}{\sqrt{11}}$

$\Rightarrow u_4 = \frac{e_4'}{\|e_4'\|} = \frac{\sqrt{11}}{3} \cdot \frac{3}{11}(-1, 0, -1, 3) = \frac{1}{\sqrt{11}}(-1, 0, -1, 3)$.

$\rightsquigarrow (u_1, u_2, u_3, u_4)$ BASE O.N. DI $\mathbb{R}^4$ ✓✓✓.

(4) $x_0 = (1 + \sqrt{2}) e_1 + e_3 = (1 + \sqrt{2}, 0, 1)$, $y_0 = -e_1 + e_2 = (-1, 1, 0) \in \mathbb{R}^3$.

$\langle x, y \rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 2 x_2 y_2 - x_1 y_3 - x_3 y_1 + 3 x_3 y_3$.

$\|x_0\|$? ANGOLO CONVESSO TRA x_0 E y_0?

• $\|x_0\| = \sqrt{\langle x_0, x_0 \rangle} = \sqrt{(1+\sqrt{2})^2 + 2(1+\sqrt{2}) \cdot 0 + 2 \cdot 0^2 - 2(1+\sqrt{2}) \cdot 1 + 3 \cdot 1^2} =$

$= \sqrt{1 + 2 + 2\sqrt{2} - 2 - 2\sqrt{2} + 3} = \sqrt{4} = 2$. ✓✓

• COSENO DELL'ANGOLO TRA x_0 E y_0 $= \frac{\langle x_0, y_0 \rangle}{\|x_0\| \cdot \|y_0\|}$.

$\|y_0\| = \sqrt{\langle y_0, y_0 \rangle} = \sqrt{(-1)^2 + 2 \cdot (-1) \cdot 1 + 2 \cdot 1^2 - 2(-1) \cdot 0 + 3 \cdot 0^2} =$

$= \sqrt{1 - 2 + 2 + 0 + 0} = 1$

$\langle x_0, y_0 \rangle = (1+\sqrt{2}) \cdot (-1) + (1+\sqrt{2}) \cdot (1) + 0 \cdot (-1) + 2 \cdot 0 \cdot 1$
$- (1+\sqrt{2}) \cdot 0 - 1 \cdot (-1) + 3 \cdot 1 \cdot 0 = 1$

$\rightarrow \cos(x_0, y_0) = \frac{1}{2 \cdot 1} = \frac{1}{2} \rightarrow$ ANGOLO$(x_0, y_0) = \frac{\pi}{3}$ ✓✓✓.

(5) $\exists$ PRODOTTO SCALARE SU $\mathbb{R}^2$ t.c.:

$$\begin{cases} \langle (1, -1), (2, 3) \rangle = 5 \\ \langle (4, -4), (-4, -6) \rangle = -1 \end{cases} \quad ?$$

SUPPONIAMO ESISTA. ALLORA:

$-1 = \langle \underbrace{(4, -4)}_{= 4 \cdot (1, -1)}, \underbrace{(-4, -6)}_{= -2 \cdot (2, 3)} \rangle = 4 \cdot (-2) \cdot \langle (1, -1), (2, 3) \rangle =$

$= 4 \cdot (-2) \cdot 5 = -40$. CHIARAMENTE ASSURDO ✗