Ex5

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$$\langle x,y\rangle := x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 - x_1y_3 - x_3y_1 + 3x_3y_3$$

PRODOTTO SCALARE SU $\mathbb{R}^3$.

1) $v_1 = (1+\sqrt{2})e_1 + e_3$, $v_2 = -e_1 + e_2$.
$\rightarrow \|v_1\|, \|v_2\|$ $\theta(v_1,v_2)$?

2) $W = \langle (1,-1,1), (3,1,1) \rangle$.
BASE O.N. DI W? COMPLETARE A BASE O.N. DI $\mathbb{R}^3$

PRELIMINARE:

$\mathcal{E}$ BASE CANONICA

$$\Rightarrow M_{\mathcal{E}}(\langle \cdot,\cdot \rangle) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \left(\langle e_i, e_j \rangle \right)_{i,j}$$

1) $$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{\langle (1+\sqrt{2})e_1 + e_3, (1+\sqrt{2})e_1 + e_3 \rangle} =$$

$$= \sqrt{\langle (1+\sqrt{2})e_1, (1+\sqrt{2})e_1\rangle + 2\langle (1+\sqrt{2})e_1, e_3\rangle + \langle e_3, e_3 \rangle} =$$

$$= \sqrt{(1+\sqrt{2})^2 \underbrace{\langle e_1, e_1\rangle}_{=1} + 2(1+\sqrt{2})\underbrace{\langle e_1, e_3\rangle}_{=-1} + \underbrace{\langle e_3, e_3\rangle}_{=3}} =$$

$$= \sqrt{3+2\sqrt{2} - 2 - 2\sqrt{2} + 3} = \sqrt{4} = 2.$$

$$\|v_2\| = \sqrt{\langle e_1 - e_2, e_1 - e_2\rangle} = \sqrt{\underbrace{\langle e_1, e_1\rangle}_{=1} - 2\underbrace{\langle e_1, e_2\rangle}_{=1} + \underbrace{\langle e_2, e_2\rangle}_{=2}}$$

$$= \sqrt{1-2+2} = \sqrt{1} = 1.$$

$$\cos\theta(v_1,v_2) = \frac{\langle v_1, v_2\rangle}{\|v_1\| \cdot \|v_2\|} = \frac{1}{2\cdot 1}\langle v_1, v_2\rangle = \frac{1}{2}\langle v_1, v_2\rangle.$$

$$\langle v_1, v_2\rangle = \langle (1+\sqrt{2})e_1 + e_3, e_1 - e_2\rangle =$$

$$= (1+\sqrt{2})\underbrace{\langle e_1, e_1\rangle}_{=1} - (1+\sqrt{2})\underbrace{\langle e_1, e_2\rangle}_{=1} + \underbrace{\langle e_1, e_3\rangle}_{=-1} - \underbrace{\langle e_2, e_3\rangle}_{=0}$$

$$= -1$$

$$\Rightarrow \cos\theta(v_1,v_2) = -\frac{1}{2} \Rightarrow \theta(v_1,v_2) = \frac{2\pi}{3}. ✓✓✓$$

2) $W = \langle w_1, w_2 \rangle$. $w_1 = (1,-1,1)$, $w_2 = (3,1,1)$.

ORTONORMALIZZIAMO:

CALCOLIAMO I PRODOTTI SCALARI CHE CI SERVIRANNO:

- $\langle w_1, w_1\rangle = 1\cdot 1 + 1\cdot(-1) + (-1)\cdot 1 + 2(-1)\cdot(-1) - 1\cdot 1 - 1\cdot 1 + 3\cdot 1\cdot 1 =$
$= 1 - 1 - 1 + 2 - 1 - 1 + 3 = 2$
- $\langle w_1, w_2\rangle = 1\cdot 3 + 1\cdot 1 + (-1)\cdot 3 + 2(-1)\cdot 1 - 1\cdot 1 - 1\cdot 3 + 3\cdot 1\cdot 1$
$= 3 + 1 - 3 - 2 - 1 - 3 + 3 = -2$
- $\langle w_2, w_2\rangle = 3^2 + 2\cdot 3\cdot 1 + 2\cdot 1^2 - 2\cdot 3\cdot 1 + 3\cdot 1^2 =$
$= 9 + 6 + 2 - 6 + 3 = 14$

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{(1,-1,1)}{\sqrt{\langle w_1, w_1\rangle}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

$$w_2' = w_2 - \langle w_2, u_1\rangle u_1 = w_2 - \frac{1}{\|w_1\|^2}\langle w_2, w_1\rangle w_1$$

$$= (3,1,1) - \frac{1}{2}\cdot(-2)\cdot(1,-1,1) = (3,1,1) + (1,-1,1)$$

$$= (4,0,2)$$

$$\rightarrow u_2 = \frac{w_2'}{\|w_2'\|}.$$

$$\|w_2'\| = \sqrt{\langle w_2', w_2'\rangle} = \sqrt{4^2 + 2\cdot 4\cdot 0 + 2\cdot 0^2 - 2\cdot 4\cdot 2 + 3\cdot 2^2} =$$

$$= \sqrt{16 + 0 + 0 - 16 + 12} = \sqrt{12} = 2\sqrt{3}$$

$$\Rightarrow u_2 = \left(\frac{4}{\sqrt{12}}, 0, \frac{2}{\sqrt{12}}\right) = \left(\frac{2}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right).$$

COMPLETIAMO A UNA BASE O.N. DI $\mathbb{R}^3$?

$$W^\perp = \left(\text{SOLUZIONI DI } \begin{cases} \langle x, w_1\rangle = 0 \\ \langle x, w_2\rangle = 0 \end{cases} \right).$$

$$\langle x, \underset{(1,-1,1)}{w_1}\rangle = x_1 - x_1 + x_2 - 2x_2 - x_1 - x_3 + 3x_3$$
$$= -x_1 - x_2 + 2x_3$$

$$\langle x, \underset{(3,1,1)}{w_2}\rangle = 3x_1 + x_1 + 3x_2 + 2x_2 - x_1 - 3x_3 + 3x_3$$
$$= 3x_1 + 5x_2$$

$$\Rightarrow \begin{cases} -x_1 - x_2 + 2x_3 = 0 \\ 3x_1 + 5x_2 = 0 \end{cases} \rightarrow \begin{cases} x_3 = \frac{x_1}{2} + \frac{x_2}{2} \\ x_2 = -\frac{3}{5}x_1 \end{cases}$$

$$\rightarrow \begin{cases} x_3 = \frac{5}{10}x_1 - \frac{3}{10}x_1 = \frac{2}{10}x_1 = \frac{1}{5}x_1 \\ x_2 = -\frac{3}{5}x_1 \end{cases}$$

$$\Rightarrow W^\perp = \left\{ \left(x_1, -\frac{3}{5}x_1, \frac{1}{5}x_1\right) \in \mathbb{R}^3 \,\middle|\, x_1 \in \mathbb{R} \right\}$$

$$= \left\langle \left(1, -\frac{3}{5}, \frac{1}{5}\right) \right\rangle = \langle \underbrace{(5,-3,1)}_{w_3} \rangle.$$

$$\|w_3\| = \sqrt{5^2 + 2\cdot(-3)\cdot 5 + 2\cdot(-3)^2 - 2\cdot 1\cdot 5 + 3\cdot 1^2} =$$

$$= \sqrt{25 - 30 + 18 - 10 + 3} = \sqrt{6}$$

$$\Rightarrow u_3 = \frac{w_3}{\|w_3\|} = \left(\frac{5}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

$$\Rightarrow \{ \underbrace{u_1, u_2}_{\text{BASE O.N. DI } W}, \underbrace{u_3}_{\text{BASE O.N. DI } W^\perp} \} \text{ BASE O.N. DI } \mathbb{R}^3 \text{ ///}$$

c) GIUSTIFICARE L'AFFERMAZIONE CHE SI TRATTA DI UN PRODOTTO SCALARE

$\langle x,y\rangle$ É SCRITTO IN COMPONENTI COME $\sum a_{ij} x_i y_j$, QUINDI É BILINEARE. ✓

INOLTRE, LA SUA MATRICE ASSOCIATA VISTA ALL'INIZIO É SIMMETRICA, QUINDI $\langle x,y\rangle$ É SIMMETRICO ✓✓

INFINE, DIMOSTRIAMO CHE $\langle x,x\rangle \geq 0$ E $\langle x,x\rangle = 0 \iff x = \underline{0}$:

$$\langle x,x\rangle = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 3x_3^2 =$$

$$= x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 3x_3^2 - 2x_2x_3 + 2x_2x_3 =$$

$$= (x_1 + x_2 - x_3)^2 + x_2^2 + 2x_3^2 = \otimes.$$

$\otimes \geq 0$ ✓

$$\otimes = 0 \iff \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \iff x_1 = x_2 = x_3 = 0 \iff x = \underline{0} \text{ ✓✓✓.}$$