Image Processing for Physicists

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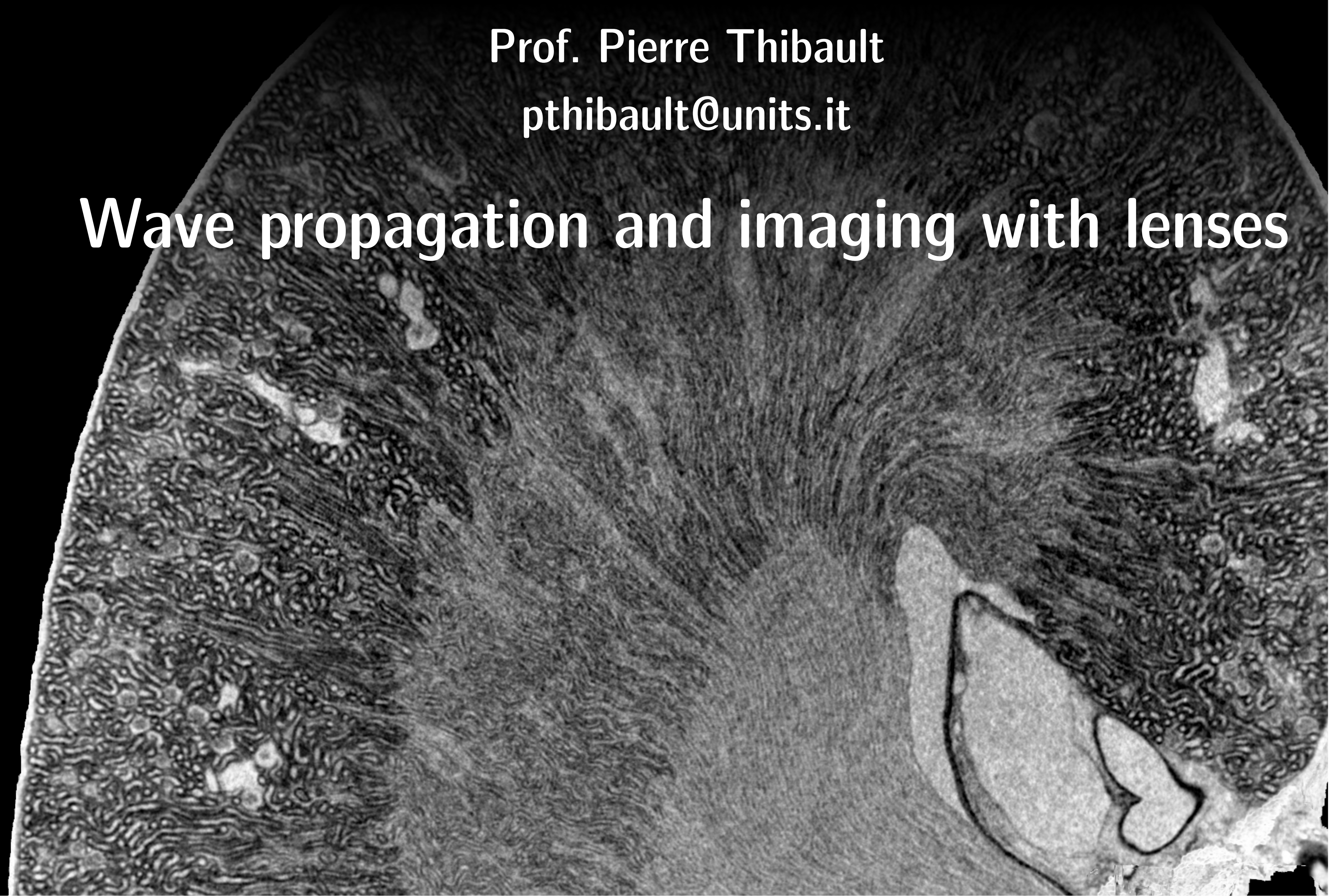
Wave propagation and imaging with lenses



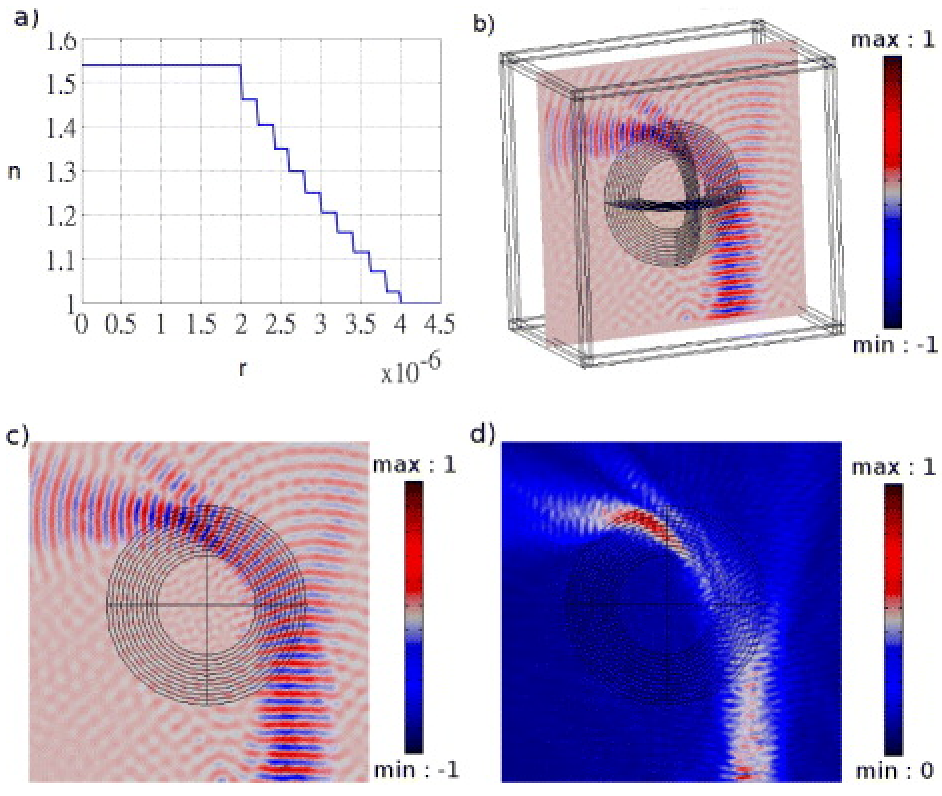
Overview

- Propagation modelization
- Wave propagation:
  - Near-field regime
  - Far-field regime

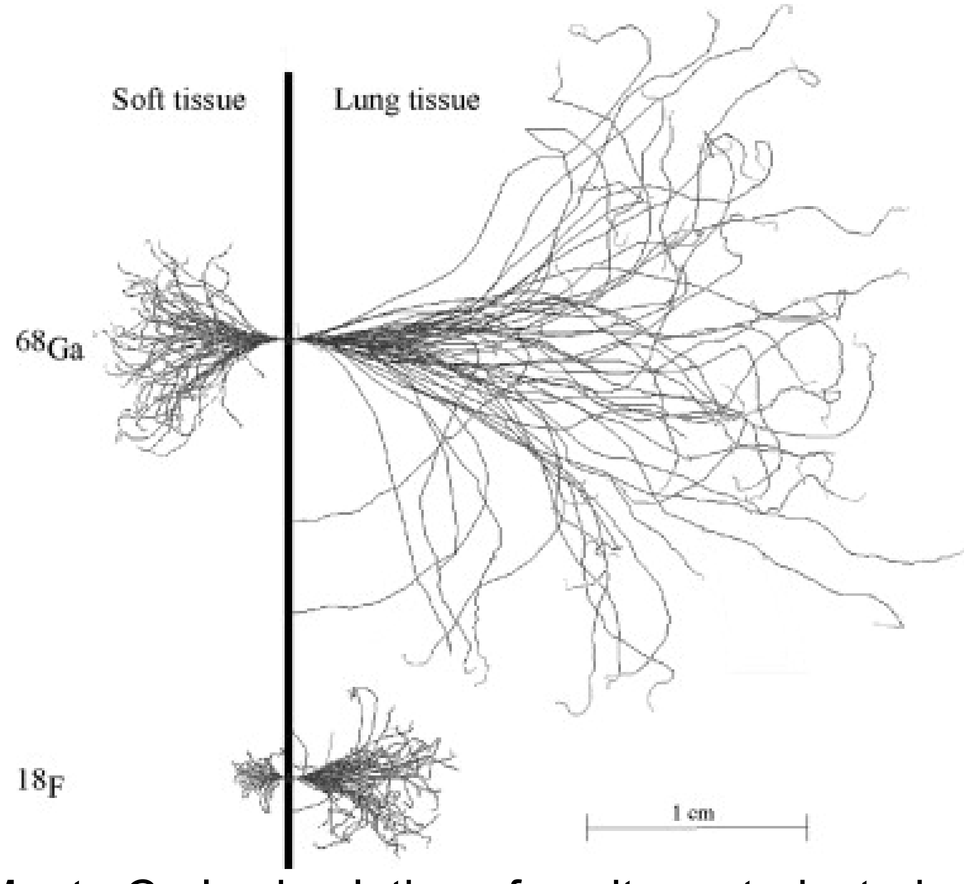
Propagation modeling

- Motivations:

1. Validation



Finite element simulation of an electro-magnetic field in a dielectric



Monte Carlo simulation of positrons trajectories resulting from ^{68}Ga and ^{18}F decay.

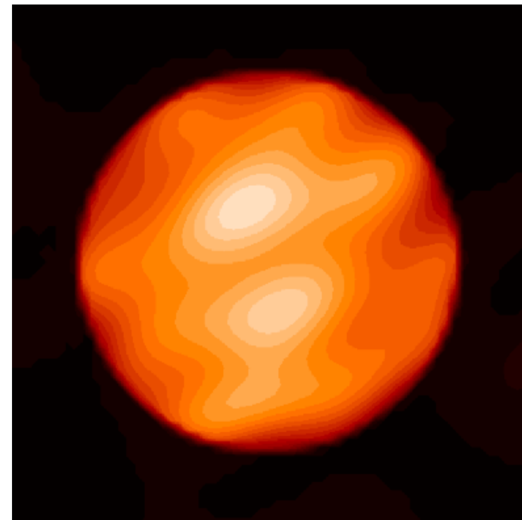
sources: T.M. Chang *et al.* New J. Phys. (2012)
A. Sanchez-Crespo, Appl. Rad. Isotopes (2012)

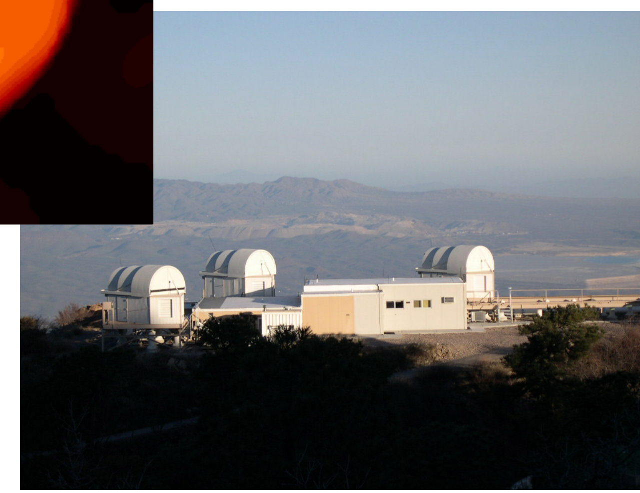
Propagation modeling

- Motivations:

  2. Inversion

Image reconstruction from sound wave propagation (ultrasonography)





The surface of Betelgeuse reconstructed from interferometric data (IOTA)

sources: wikipedia
Haubois *et al. Astronom. & Astrophys.* (2009)

Propagation modeling

- Particles
  - Model particle tracks (rays) through different media
  - Model may include: refraction, force fields, particle decay and interactions
  - Not included: diffraction
- Wave
  - Model the interaction of a field with a medium
  - Can be very complicated $\rightarrow$ approximations are needed

Propagation modeling

Starting point: Helmholtz equation

- for EM field: neglect polarization (scalar wave approximation)

- for electron wave, assume high energy electrons

index of refraction (spatially dependent)

Maxwell eq. $\rightsquigarrow$ $\nabla^2 \psi + \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0$

ψ: complex-valued scalar field

n: index of refraction

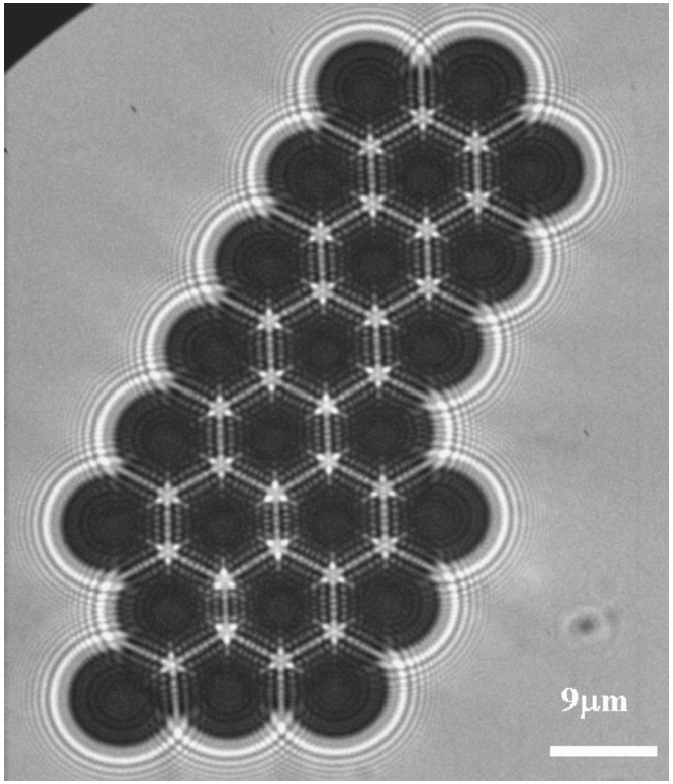
c: speed of light

constant n: plane wave solutions $\psi = \psi_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ with $k^2 = \frac{n^2 \omega^2}{c^2}$

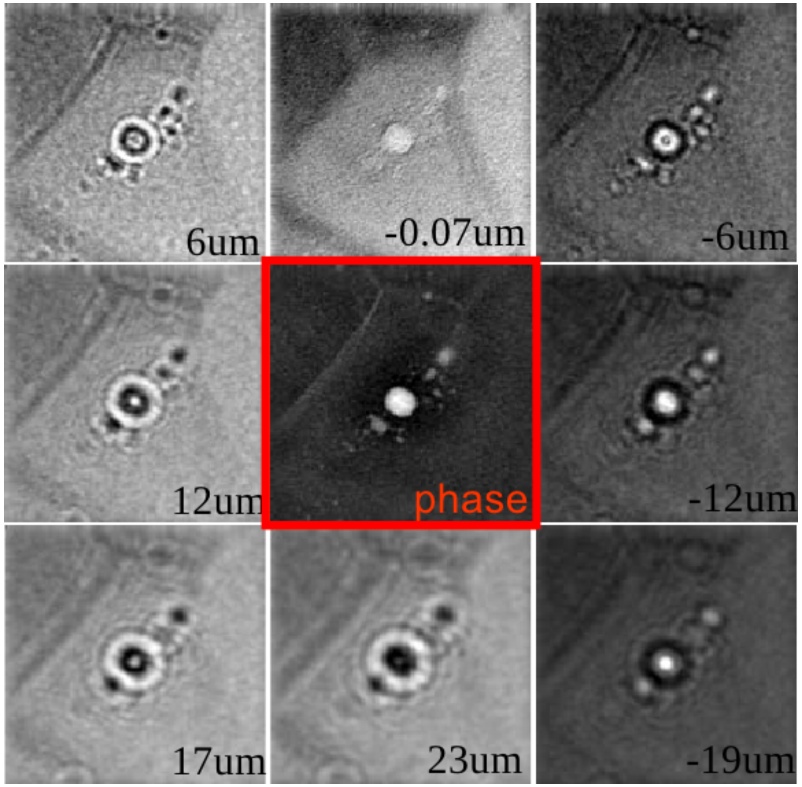
Propagation modeling

- Useful to:
  - better understand optical systems
  - understand diffraction, holography, phase contrast, interferometry, ...

X-ray hologram



TEM through-focus series



sources: Mayo *et al.* Opt. Express (2003)
http://www.christophtkoch.com/Vorlesung/

The physics of propagation

In free space $(n=1)$ General solution is

$$\psi(\vec{r},t) = \sum_{\omega} \sum_{\vec{k}} A_{\omega\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\uparrow$ $|\vec{k}|^2 = \frac{\omega^2}{c^2}$

wave vector $\downarrow$

$|\vec{k}| = \frac{2\pi}{\lambda}$ ← wavelength

$\updownarrow$

u, v

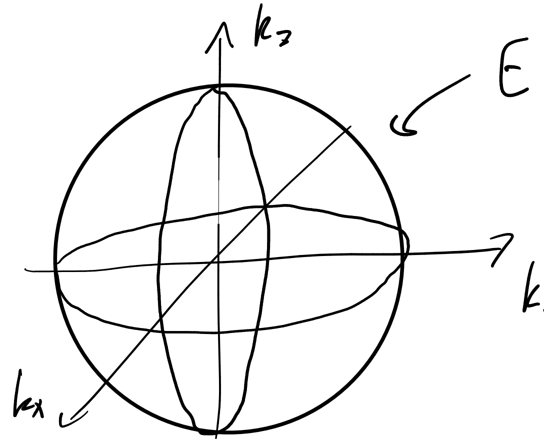
Commonly: fix ω and solve monochromatic case

$$\Rightarrow \psi(\vec{r},t) = \psi(\vec{r})\, e^{i\omega t}$$

$$\psi(\vec{r}) = \boxed{\sum_{\vec{k}} A_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}}$$ ← is this a Fourier transform?

(such that $|\vec{k}| = \omega/c$)

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$



Ewald sphere

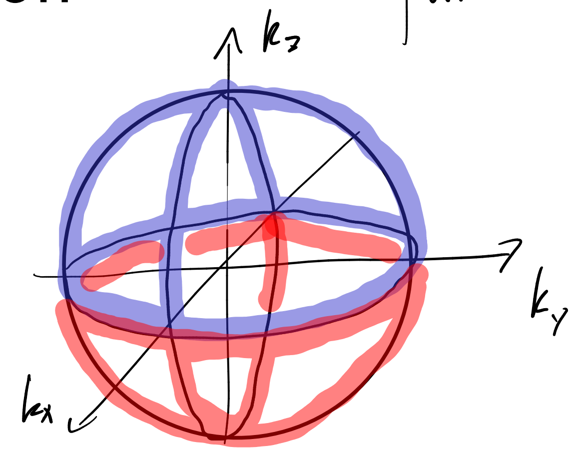
Only wavevectors lying on the surface of this sphere are part of the sum

The physics of propagation

Angular spectrum representation

main propagation direction

$$k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$



$$\psi(\vec{r}) = \sum_{k_x k_y} A^+_{k_x k_y}\, e^{i\left(k_x x + k_y y + \sqrt{k^2 - k_x^2 - k_y^2}\, z\right)} + \cancel{A^-_{k_x k_y}\, e^{i\left(k_x x + k_y y - \sqrt{k^2 - k_x^2 - k_y^2}\, z\right)}}$$

because we are interested in forward propagation only

$$\psi(\vec{r}_\perp; z) = \sum_{\vec{k}_\perp} \underbrace{A_{\vec{k}_\perp} \exp(i\vec{k}_\perp \cdot \vec{r}_\perp)}_{\text{2D Fourier transform}} \exp\left(i\sqrt{k^2 - k_\perp^2}\, z\right)$$

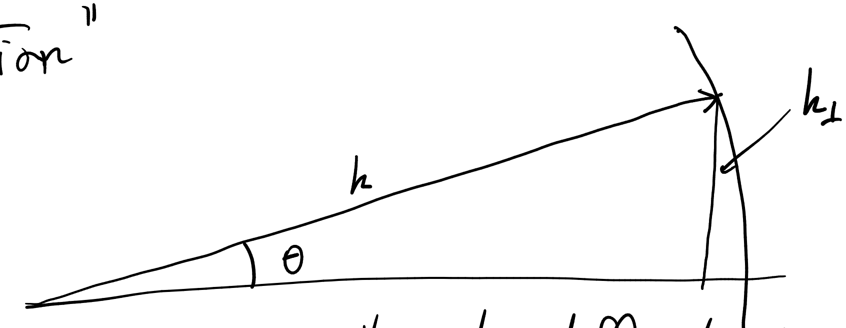
$\vec{r}_\perp = (x, y)$ $\quad$ (k_x, k_y)

Forward propagation

$z=0$: $\psi(\vec{r}_\perp; z=0) = \sum_{\vec{k}_\perp} A_{\vec{k}_\perp} \exp(i\vec{k}_\perp \cdot \vec{r}_\perp)$

$\hookrightarrow A_{\vec{k}_\perp} = \mathcal{F}\{\psi(\vec{r}_\perp; z=0)\}$

Often: $|\vec{k}_\perp| \ll k$ "paraxial approximation"



(small angle diffraction)

$$\sqrt{k^2 - k_\perp^2} = k\sqrt{1 - \frac{k_\perp^2}{k^2}} \simeq k\left(1 - \frac{k_\perp^2}{2k^2}\right)$$

$$= k - \frac{k_\perp^2}{2k}$$

$$\exp\left(i\sqrt{k^2 - k_\perp^2}\, z\right) \simeq \exp(ikz) \exp\left(\frac{-izk_\perp^2}{2k}\right)$$

parabolic approximation of a sphere

"Fresnel propagator"

Forward propagation

$$\psi(\vec{r}_\perp; z) = \mathcal{F}^{-1}\left\{\mathcal{F}\{\psi(r_\perp; z=0)\}\exp(-\pi i \lambda z \vec{u}^2)\right\}$$

"Trick" #1: F.T. $e^{i\vec{k}\cdot\vec{r}} \rightarrow e^{2\pi i \vec{u}\cdot\vec{r}}$

$k_\perp^2 = (2\pi u)^2 = 4\pi^2 u^2$

$\frac{1}{2k} = \frac{\lambda}{4\pi}$

"Trick" #2: D.F.T. $e^{2\pi i \, n\cdot m/N}$

$u \cdot x = n \cdot m/N$

$x = n \cdot \Delta x$

$u = m \cdot \Delta u$

$nm \, \Delta x \Delta u = nm/N$

$\Rightarrow \Delta x \Delta u = \frac{1}{N}$

$$\exp(-i\pi\lambda z u^2) = \exp\left(-i\pi\lambda z m^2 \frac{1}{\Delta x^2 N^2}\right)$$

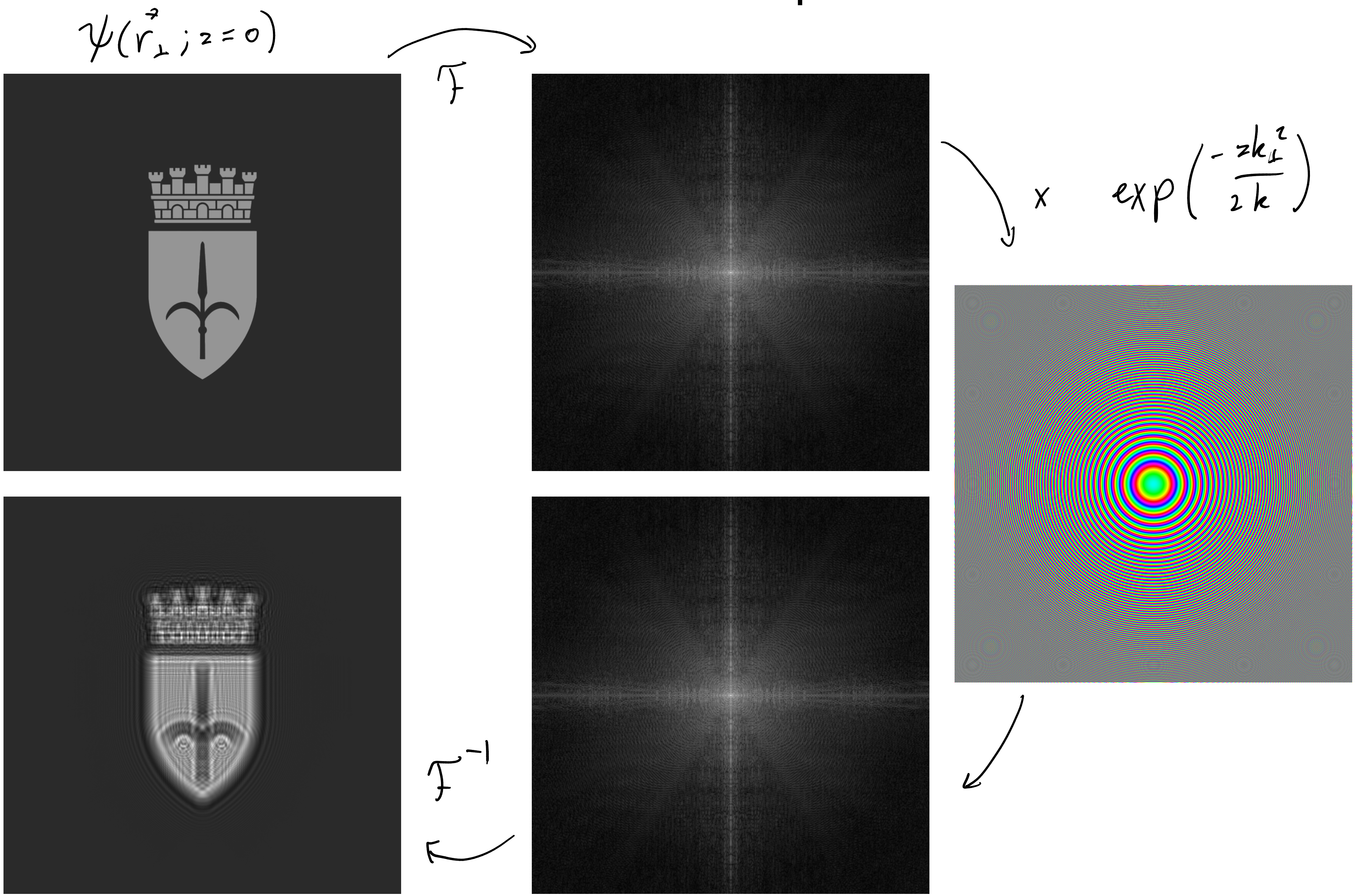
$$= \exp\left(-i\pi \frac{\lambda}{\Delta x} \cdot \frac{z}{\Delta x} \cdot \left(\frac{m}{N}\right)^2\right)$$

assuming that Δx is known

$\frac{m}{N}$ ← numpy.fft.fftfreq

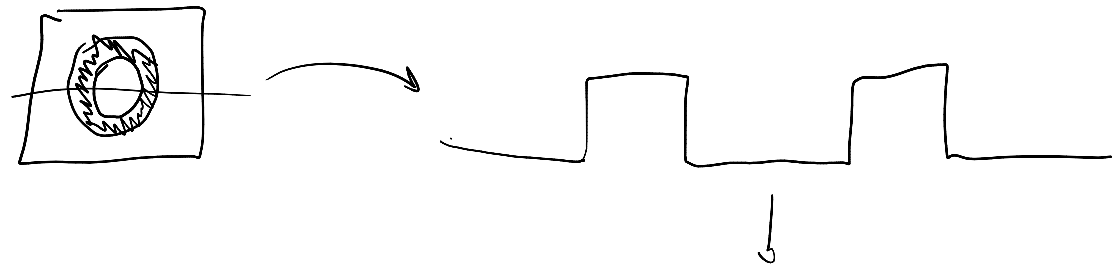
Forward propagation

A numerical recipe

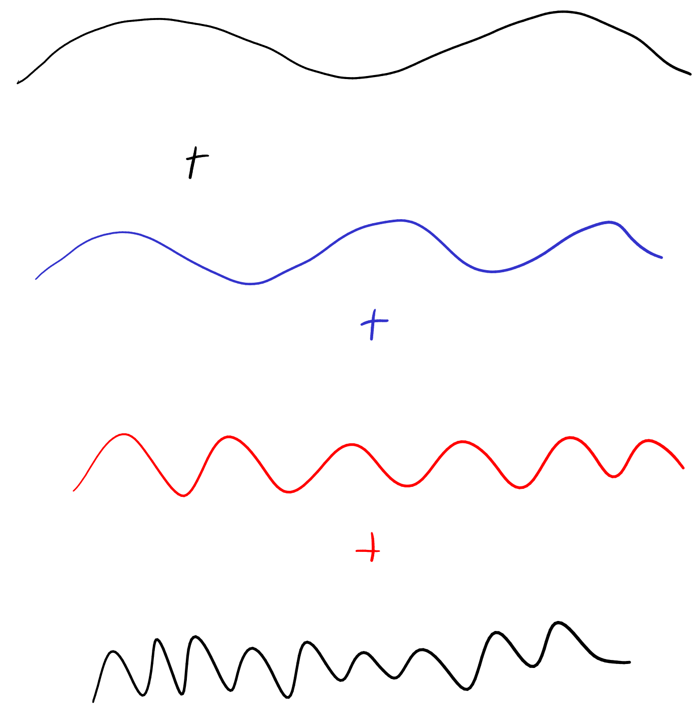


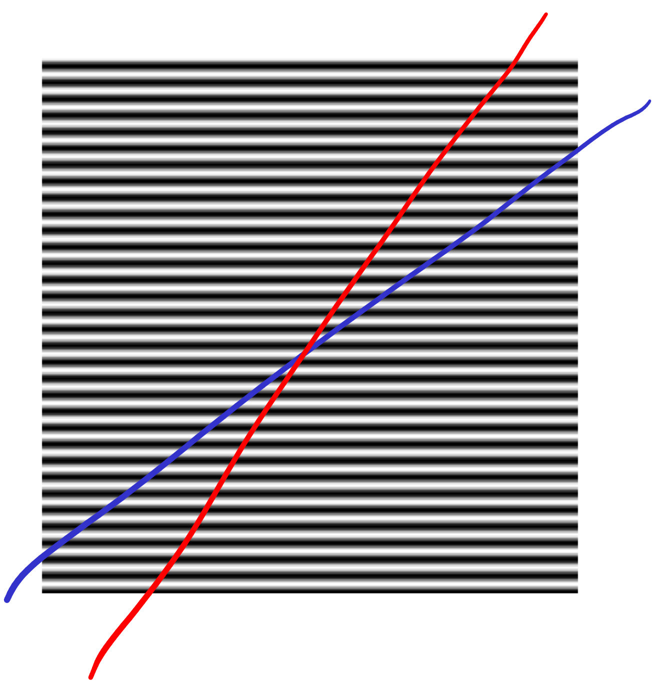
Near field, far field

Image e.g.



Wave carrying information has a fixed wavelength





Near field, far field

$k_\perp^2 = 4\pi \vec{u}^2$

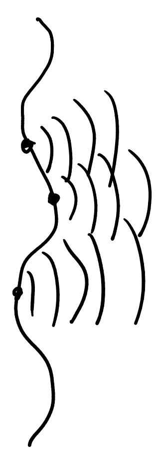
$$\psi(r_\perp; z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{\psi(r_\perp; z=0)\} \exp\left(\frac{-i z k_\perp^2}{2k}\right)\right\}$$

$k = 2\pi/\lambda$

convolution!

$$= \psi(r_\perp; z=0) * P_z(\vec{r}_\perp)$$

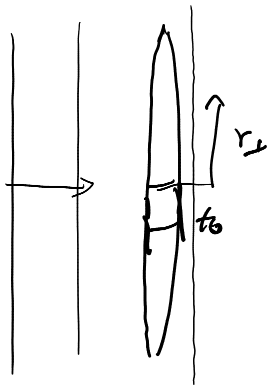
$$P_z(\vec{r}_\perp) = \mathcal{F}^{-1}\left\{\exp\left(-\pi i \lambda z \vec{u}^2\right)\right\} = \frac{-2\pi i}{\lambda z} \exp\left(\frac{i k r_\perp^2}{2z}\right)$$



$$\psi(\vec{r}, z) = \frac{-i2\pi}{\lambda z} \int d^2 r'_\perp \, \psi(\vec{r}\,'_\perp; z=0) \exp\left(\frac{ik(\vec{r}_\perp - \vec{r}\,'_\perp)^2}{2z}\right)$$

"Fresnel - Huygens integral"

Back focal plane of a lens



* model for thin lens thickness profile:

$$t(r) = t_0 - \alpha r_\perp^2$$

← total thickness decreases quadratically with distance from center

* phase

$$\phi(r_\perp) = \frac{2\pi}{\lambda}(n-1)\, t(r_\perp)$$

$$= k(n-1)t_0 - k(n-1)\alpha r_\perp^2$$

"kx"

at exit of lens: $\psi(\vec{r}_\perp) = \psi_0(\vec{r}_\perp) \cdot \exp\left(-ik(n-1)\alpha r_\perp^2\right)$

↑ before lens

Propagate $\psi(r_\perp)$ using Fresnel–Huygens integral:

$$\psi(\vec{r}_\perp; z) = \frac{-ik}{z}\int d^2r_\perp'\ \psi_0(\vec{r}_\perp')\ \underbrace{\exp\left(-ik(n-1)\alpha r_\perp'^2\right)\exp\left(\frac{ik(r_\perp - r_\perp')^2}{2z}\right)}$$

$$\frac{1}{2f} = (n-1)\alpha$$

$$\exp\left[ik\left(-\frac{r_\perp'^2}{2f} + \frac{r_\perp^2}{2z} + \frac{r_\perp'^2}{2z} - \frac{\vec{r}_\perp \cdot \vec{r}_\perp'}{z}\right)\right]$$

$$= \exp\left[\frac{-ik\vec{r}_\perp}{z}\cdot\vec{r}_\perp'\right]\exp\left(\frac{ikr_\perp'^2}{2}\left(\frac{1}{z} - \frac{1}{f}\right)\right)\exp\left(\frac{ikr_\perp^2}{2z}\right)$$

Back focal plane of a lens

$$\psi(\vec{r}_\perp; z) = -\frac{ik}{z}\exp\left(\frac{ikr_\perp^2}{2z}\right)\int \psi_0(\vec{r}_\perp{}') \exp\left(-i\frac{k\vec{r}_\perp}{z}\cdot\vec{r}_\perp{}'\right) \cdot \exp\left(\frac{ikr_\perp'^2}{2}\left(\frac{1}{z}-\frac{1}{f}\right)\right) d^2r'$$

At $z = f$: $\frac{1}{z} - \frac{1}{f} = 0$

$$\psi(\vec{r}_\perp; z=f) = -\frac{ik}{z}\exp\left(\frac{ikr_\perp^2}{2z}\right) \mathcal{F}\left\{\psi_0(r)\right\}\left(\vec{u} = \frac{k\vec{r}}{z}\right)$$

A lens acts as a Fourier transform operator!

"back focal plane": plane where F.T. is present

Ernst Abbe imaging theory based on this realization

Plane waves, point sources

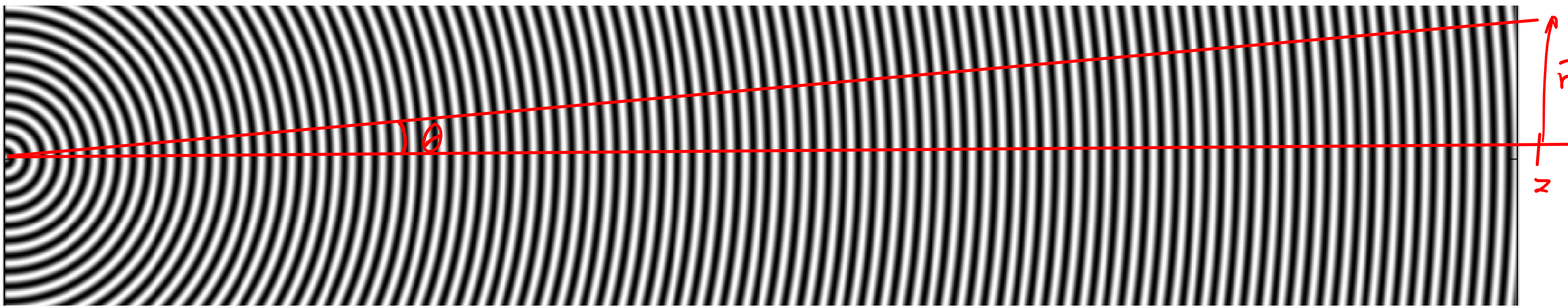
"z sinθ"

Set $f \to \infty$ in previous result:

$$\psi(\vec{r}_\perp ; z \to \infty) = \frac{ik}{z} e^{\frac{ikr^2}{2z}} \mathcal{F}\{\psi_0(\vec{r})\}\left(\vec{u} = \frac{k\vec{r}}{z}\right)$$

Far-field propagation, Fraunhofer propagation

point source

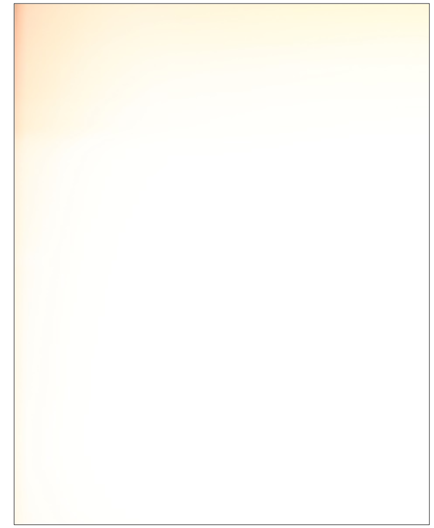


circular waves
evanescent waves
contact region

parabolic waves
near field
Fresnel region

plane waves
far field
Fraunhofer region

Why optical elements?

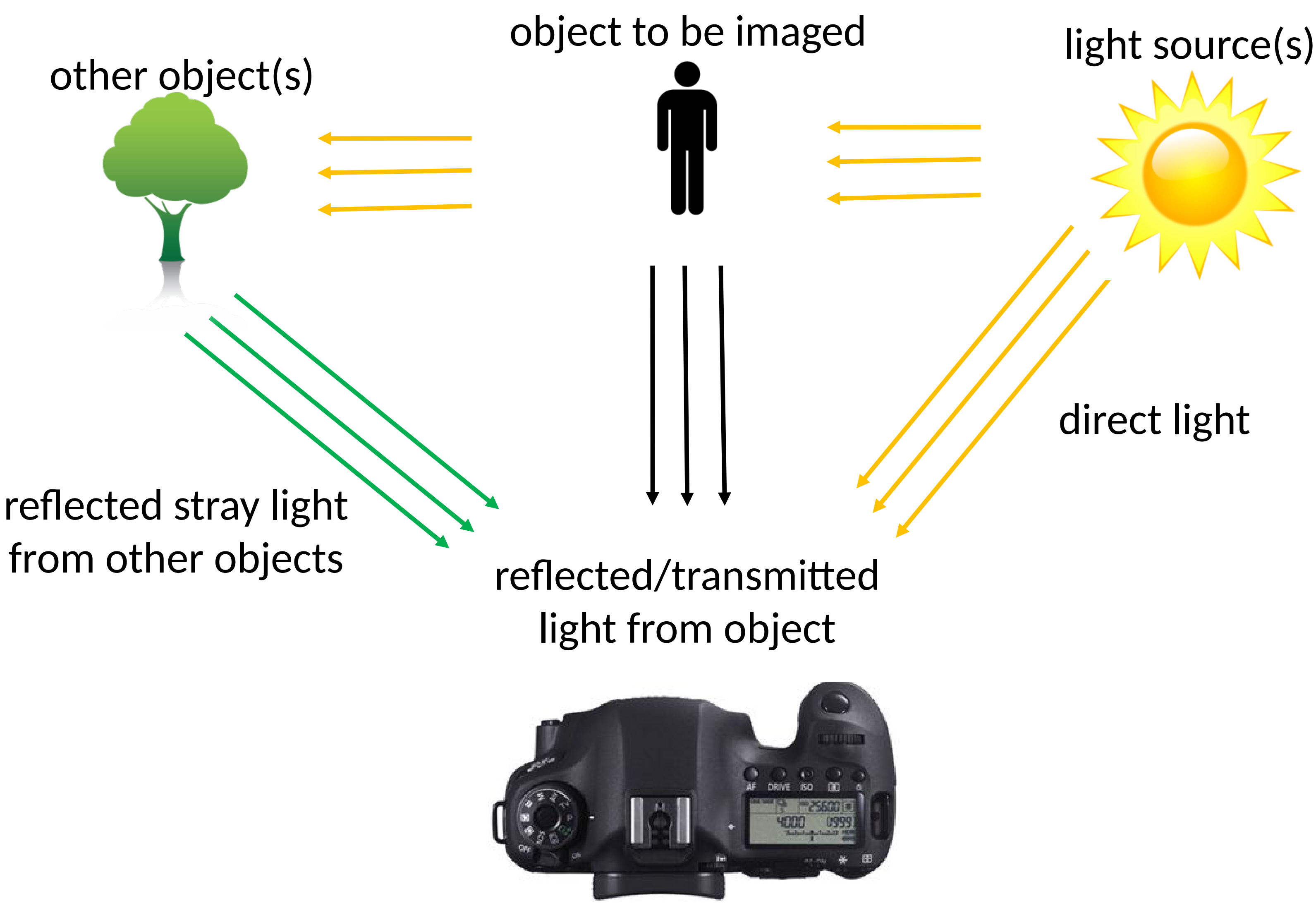


with objective lens

without objective lens

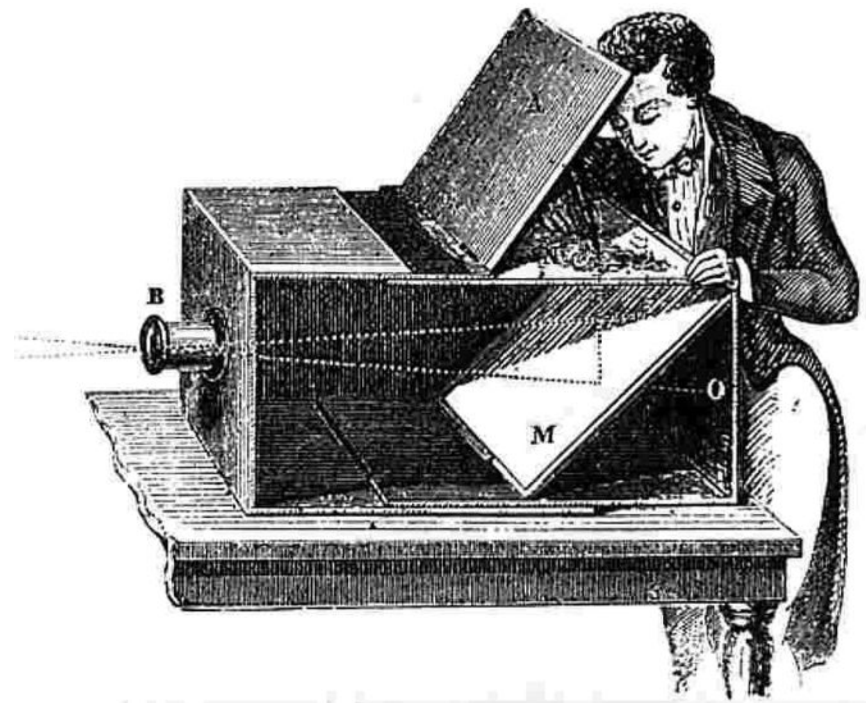
Why optical elements?

- Information from many sources overlaps in detector plane
- Need models to understand image forming systems



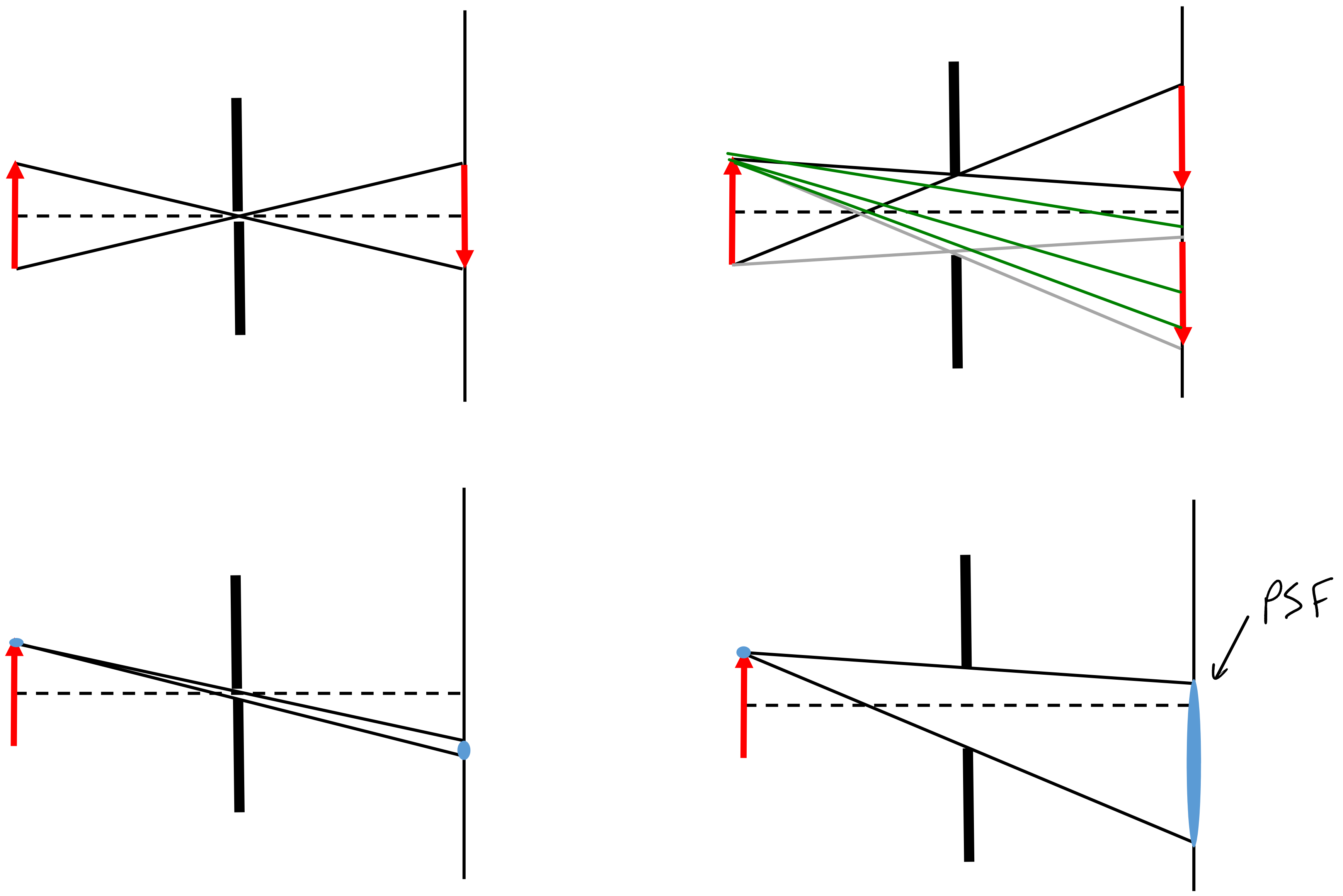
Pinhole camera model

camera obscura



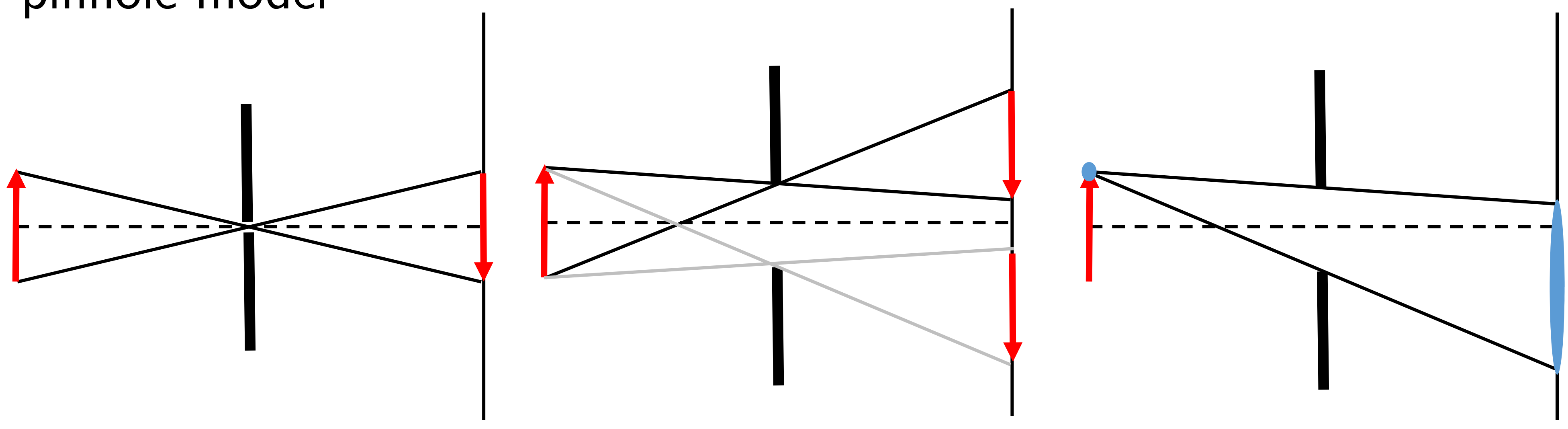
Pinhole camera model

PSF determined by aperture width

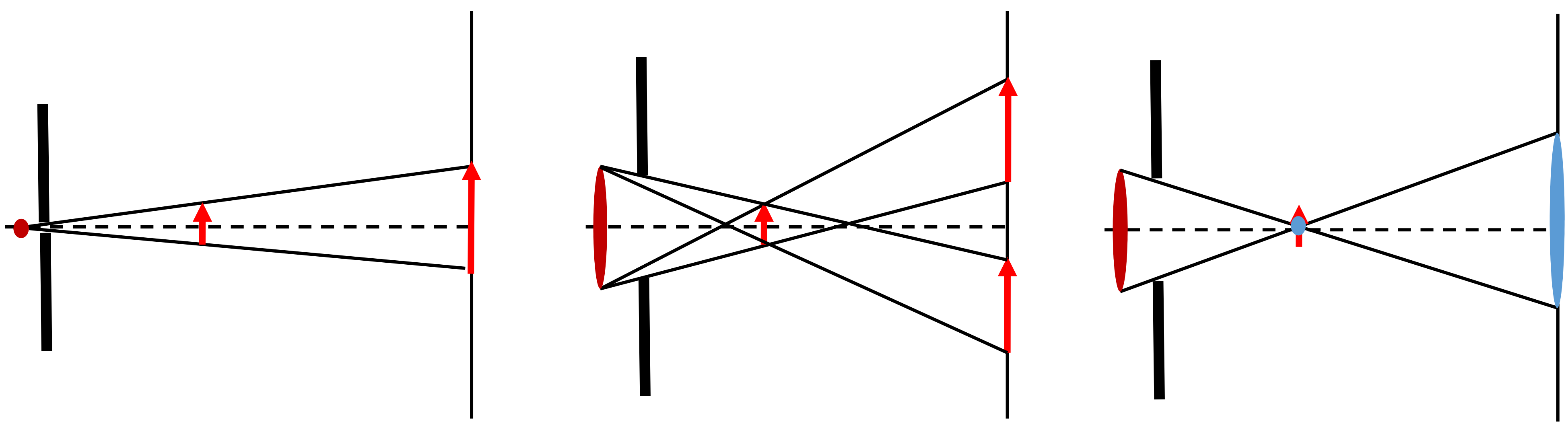


Projection model

pinhole model

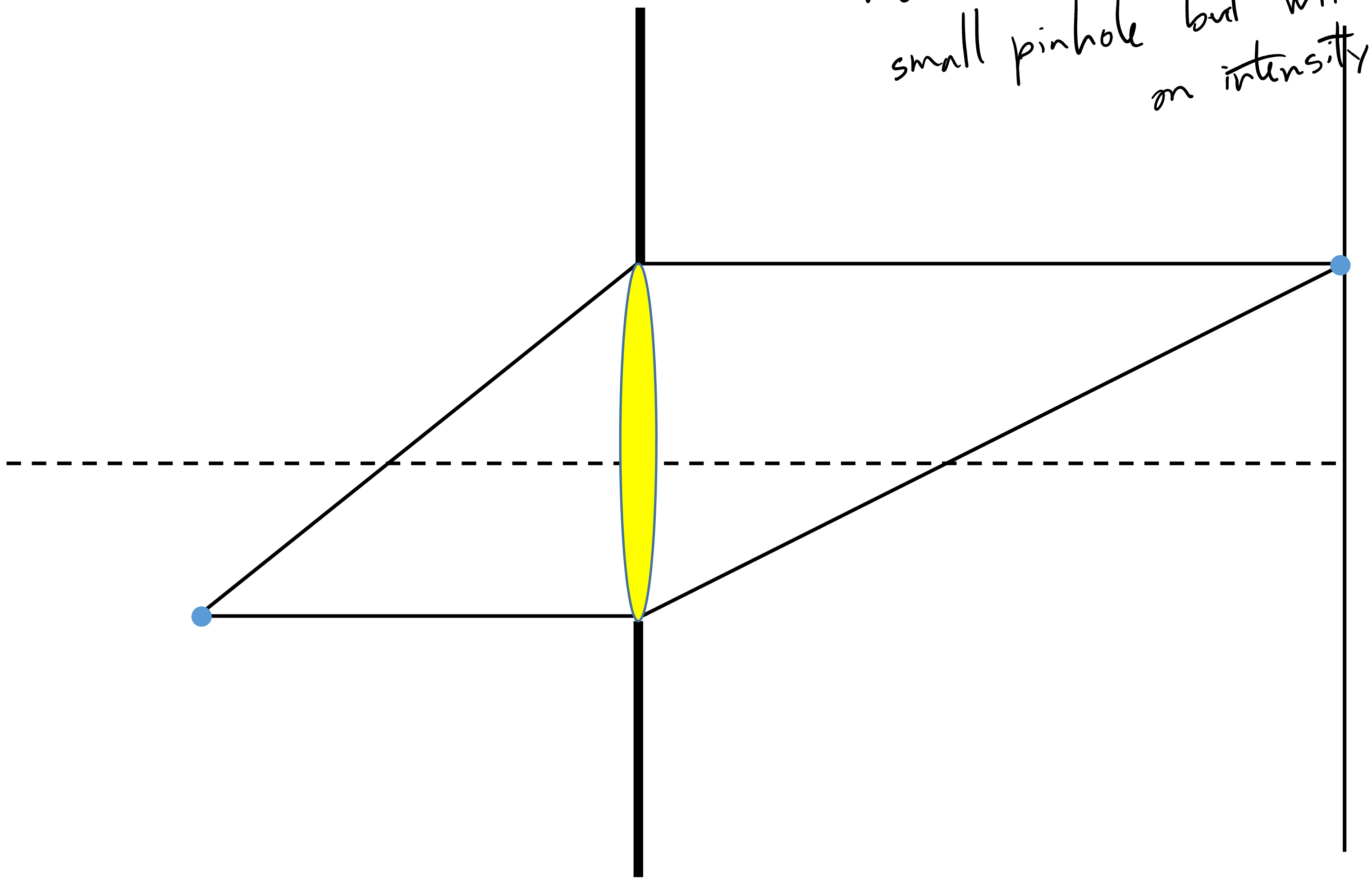


projection model



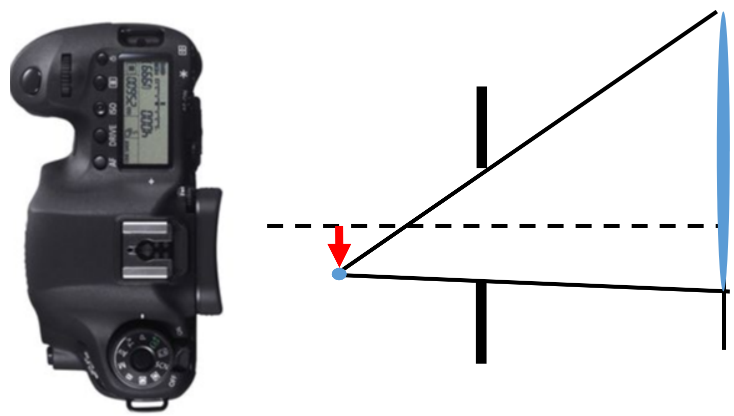
Lens camera model

Result similar to small pinhole but without compromise on intensity

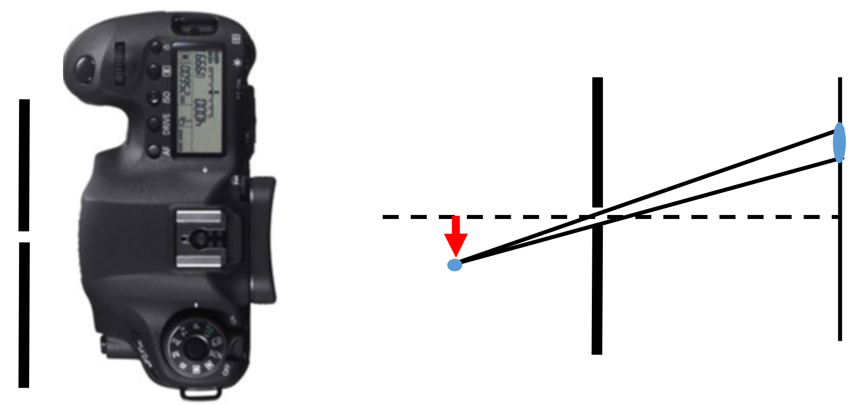


Lens camera model

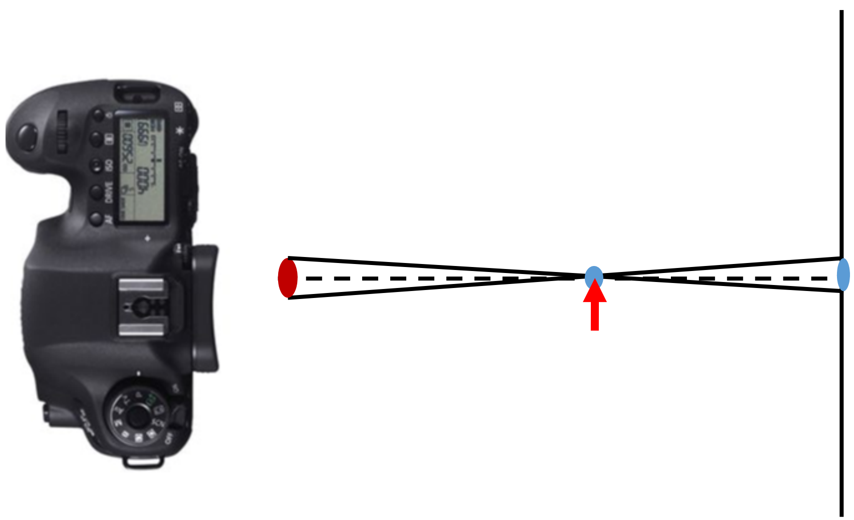
lensless model



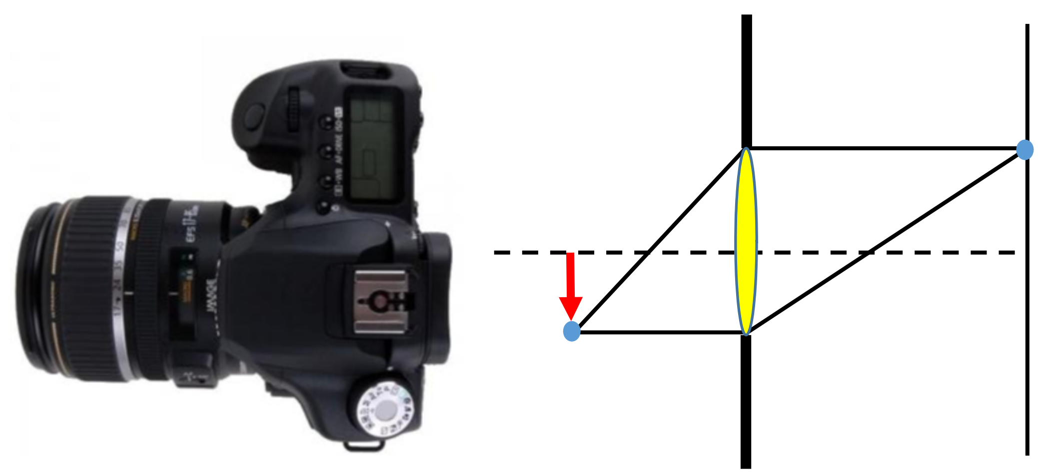
pinhole camera model



projection model



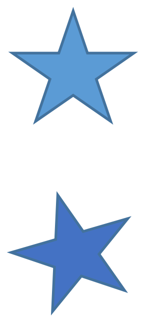
lens camera model

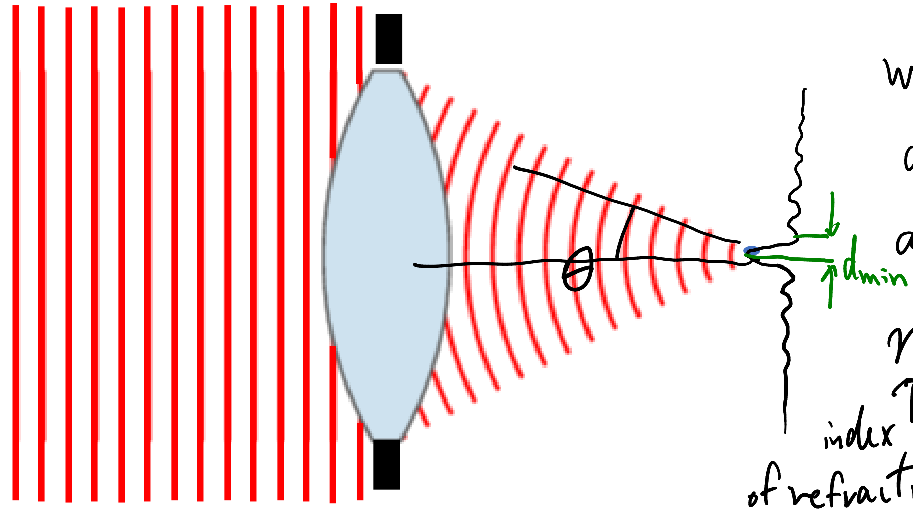


Diffraction-limited imaging systems

- Rayleigh criterion

PSF: Airy "disc" Bessel function





width of the PSF depends only on the angular opening

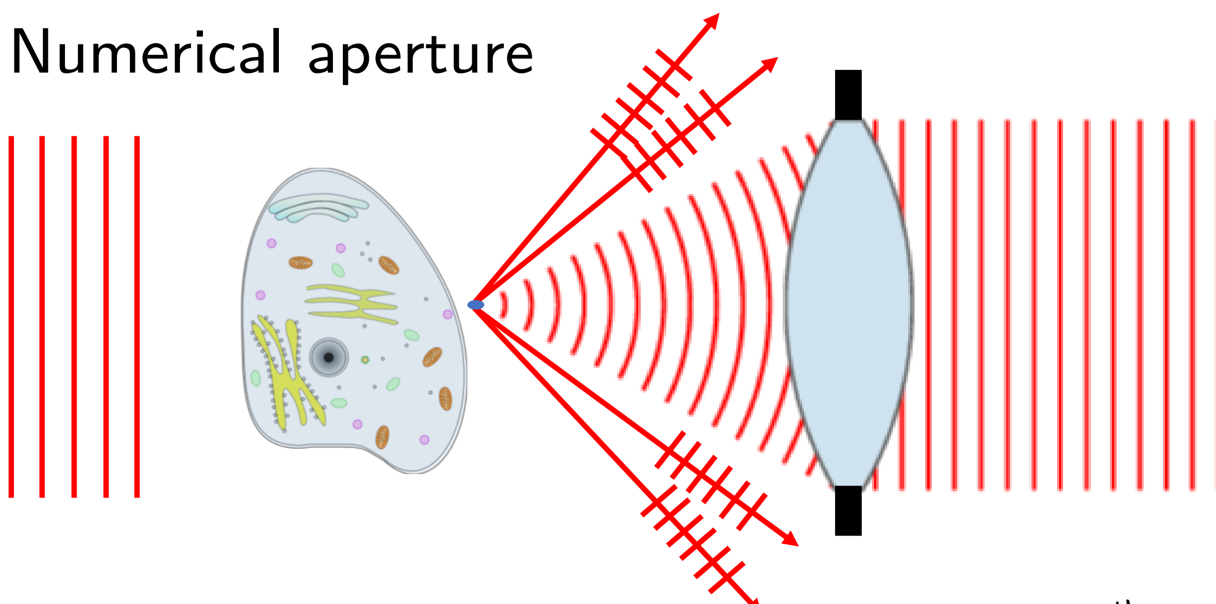
$n \cdot \sin\theta = NA$ "numerical aperture"

PSF width:

$$d_{min} = 1.22 \frac{\lambda}{2NA}$$

distance between maximum and first zero

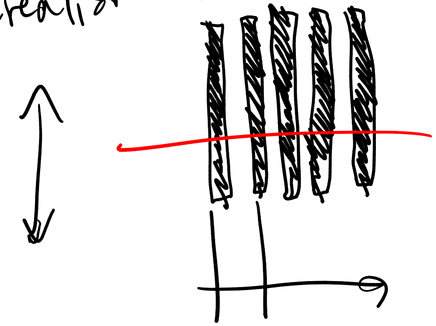
- Numerical aperture

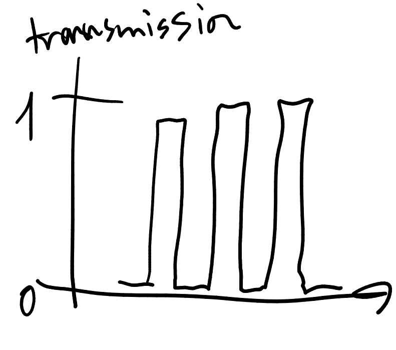


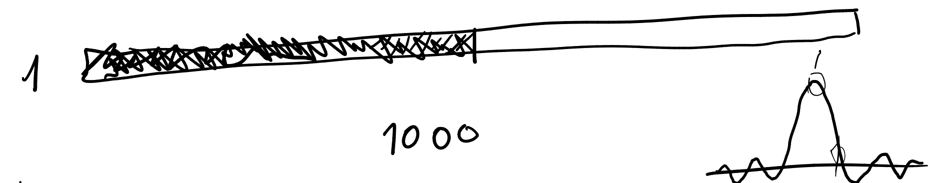
small NA: $\sin\theta \approx \frac{D}{2f}$

(sketches to explain the creation of a grating)

1 px





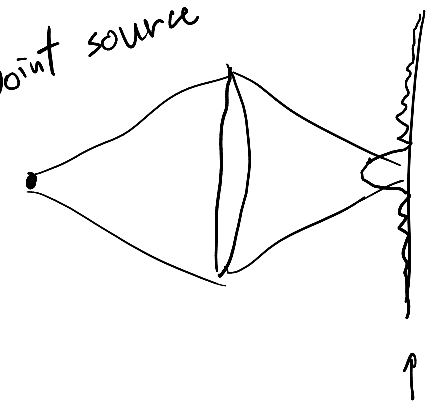


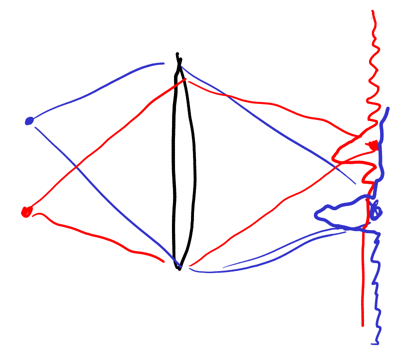
Coherent vs incoherent imaging systems

PSF: $\mathcal{F}\{\text{aperture}\}$ e.g. Bessel function

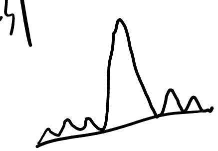
coherent:

point source





PSF: $|\mathcal{F}\{\text{aperture}\}|^2$



wavefield ψ: $\mathcal{F}(\text{aperture})$

Intensity: $|\psi|^2 = |\mathcal{F}(\text{aperture})|^2$

Coherent: intensity: $|\psi_1 + \psi_2|^2$

Incoherent: intensity: $|\psi_1|^2 + |\psi_2|^2$

Scanning systems

Transmission

- **S**canning **T**ransmission **E**lectron **M**icroscopy
- **S**canning **T**ransmission **X**-ray **M**icroscopy
- …

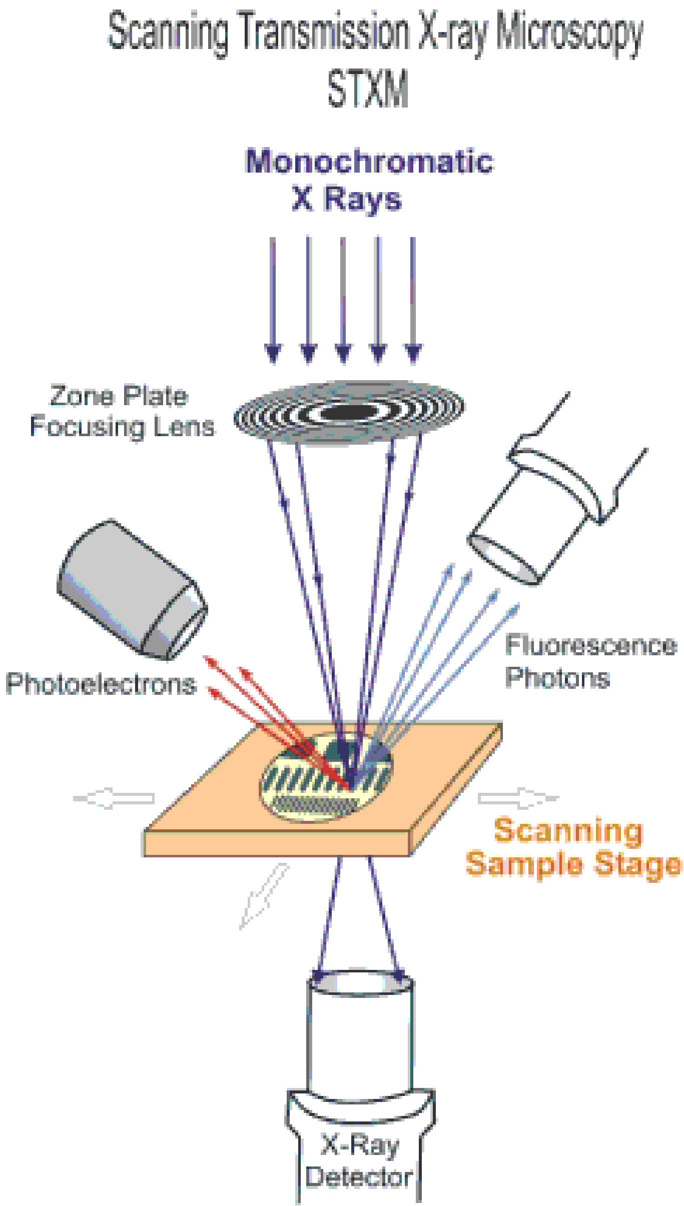
Indirect (reflection, scattering, fluorescence, …)

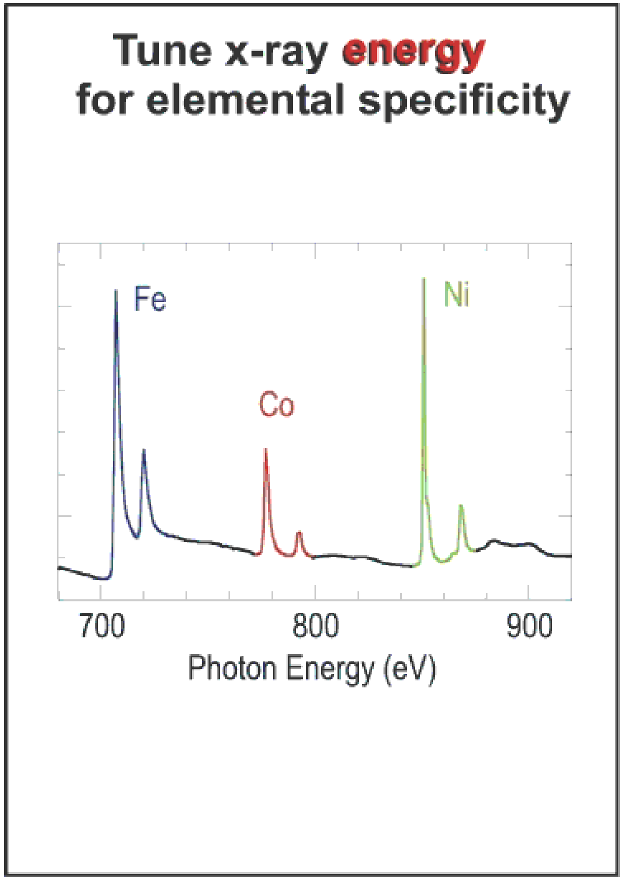
- **L**aser **S**canning **C**onfocal **M**icropsopy
- **S**canning **E**lectron **M**icroscopy
- **X**-ray **F**luorescence **M**icroscopy
- **P**hoto**E**mission **E**lectron **M**icroscopy
- ...

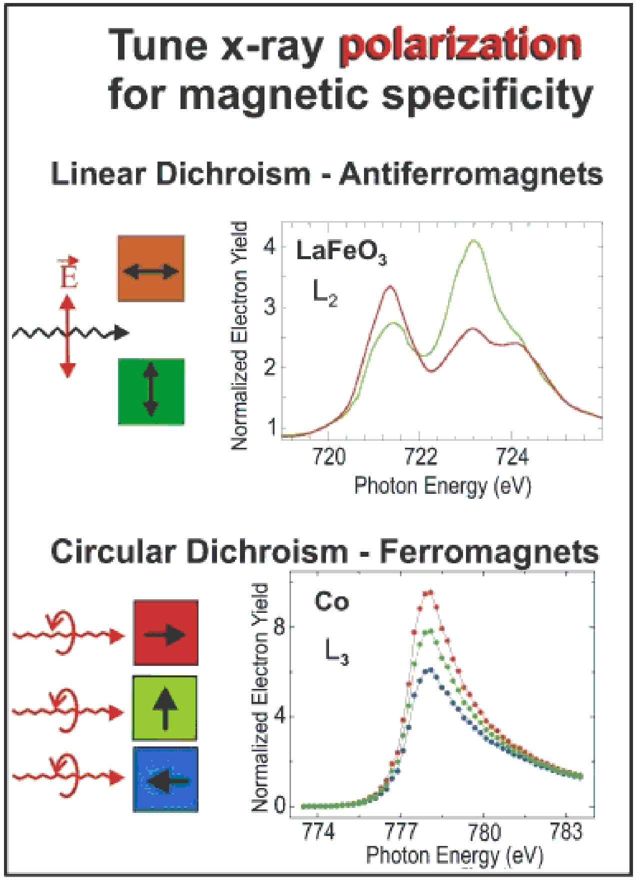
Physical probe

- **A**tomic **F**orce **M**icroscopy
- **S**canning **T**unneling **M**icroscopy
- ...

Scanning transmission X-ray microscopy

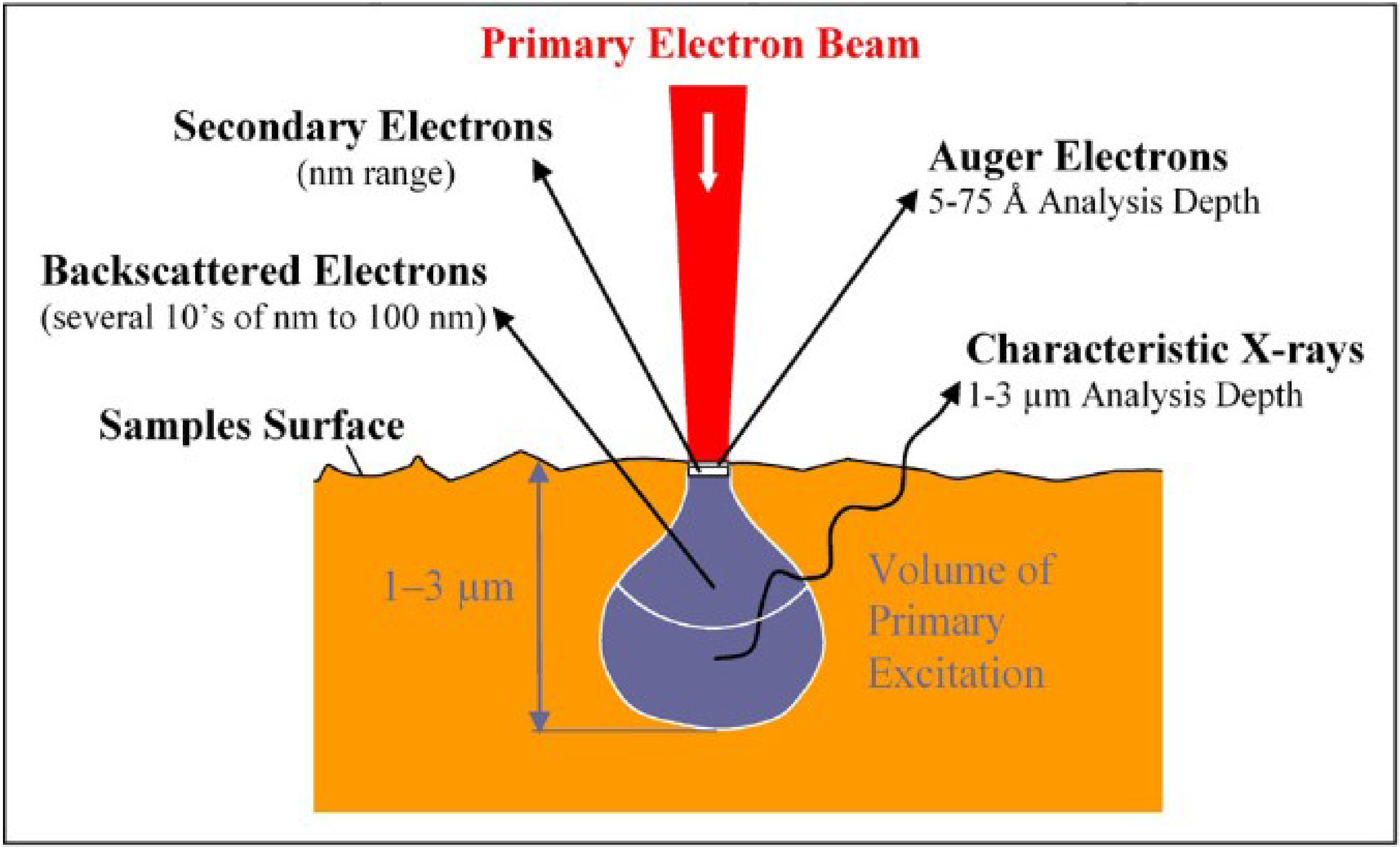




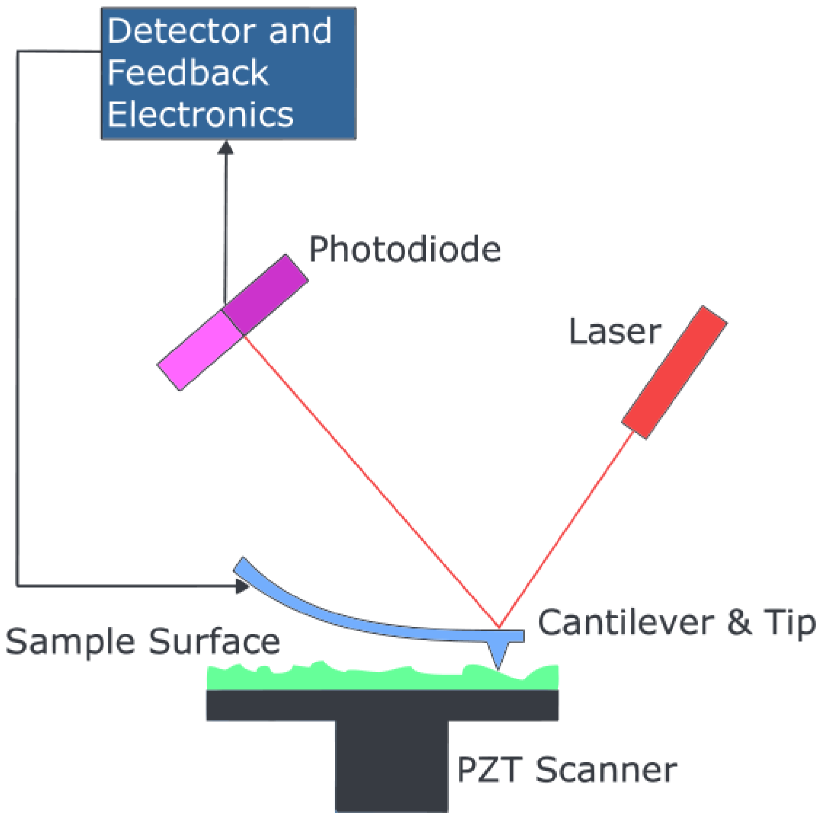


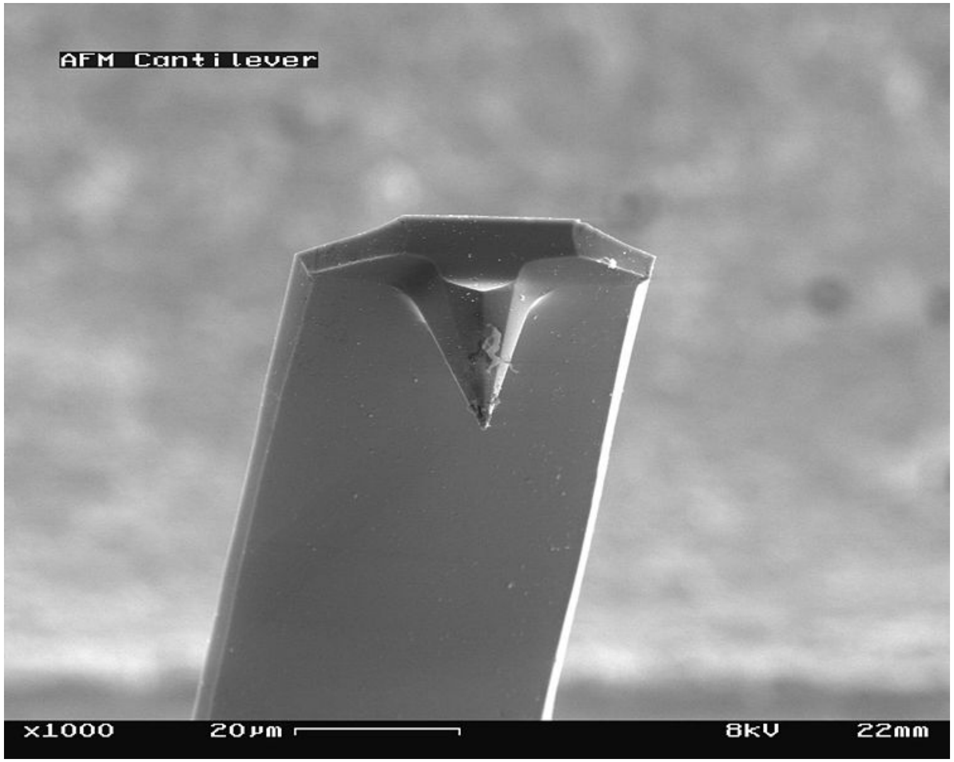
source: http://www-ssrl.slac.stanford.edu

Scanning electron microscopy



Atomic force microscopy





Resolution in scanning systems

Resolution mainly limited by probe size

Scanning vs. full field systems

Transmission probe: the reciprocity theorem

resolution is the same: limited by numerical aperture of the lens