Image Processing for Physicists

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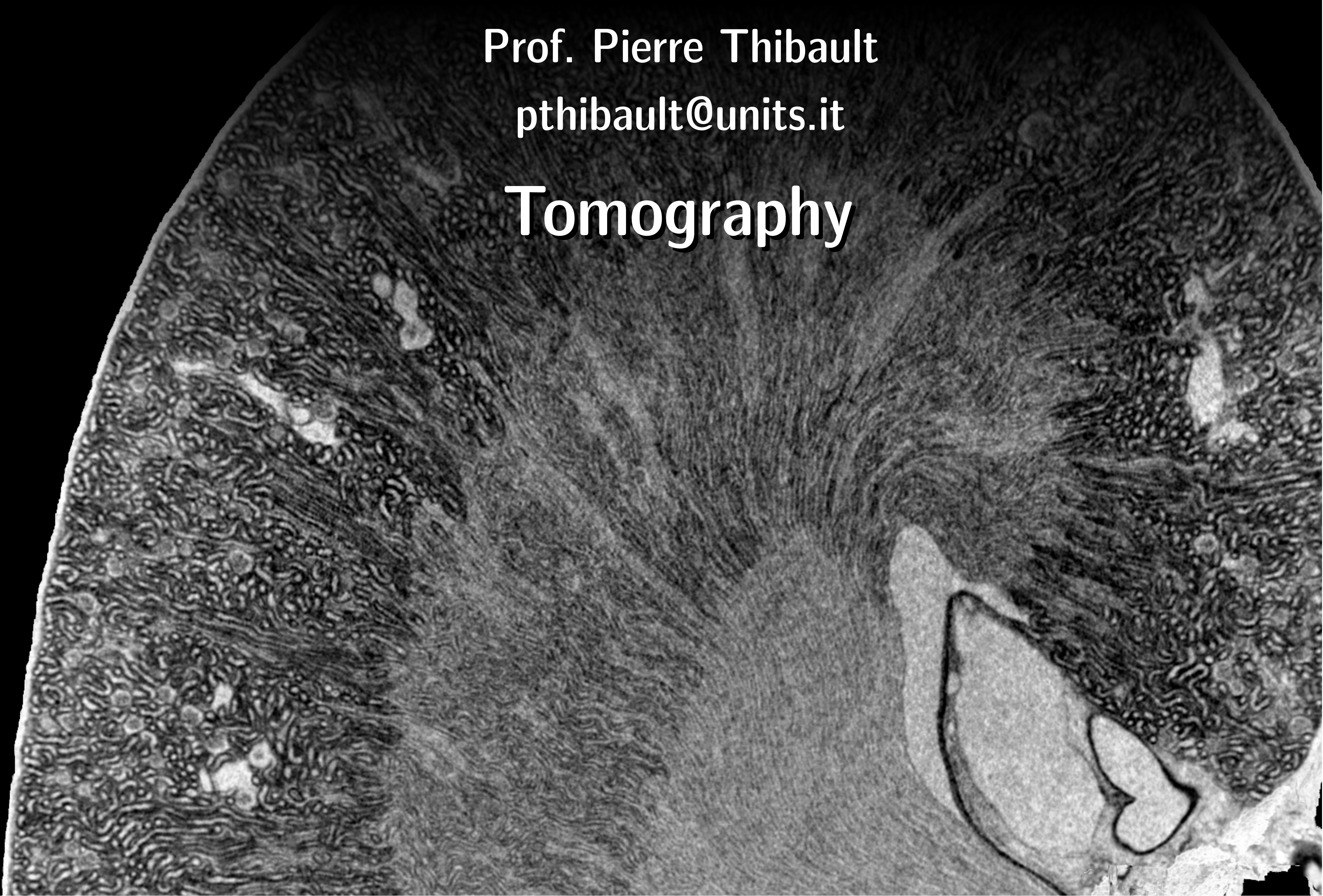
Tomography

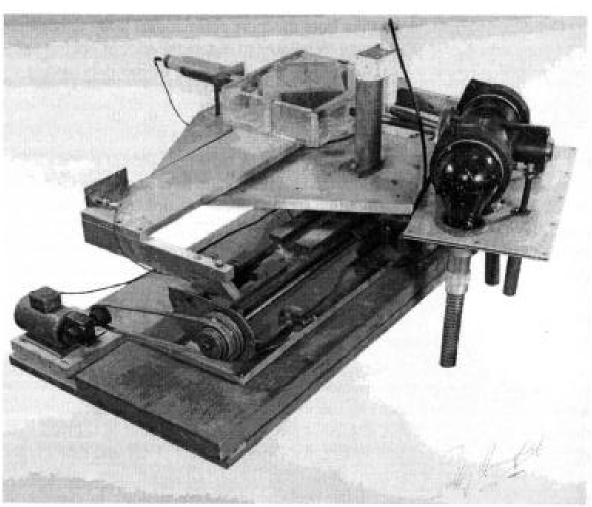


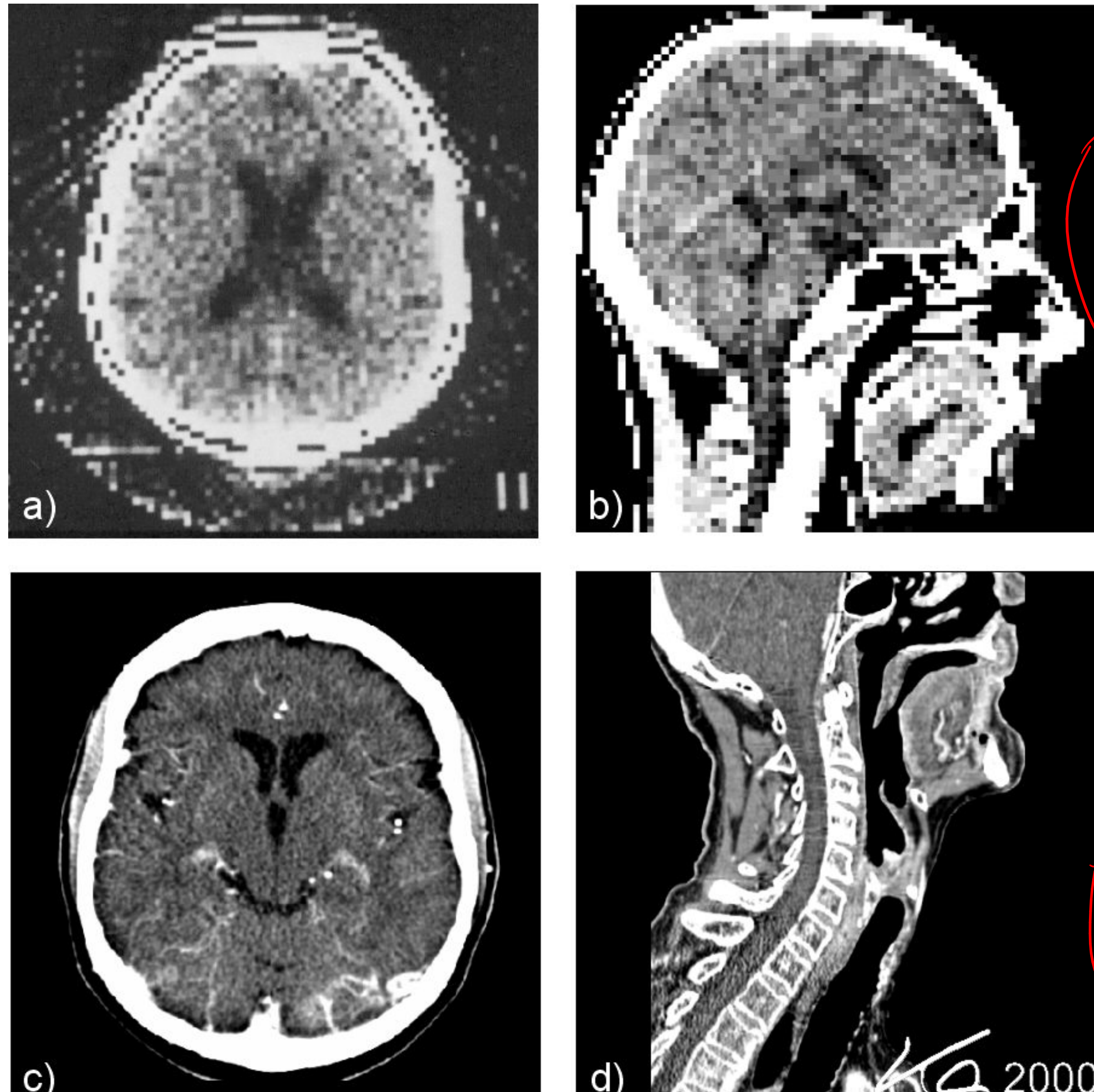
Overview

- Fundamentals of tomography
  - Physics & geometry
- Analytic formulation
  - Radon transform
  - Filtered back-projection
- Algebraic formulation

Examples of tomographic imaging

Computed (X-ray) Tomography (CT)





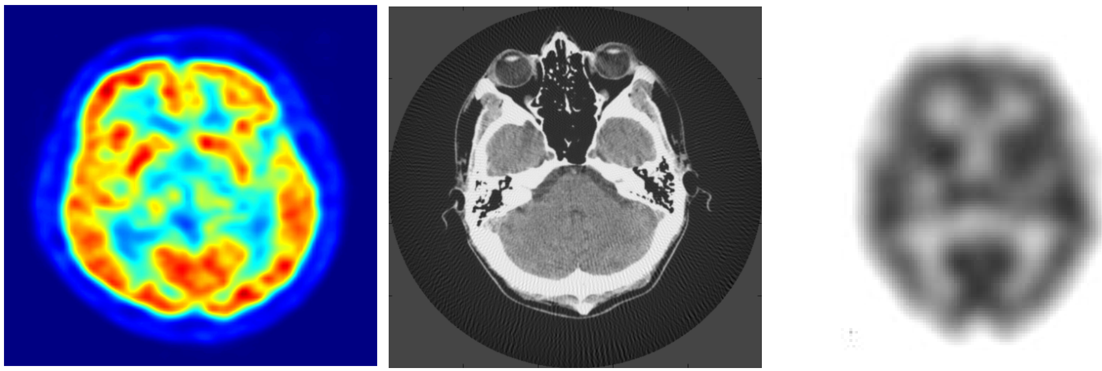
1974, 80x80 pixels

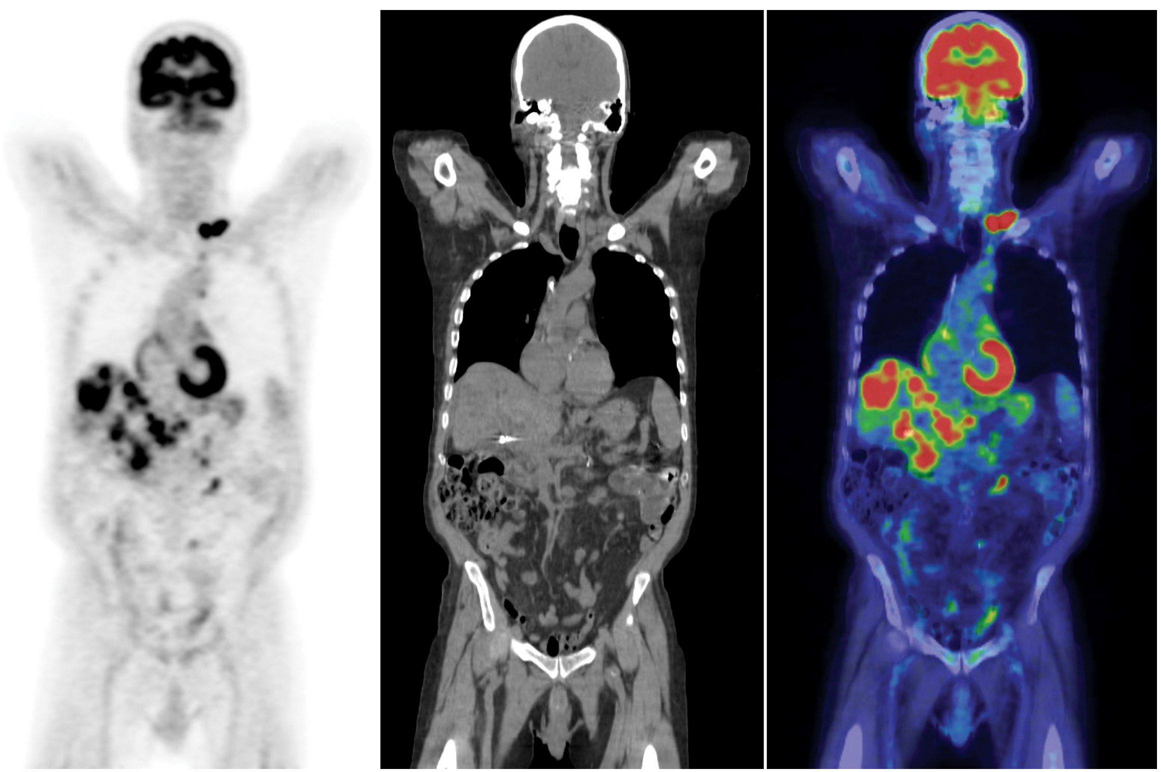
2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

Examples of tomographic imaging

Positron emission tomography (PET) + CT

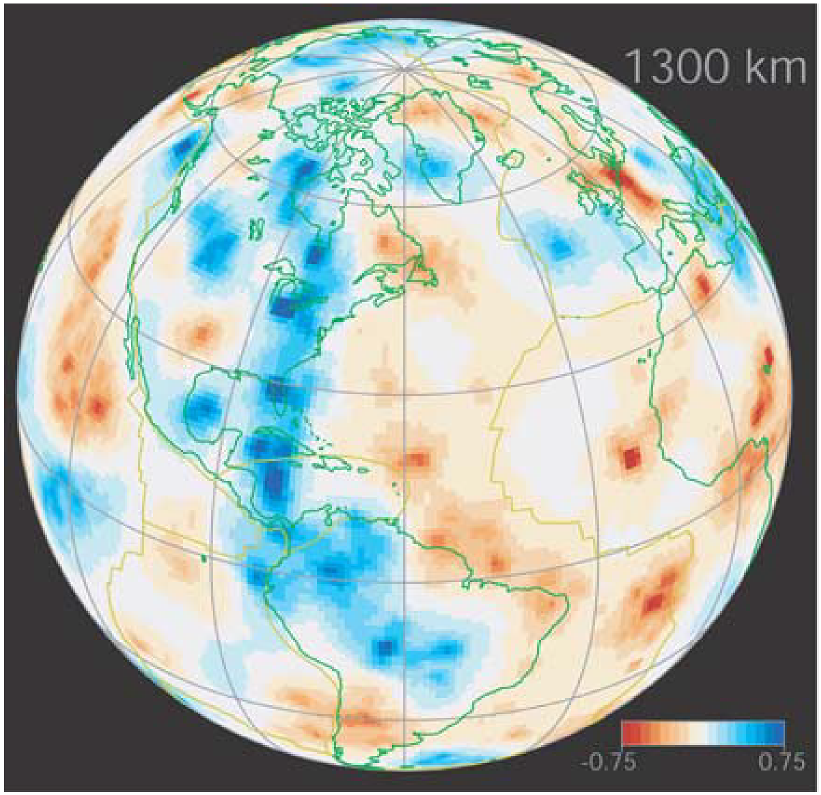


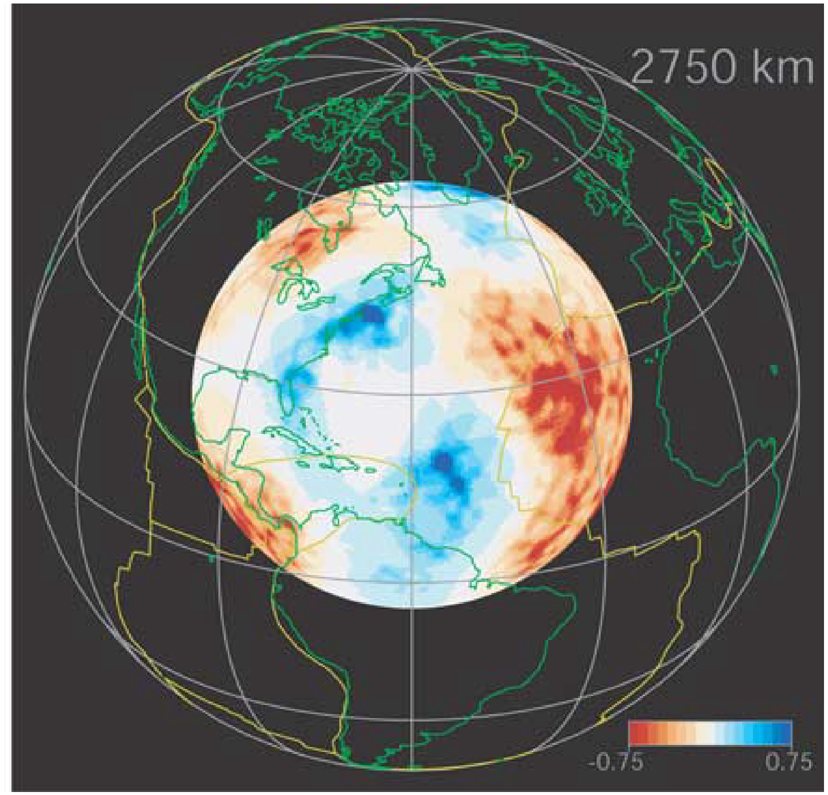


Single-Photon Emission Computed Tomography (SPECT)

Examples of tomographic imaging

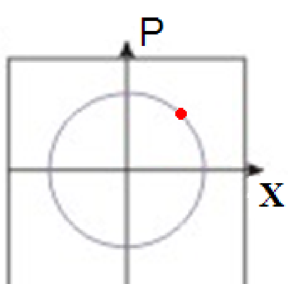
Seismic tomography

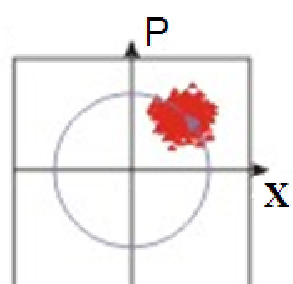


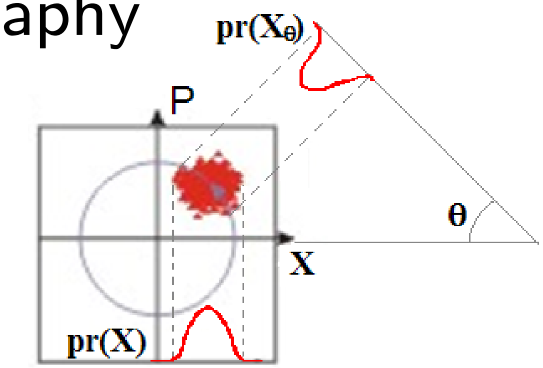


source: Sambridge et al. G3 Vol.4 Nr.3 (2003)

Quantum state tomography



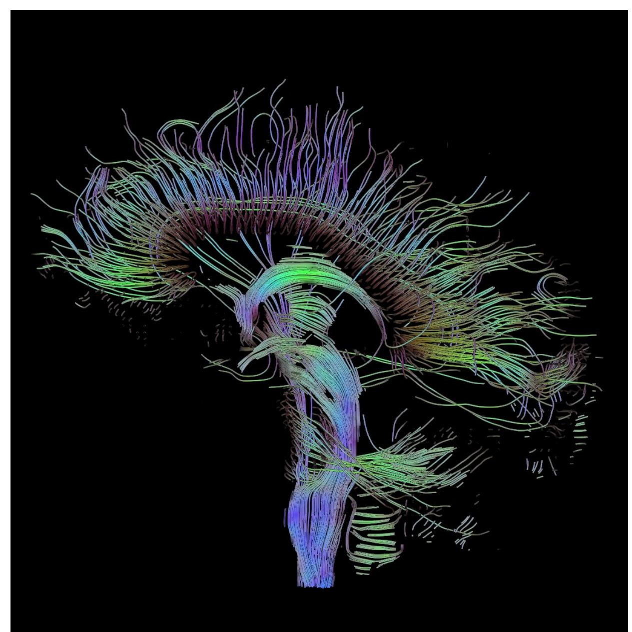




Examples of tomographic imaging

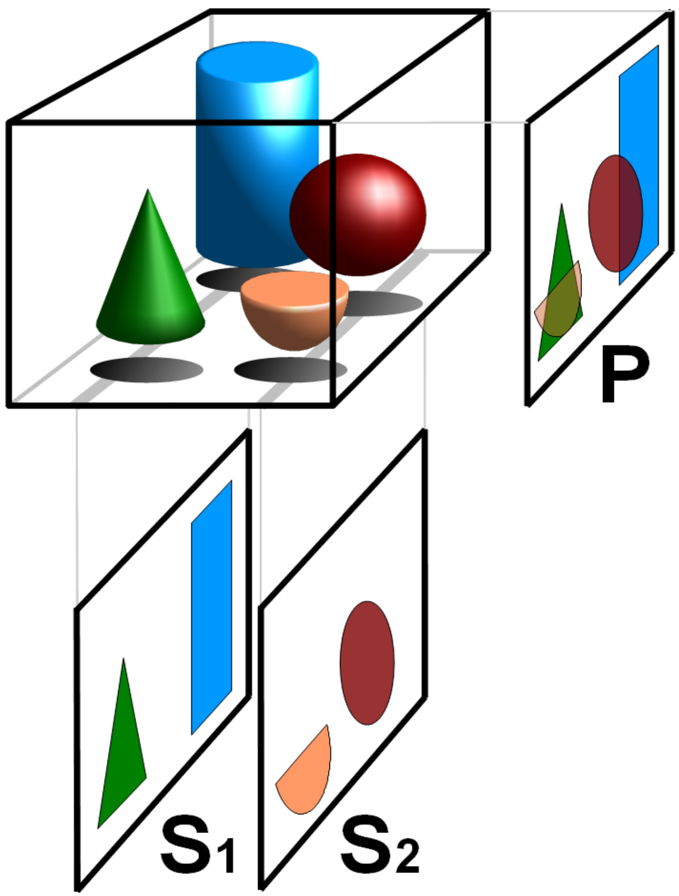
Ultrasonography/tomography (US/UST)

Magnetic resonance imaging/tomography (MRI/MRT)

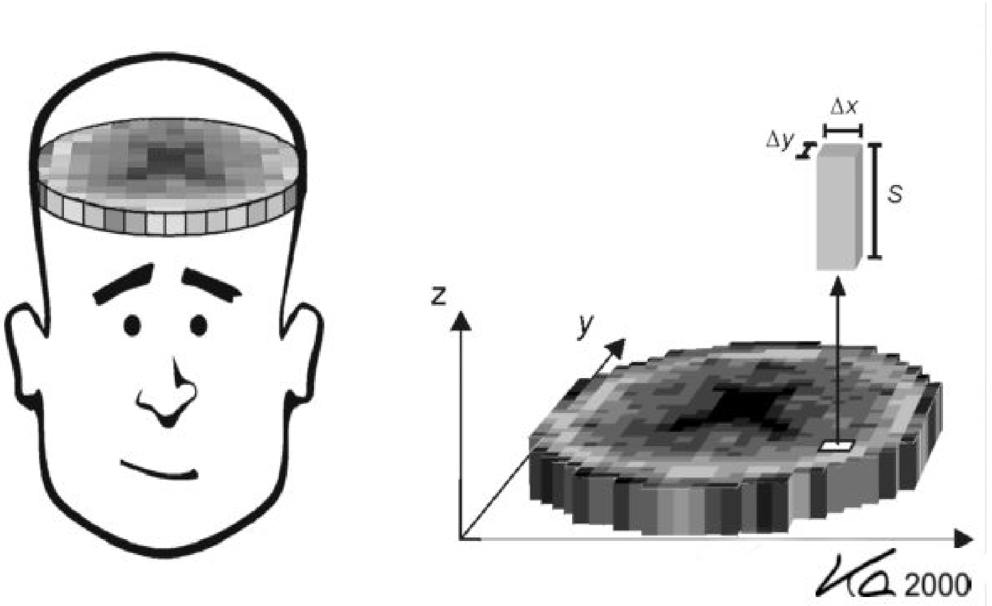


Reconstructions from projections

Reconstruction of volume from projections



Digitization into voxels



source: W. Kalender, Publicis, 3rd ed. 2011

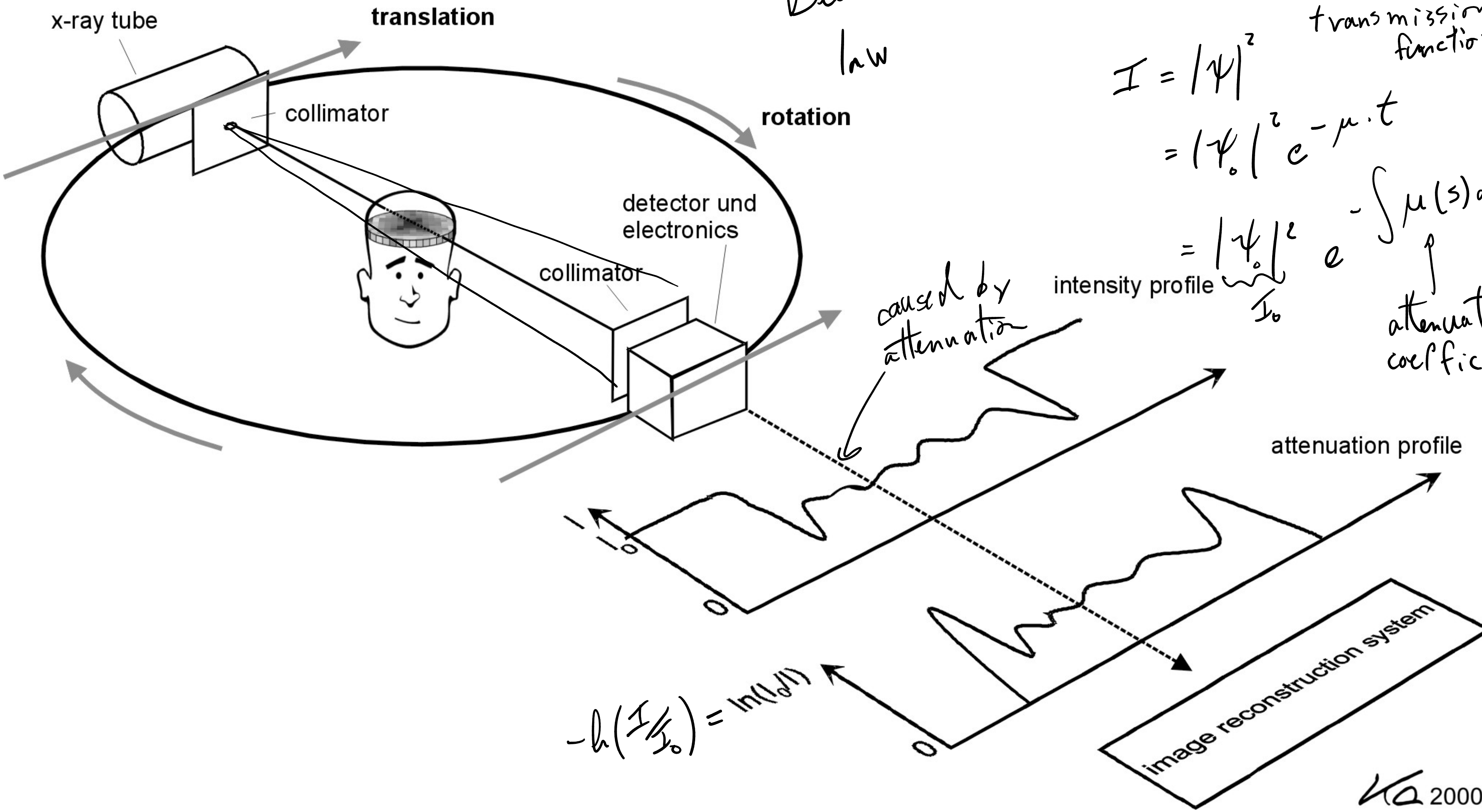
Principles of X-ray CT

Beer-Lambert law

$$\psi = \psi_0 \underbrace{e^{ik(n-1)t}}_{\text{transmission function}} \quad \leftarrow t\text{: thickness}$$

$$I = |\psi|^2 = |\psi_0|^2 e^{-\mu \cdot t} = \underbrace{|\psi_0|^2}_{I_0} e^{-\int \mu(s)\,ds}$$

$\mu(s)$: attenuation coefficient

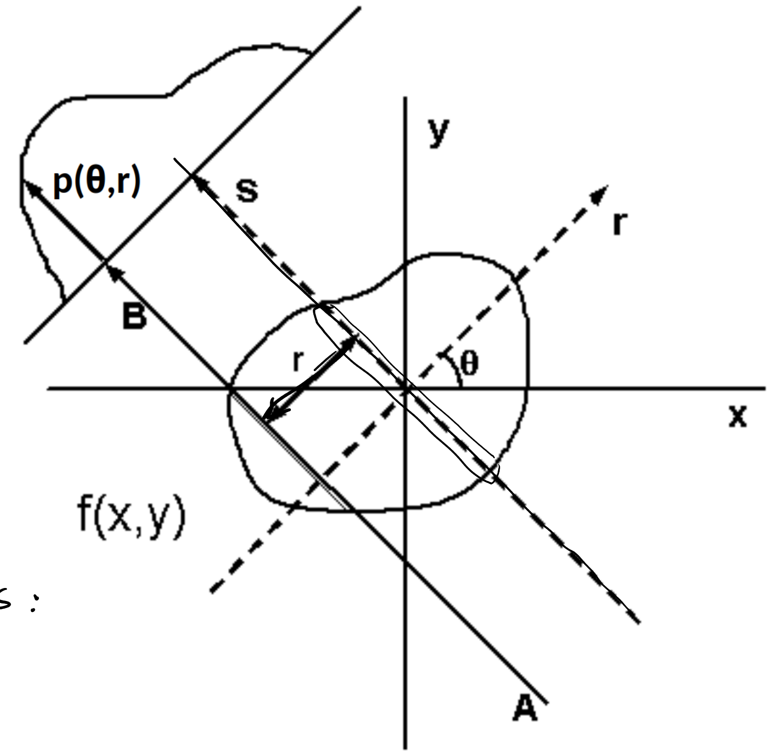


source: W. Kalender, Publicis, 3rd ed. 2011

Radon transform

Rotated coordinate system

Radon transform



$$p(r,\theta) = \iint f(x,y)\,\delta(r - (x\cos\theta + y\sin\theta))\,dx\,dy$$

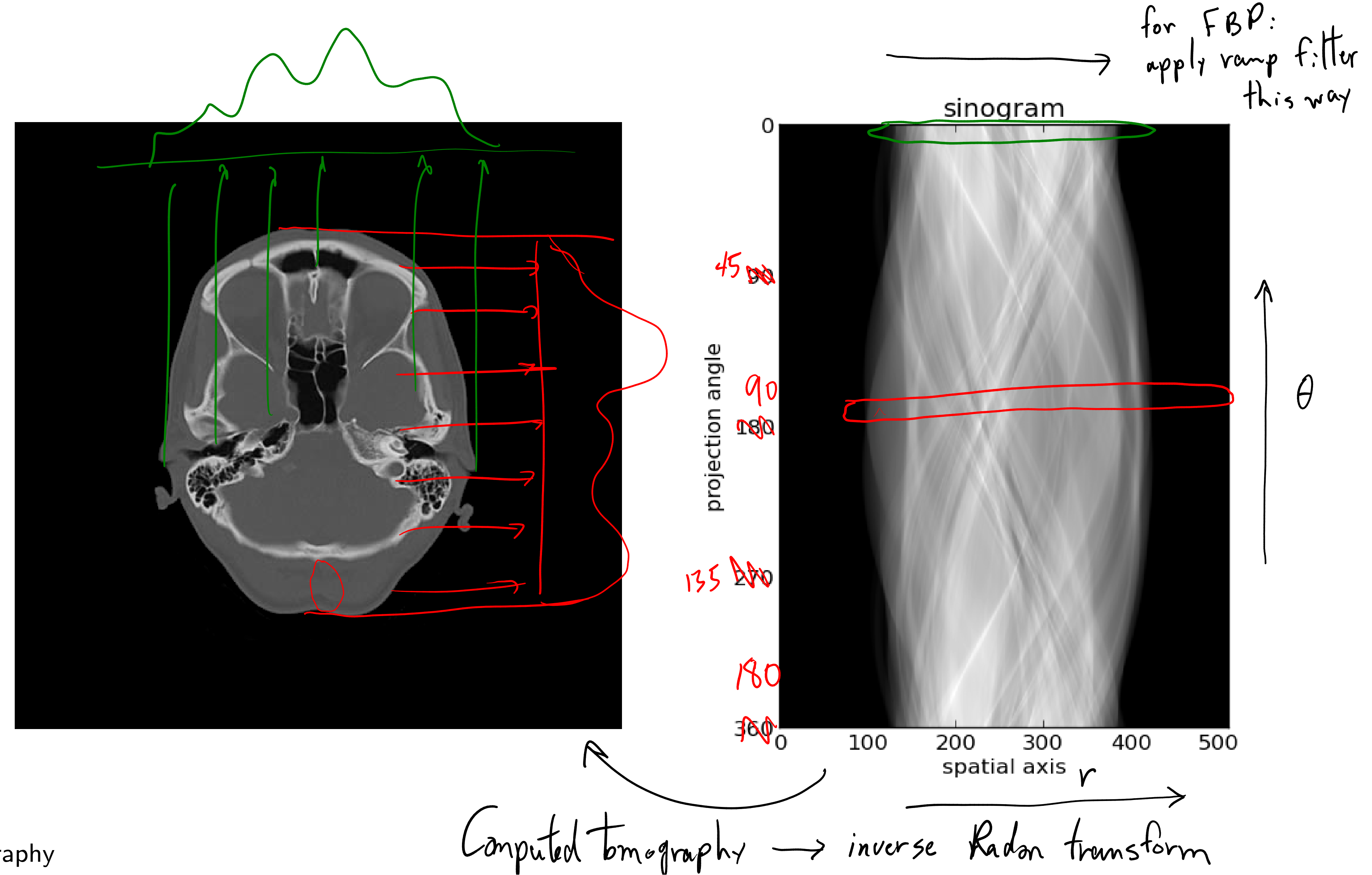
r not as in polar coordinates: can be negative

$$f(x,y) = ???\left(p(r,\theta)\right)$$

inverse Radon transform

Sinogram

Representation of projection measured by
a single detector line as a function of angle



The Fourier slice theorem

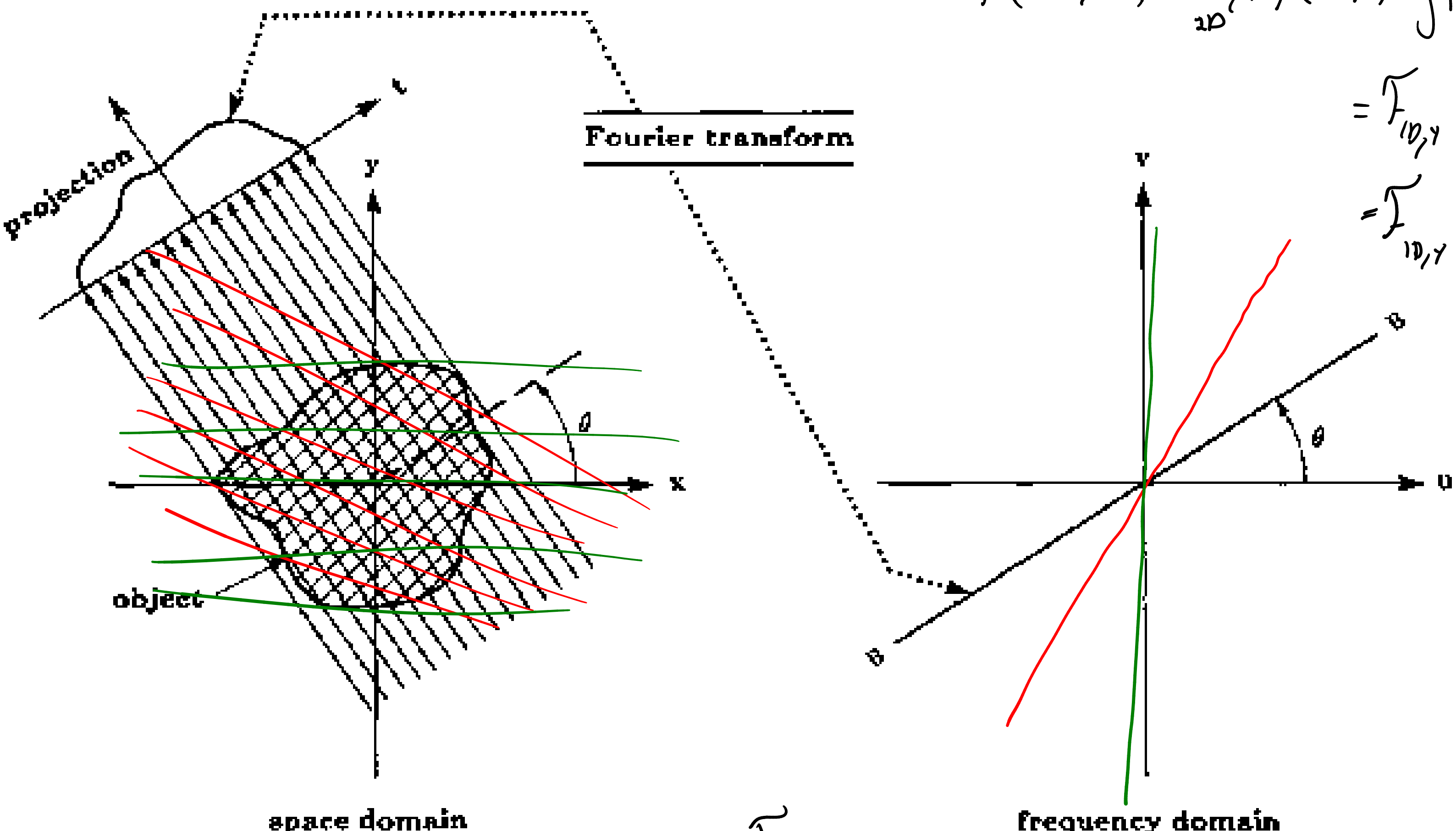
"DC term"

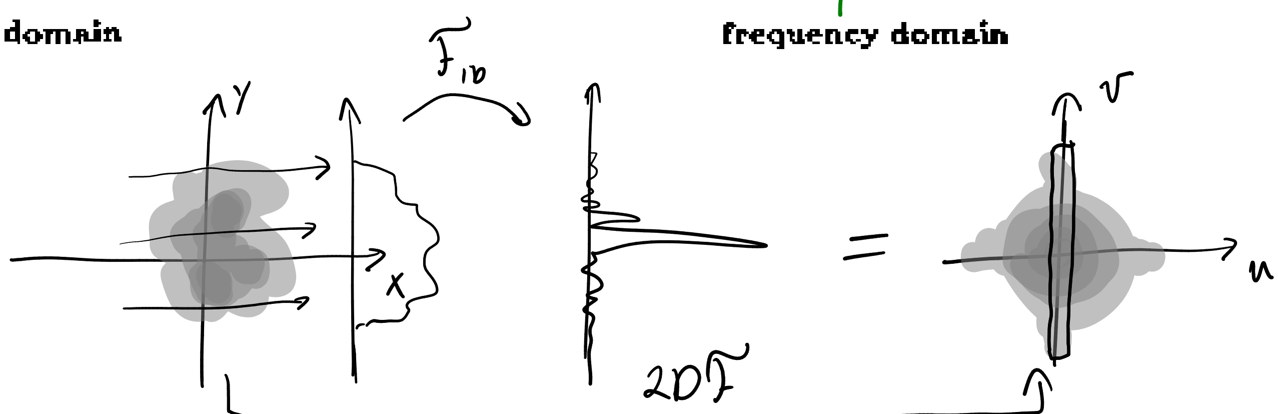
$$F(u=0) = \int f(x)\,dx$$

$$F(u=0,v) = \mathcal{F}_{2D}\{f\}(u=0,v) = \int f(x,y)\, e^{-ivy}\, dx\,dy$$

$$= \mathcal{F}_{1D,y}\{\mathcal{F}_{1D,x}\{f\}\}$$

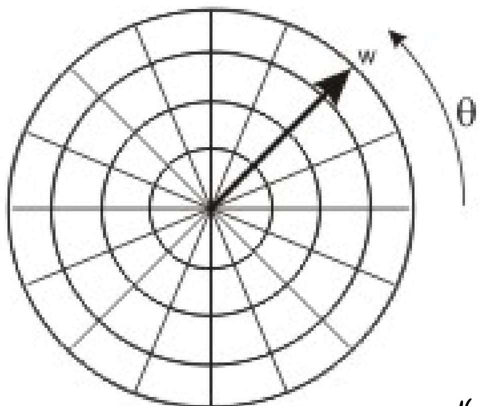
$$= \mathcal{F}_{1D,y}\{\underbrace{\int f(x,y)\,dx}_{\text{projection along x}}\}$$

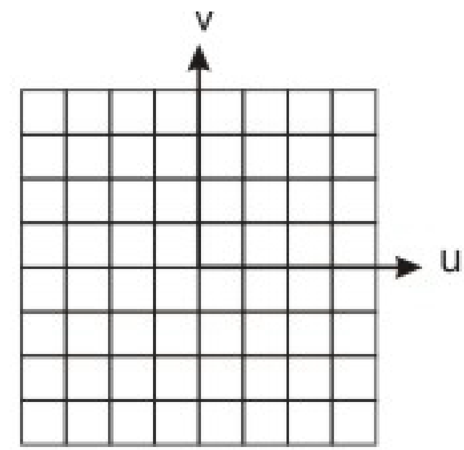




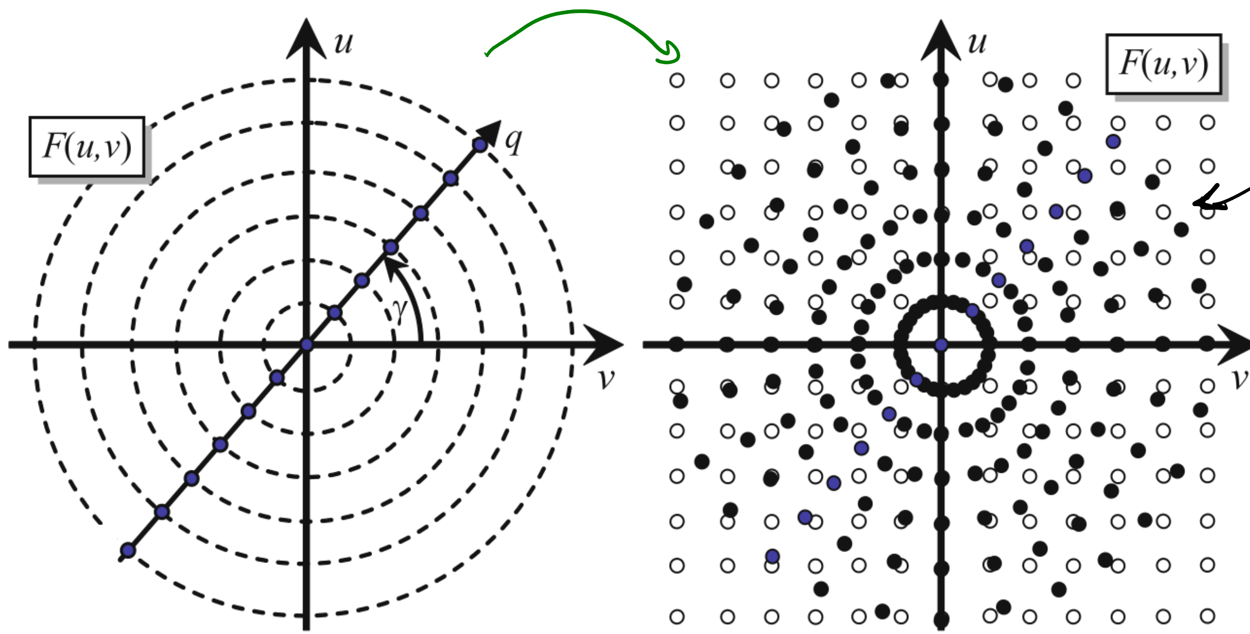
Frequency space sampling

Change of sampling grid from polar
to rectangular requires interpolation





"regridding"



interpolation required

a

b

Filtered back-projection

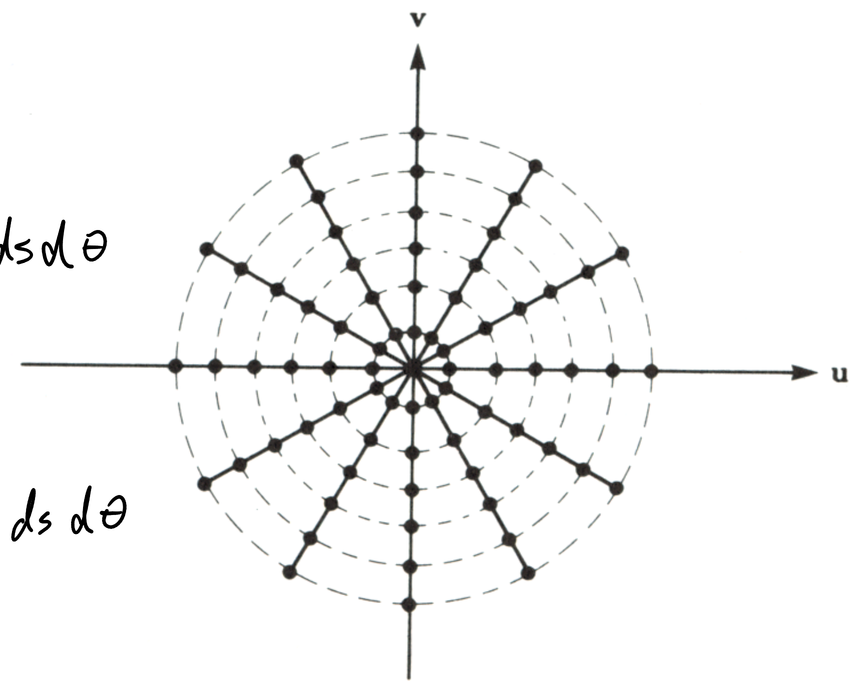
$$f(x,y) = \mathcal{F}^{-1}\left\{ F(u,v) \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i (ux+vy)}\, du\, dv$$

Polar coordinates: $u = s\cos\theta$, $v = s\sin\theta$, $du\,dv = s\,ds\,d\theta$

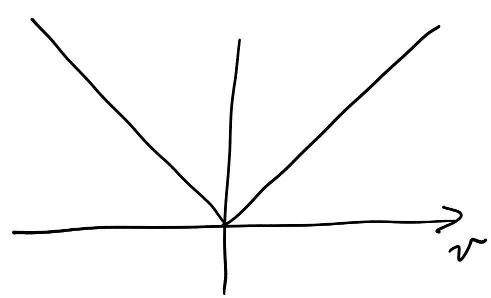
$$f(x,y) = \int_0^{\pi}\int_0^{\infty} F(s\cos\theta, s\sin\theta)\, e^{2\pi i s(x\cos\theta + y\sin\theta)}\, s\, ds\, d\theta$$

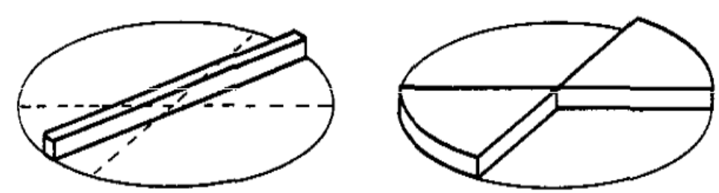
e.g. $\theta = 90^\circ$: $\int F(u=0, v)\, e^{2\pi i v y}\, v\, dv$



frequency domain

ramp filter





Filtered back-projection

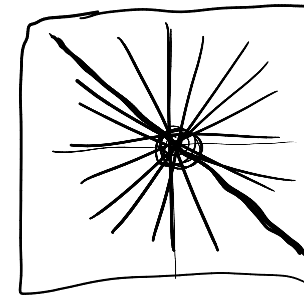
Radon transform $\longrightarrow \mathcal{F}$ along r

$$\mathcal{F}_r\{p(r,\theta)\} = F(w\cos\theta, w\sin\theta)$$

$F(u,\cdot)$: 2d F.T. of $f(x,y)$

filtered

$$FBP = \int d\theta\, \tilde{p}(x\cos\theta + y\sin\theta, \theta)$$



Filter

$$\tilde{p}(r,\theta) = \int_{-\infty}^{\infty} F(w\cos\theta, w\sin\theta)\, e^{2\pi i r w}\, w\, dw$$

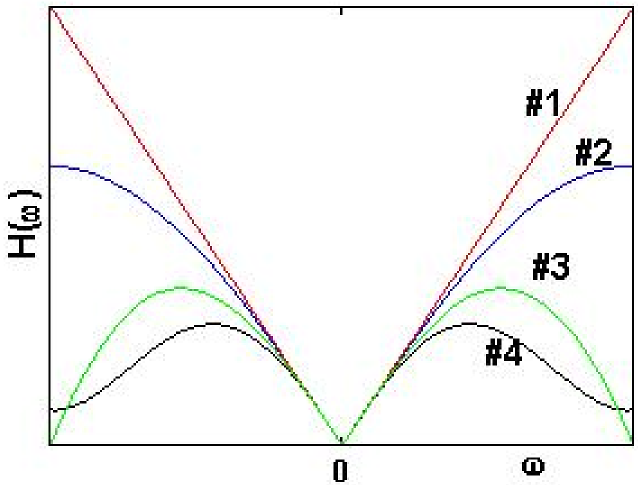
$$\tilde{p}(x\cos\theta + y\sin\theta, \theta) = \int_{-\infty}^{\infty} F(w\cos\theta, w\sin\theta)\, e^{2\pi i (x(w\cos\theta) + y w\sin\theta)}\, w\, dw$$

$$\int d\theta\, \tilde{p}(x\cos\theta + y\sin\theta, \theta) = \iint F(w\cos\theta, w\sin\theta)\, e^{2\pi i (x w\cos\theta + y w\sin\theta)}\, w\, dw\, d\theta$$

$\mathcal{F}^{-1}$ in polar coordinates! $= f(x,y)$!

Filtered back-projection

- Filter can be tuned to achieve image enhancement
- Trade-off between noise and sharpness

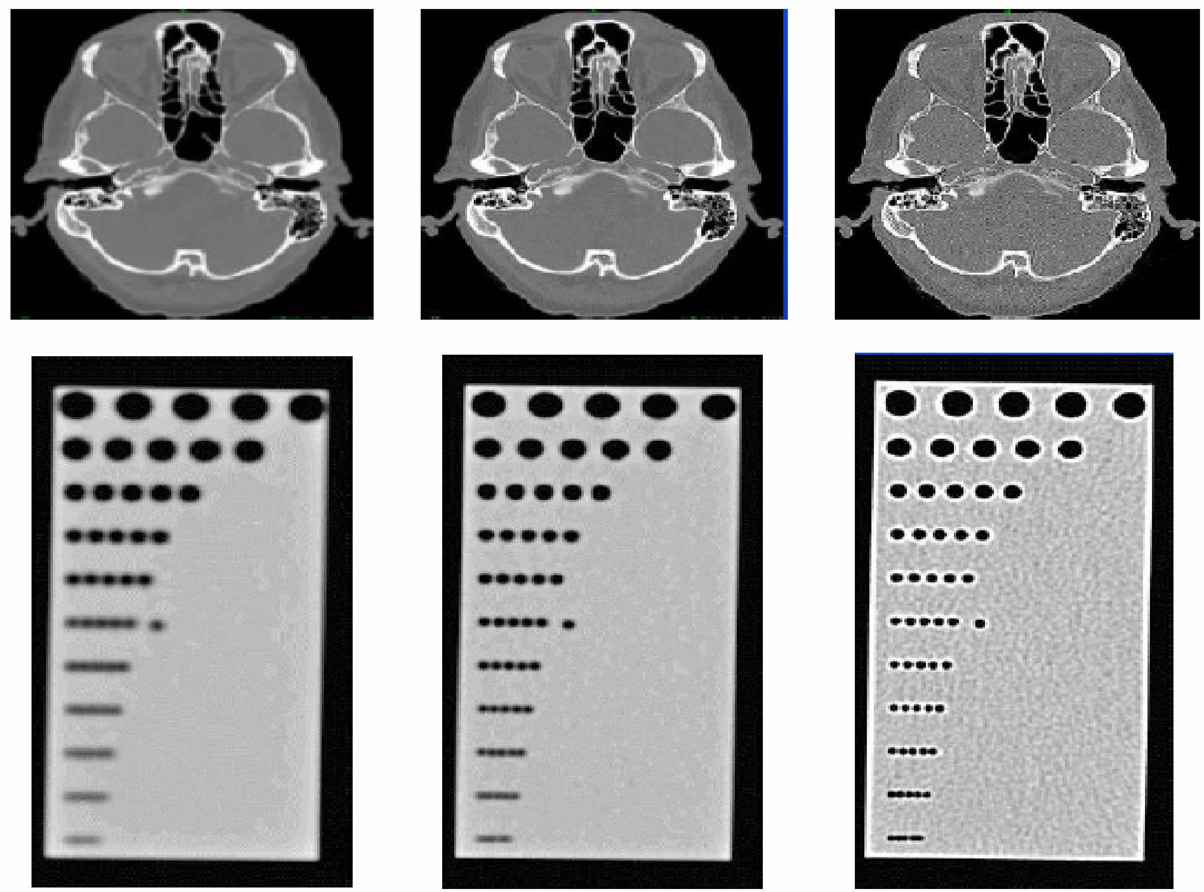


#1 ram-lak (ramp)

#2 Shepp-Logan

#3 cosine

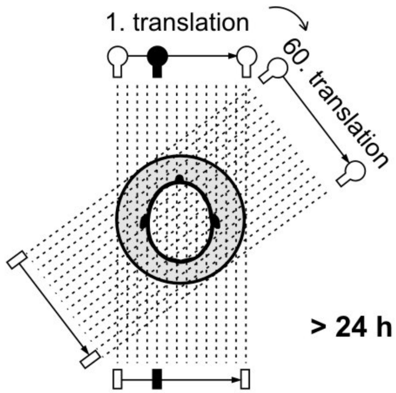
#4 Hamming



smoothing standard edge enhanced

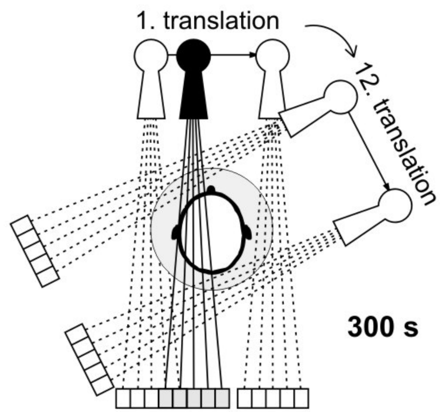
Geometries

pencil beam (1970)

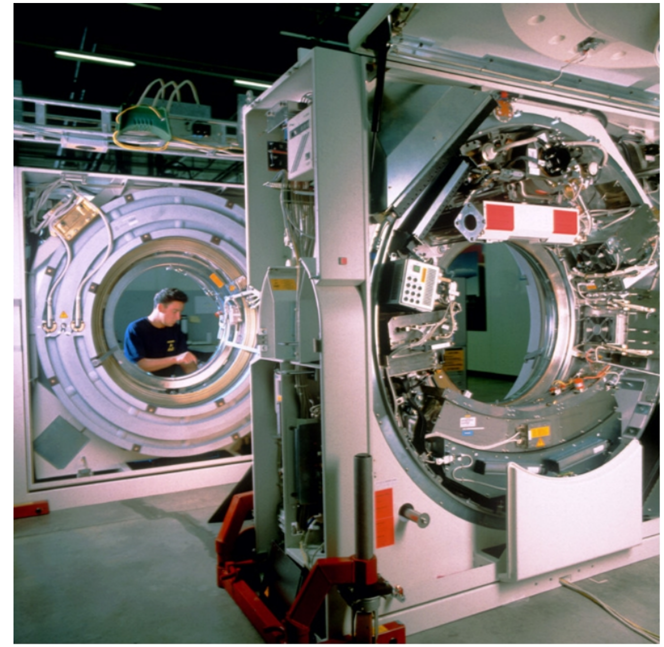


1st generation: translation / rotation

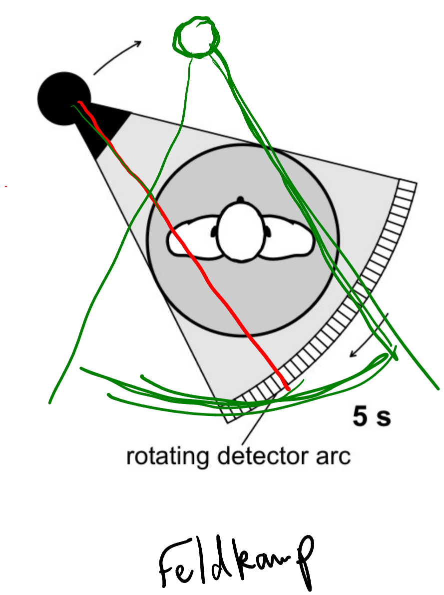
partial fan beam (1972)



2nd generation: translation / rotation

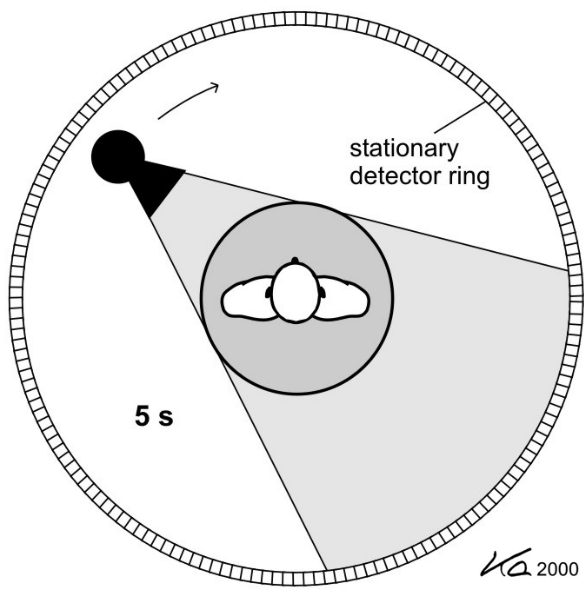


fan beam (1976)

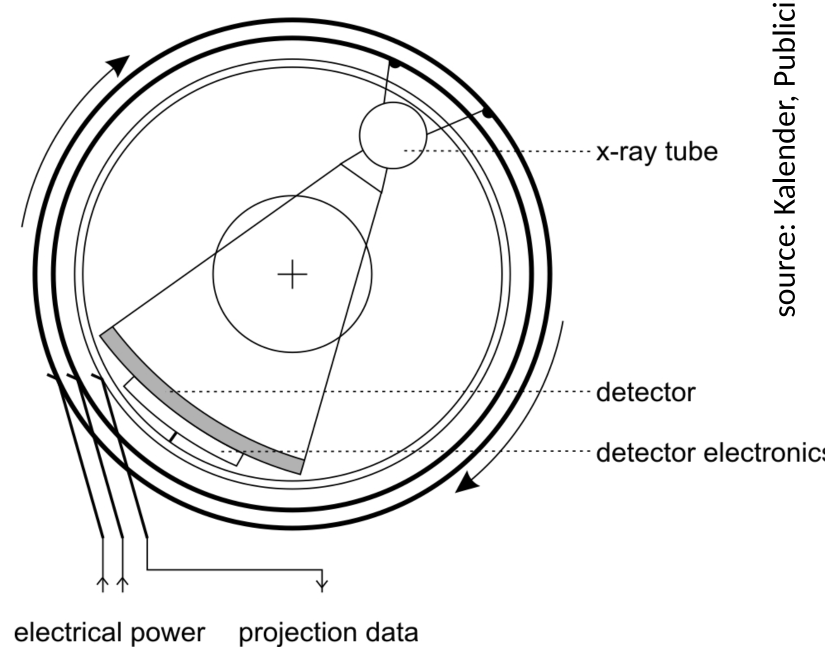


3rd generation: continuous rotation

fan beam (1978)



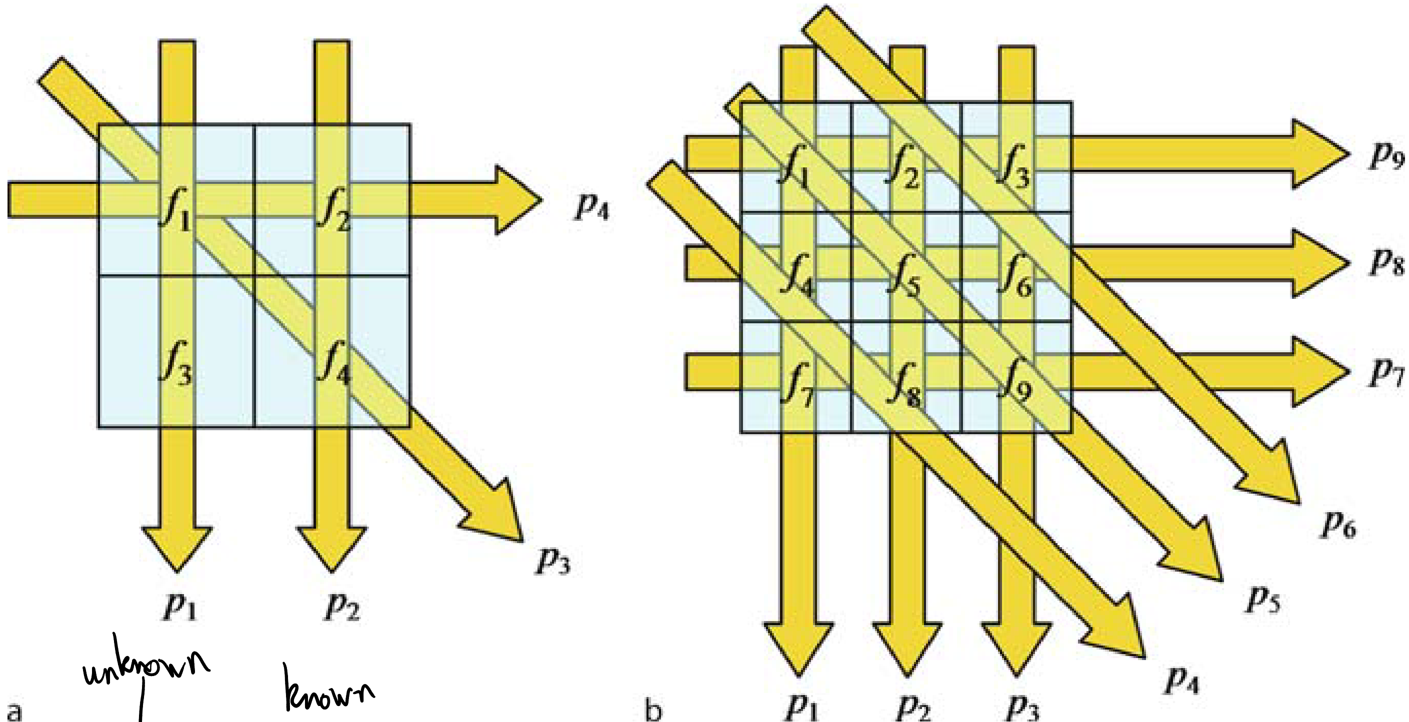
4th generation: continuous rotation



source: Kalender, Publicis, 3rd ed. 2011

Algebraic formulation

Tomography can be formulated as a set of linear equations



unknown ↓ known ↓

$f_1 + f_2 = p_4$

$f_1 + f_3 = p_1$

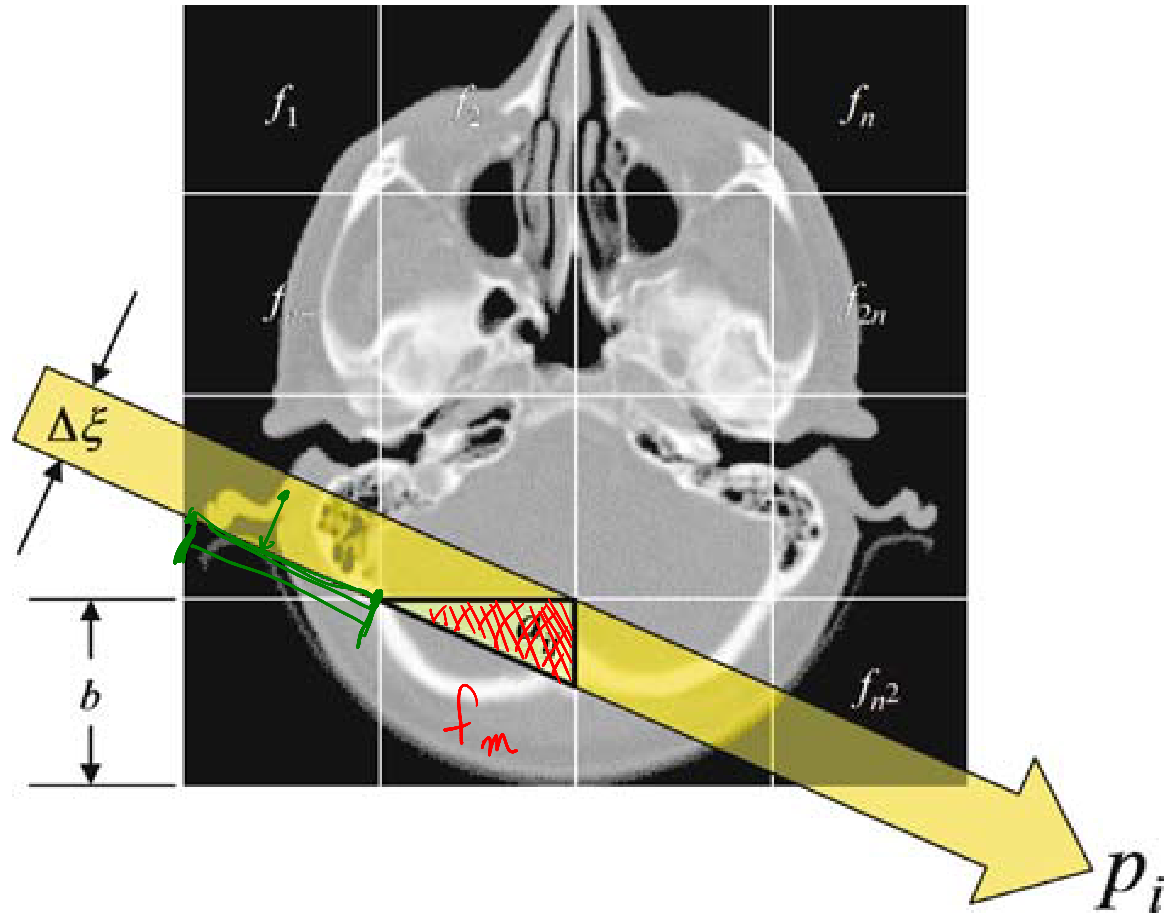
$\vdots$

source: Buzug, Springer, 1st ed. 2008

Weighting coefficients

Weighting measures:

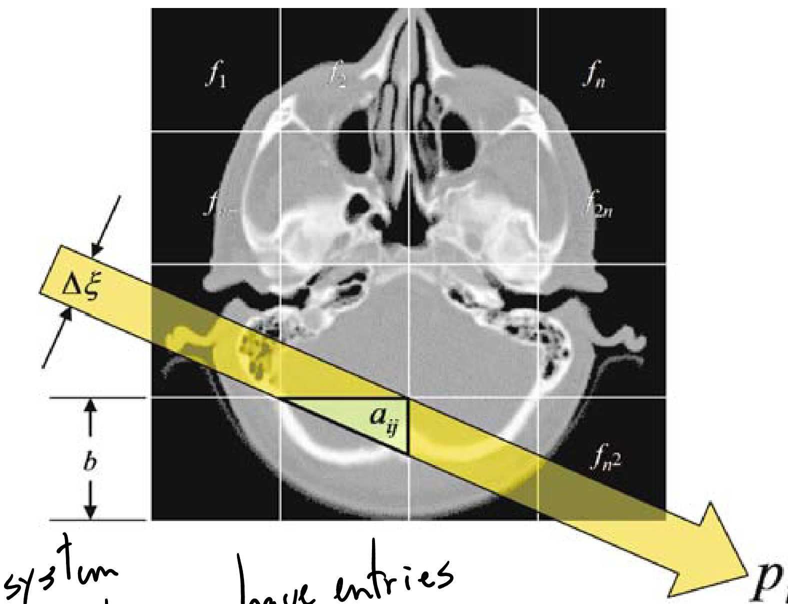
- Logic
- Area
- Path length
- Distance to pixel center



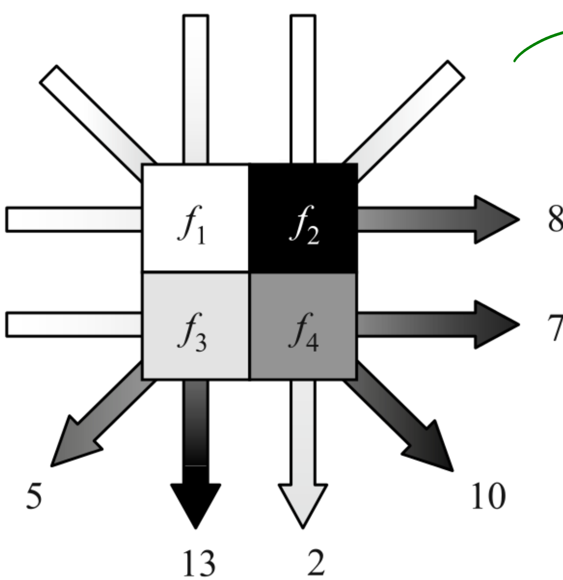
Differences in calculation effort, smoothness, noise sensitivity, ...

source: Buzug, Springer, 1st ed. 2008

System Matrix



real system matrices have entries between 0 and 1



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

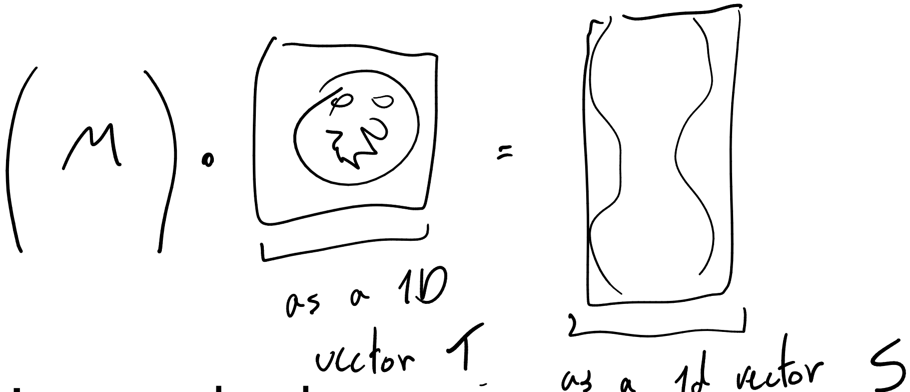
source: Buzug, Springer, 1st ed. 2008

Matrix (pseudo)-inversion

$M^{-1} = (M^T M)^{-1} M^T$

pseudo inverse
"best" solution in the least square sense

Tomographic reconstruction = linear system inversion



$M \cdot T = S$

$T \sim 1000 \times 1000 = 10^6$

$S \sim 1000 \times 1000 = 10^6$

M : 1000000 x 1000000 matrix

Iterative methods:

- ART Algebraic reconstruction technique
- SART Simultaneous algebraic reconstruction technique
- SIRT Simultaneous iterative reconstruction technique
- MART Multiplicative algebraic reconstruction technique
- MLEM Maximum likelihood expectation maximization
- OSEM Ordered subset expectation maximization
- ... and many, many more

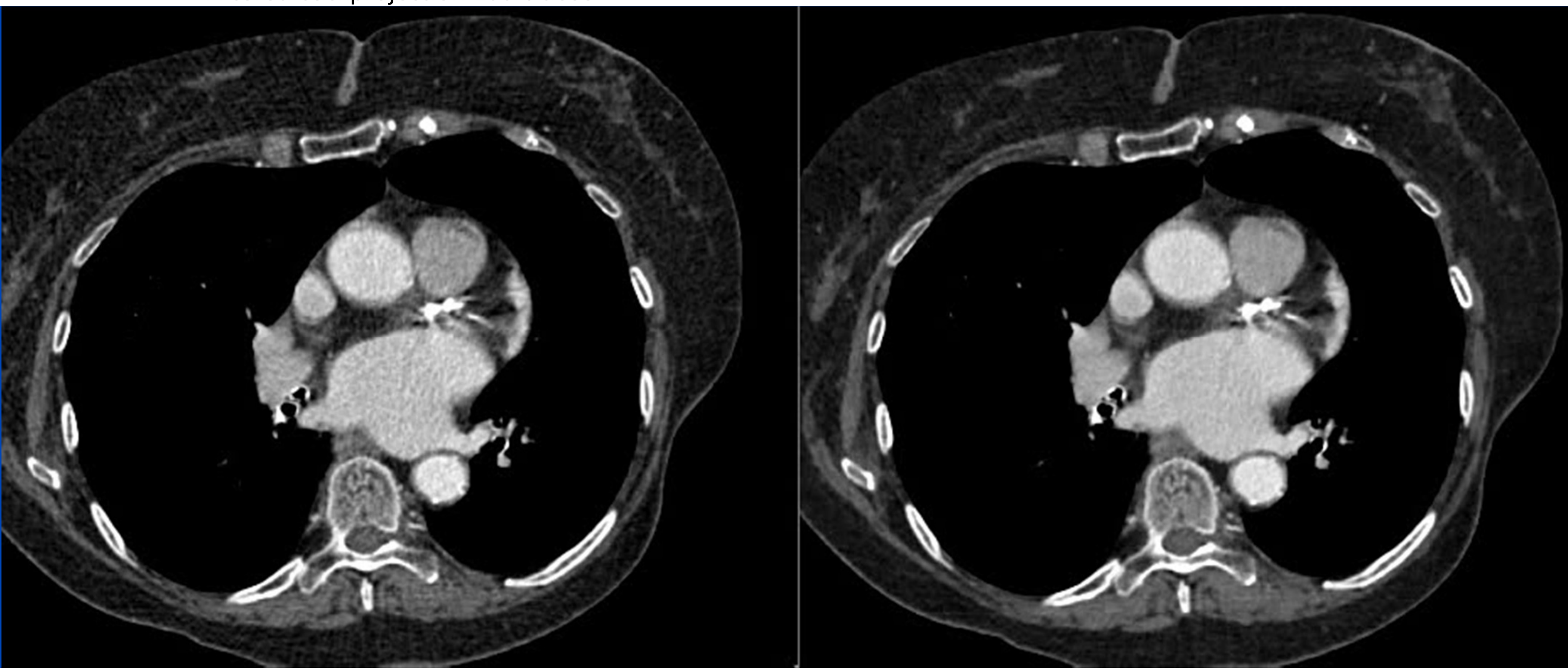
FBP vs algebraic methods

FBP

ART (or SART)

Filtered backprojection 100% dose

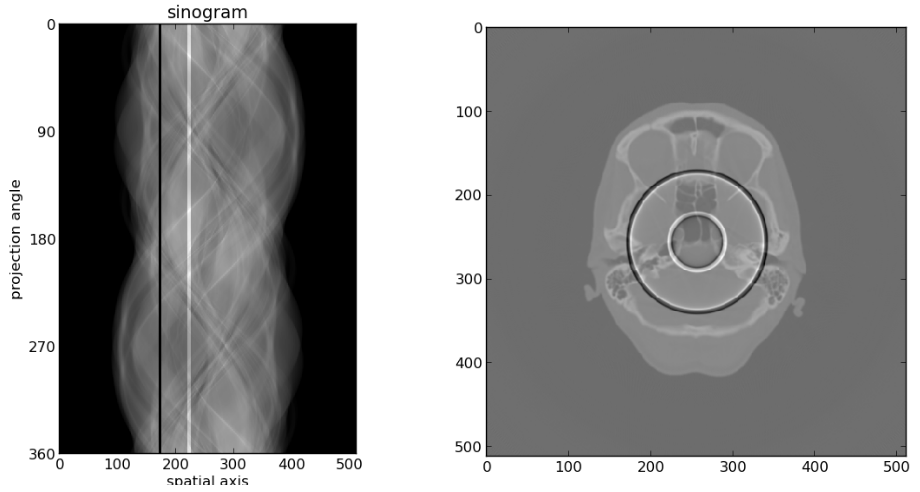
iterative 40% dose



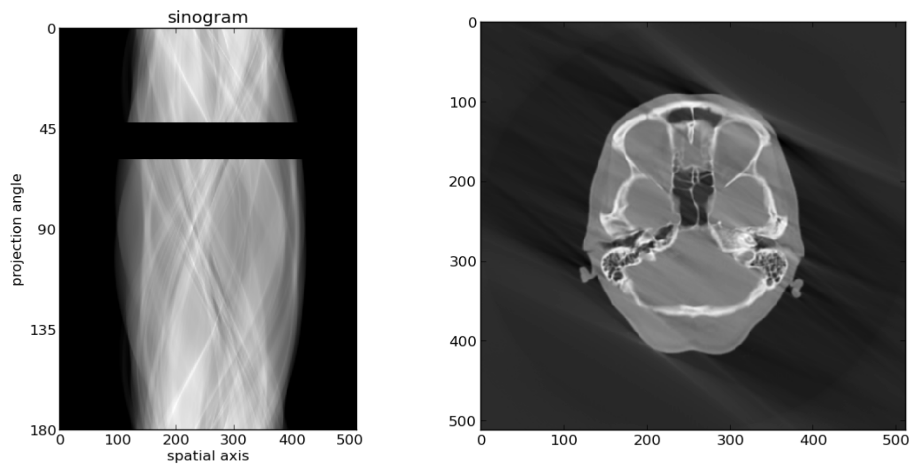
source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

Artifacts

Detector imperfections → ring artifacts

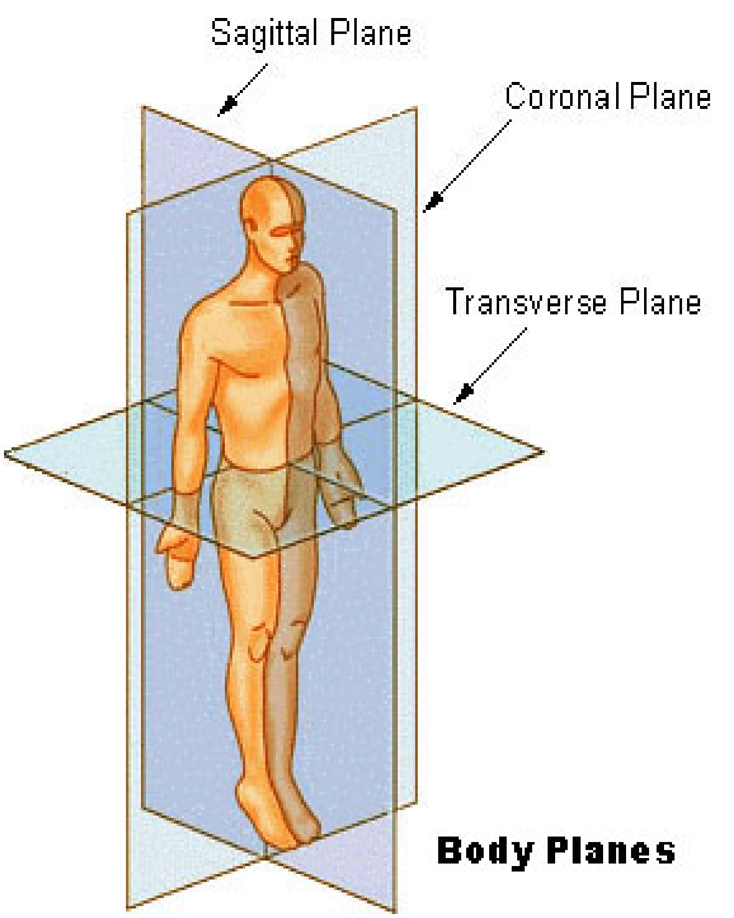


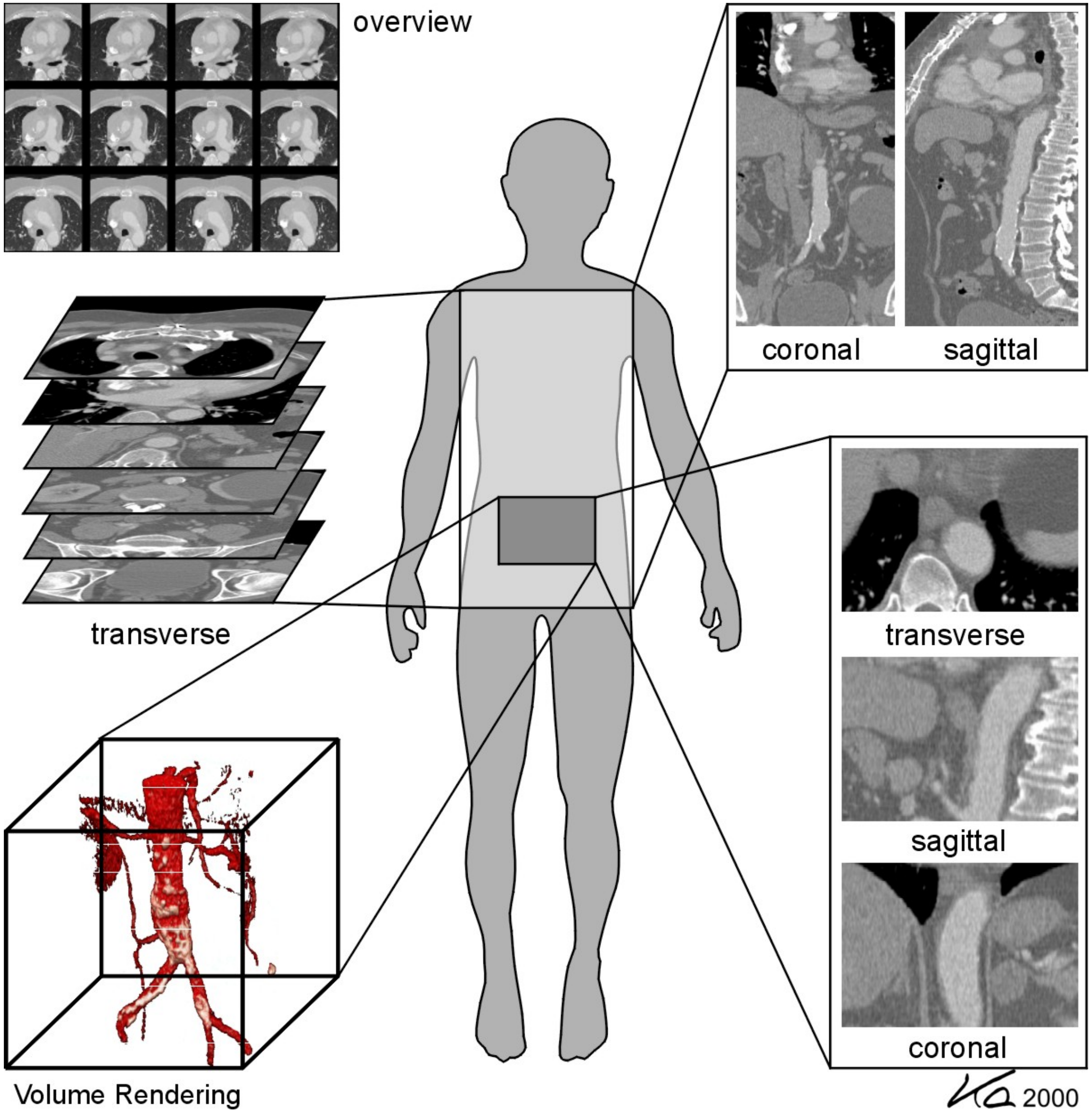
Missing projections → "streak" artifacts



Also: sample motion, beam hardening, ...

Tomographic Display

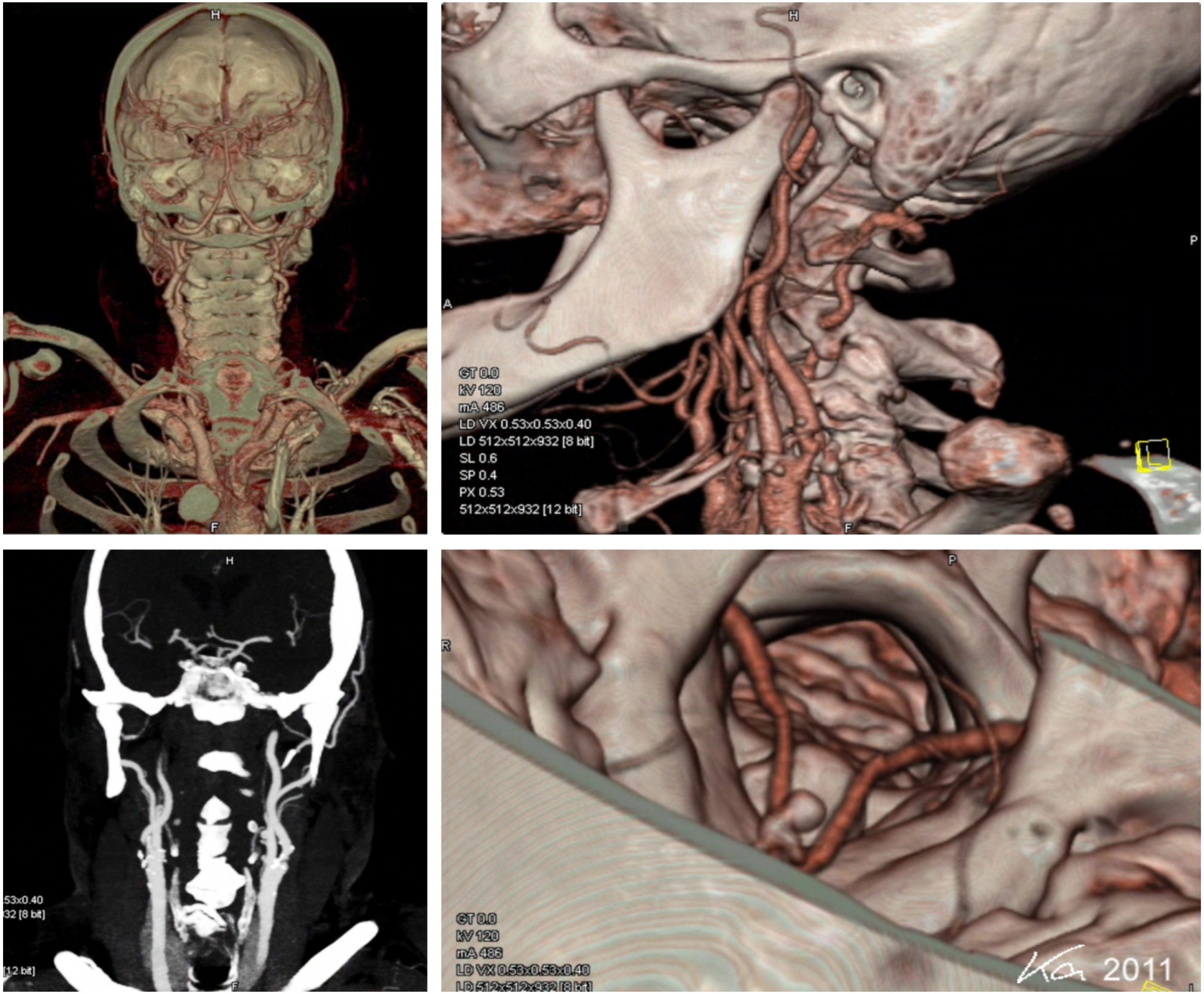




source: http://wikipedia.org

source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

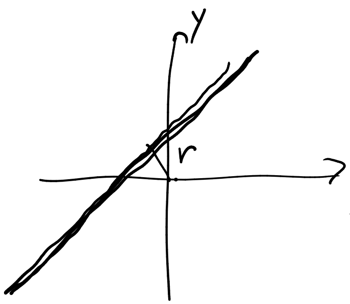
Summary

- Computed tomography: reconstruction from projections
- Analytic approach:
  - Projections and tomographic slices are related by the Fourier slice theorem
  - Standard algorithm uses filtered back-projection
- Algebraic approach:
  - Tomography as a system of linear equations
  - Iterative methods are used for large matrix inversions
  - More powerful but computationally more costly
- Imperfect data leads to artifacts

Radon transform (reminder)

w : reciprocal variable to r

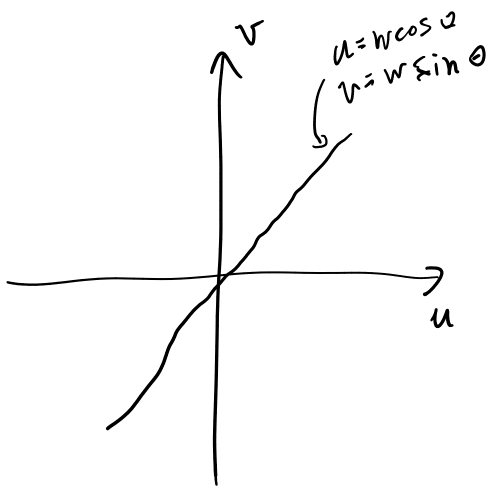
$$p(r,\theta) = \iint f(x,y)\,\delta(r-(x\cos\theta + y\sin\theta))\,dx\,dy$$



$$\mathcal{F}_r\{p(r,\theta)\} = \int_{-\infty}^{\infty} p(r,\theta)\, e^{-2\pi i r w}\, dr$$

$$= \int_{-\infty}^{\infty} \iint f(x,y)\,\delta(r-(x\cos\theta + y\sin\theta))\, e^{-2\pi i r w}\, dx\,dy\,dr$$

$$= \iint f(x,y)\exp\left(-2\pi i w(x\cos\theta + y\sin\theta)\right) dx\,dy$$



$$= \iint f(x,y)\exp\Big[-2\pi i\big(x\cdot\underbrace{w\cos\theta}_{u} + y\cdot\underbrace{w\sin\theta}_{v}\big)\Big]\,dx\,dy$$

$$= F(u = w\cos\theta,\ v = w\sin\theta)$$