Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA
PER IL SISTEMA TERRA

Harmonic oscillators

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What is an Oscillation?

Oscillation is the variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. Familiar examples include a swinging pendulum and AC power.

The term **vibration** is sometimes used more narrowly to mean a mechanical oscillation but sometimes is used to be synonymous with "oscillation".

Physical

Any motion that **repeats** itself after an interval of time

Engineering

Deals with the relationship between **forces** and oscillatory **motion** of bodies

Mechanical systems

Deal with Oscillations

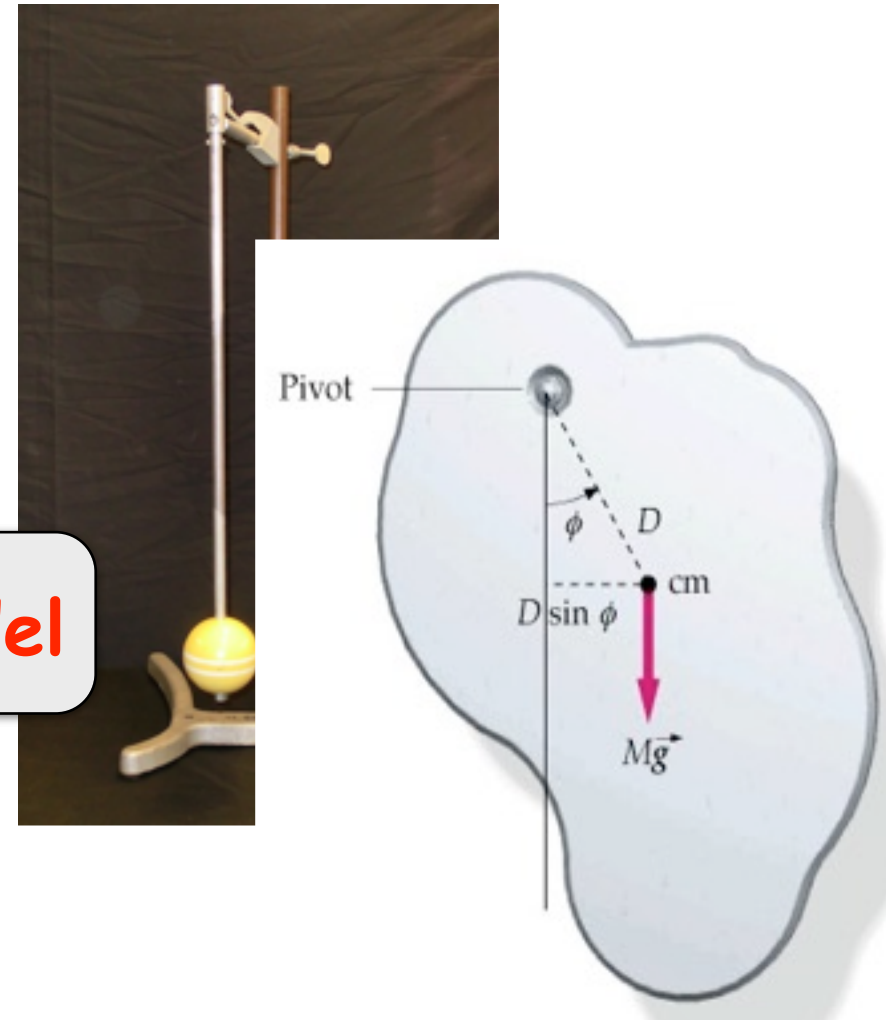
Physical system

Deal with Oscillations

Physical system

Modelling

Engineering model



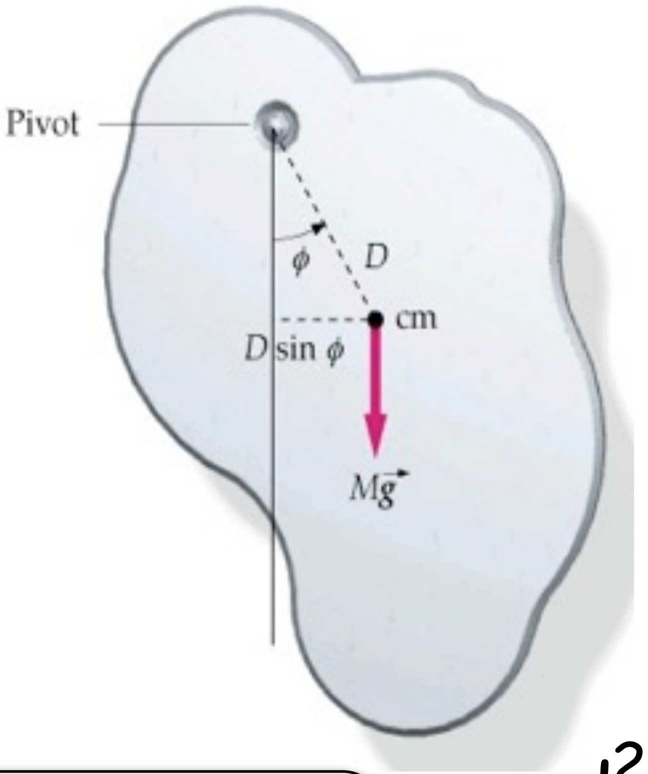
Physical system

Modelling

Engineering model

Physical law

Mathematical model



$$\frac{d^2\phi}{dt^2} = -\frac{MgD}{I}\phi$$

Deal with Oscillations

Physical system

Modelling

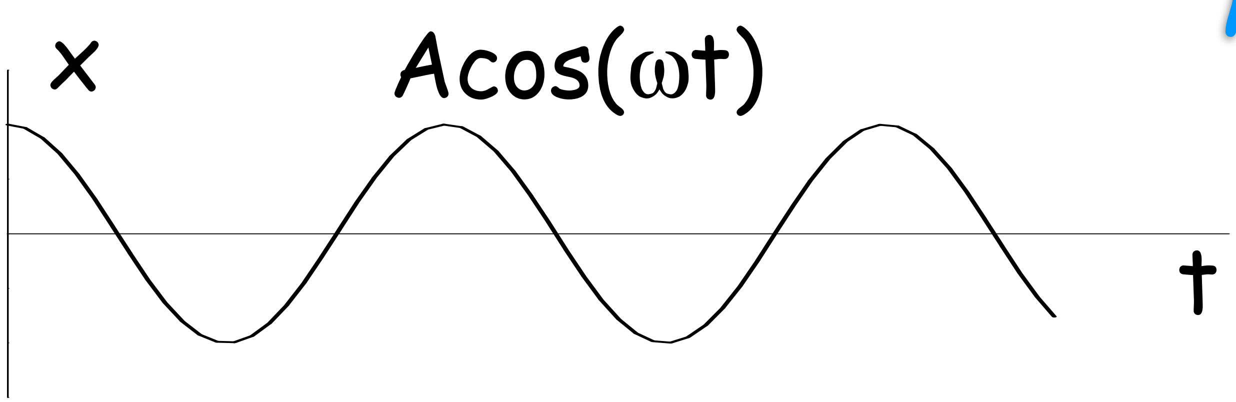
Engineering model

Physical law

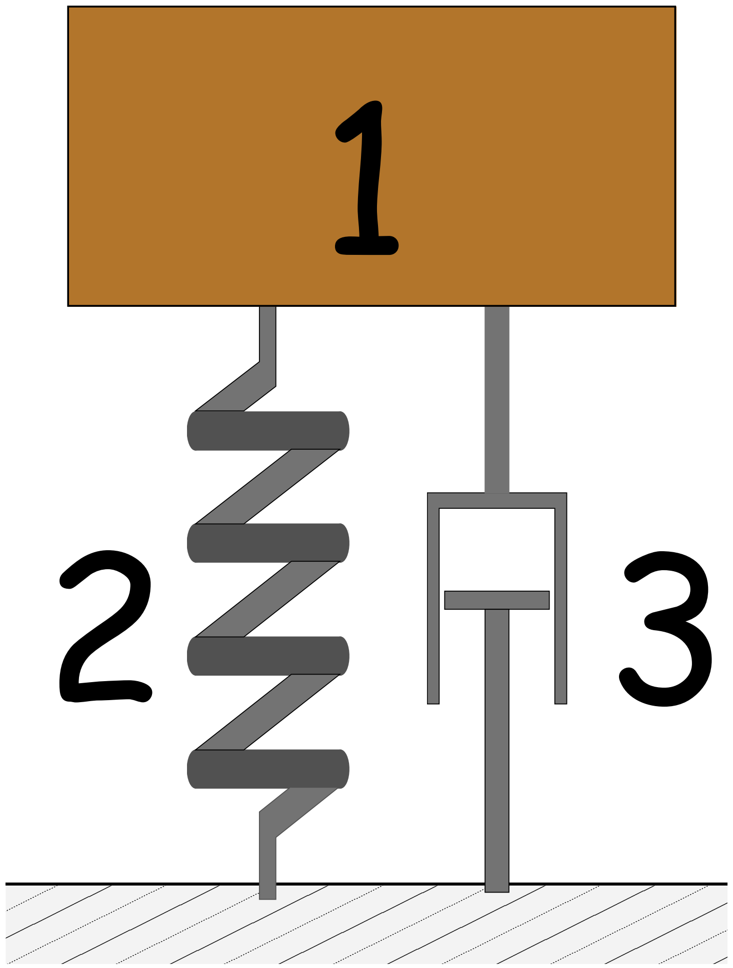
Mathematical model

$$\frac{d^2\phi}{dt^2} = -\frac{MgD}{I}\phi$$

Maths



Math solution



The Ingredients:

- **Inertia** (stores kinetic energy)
- **Elasticity** (stores potential energy)

Realistic Addition:

- Energy Dissipation

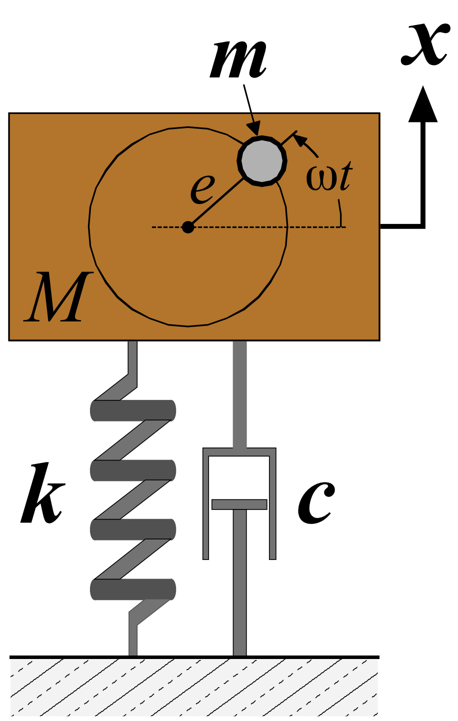
- mass
- stiffness
- damping

- to model lots of physical systems: engines, water towers, building etc...

Resonance

A vibration of large amplitude, that occurs when an object is forced near its natural frequency

Object

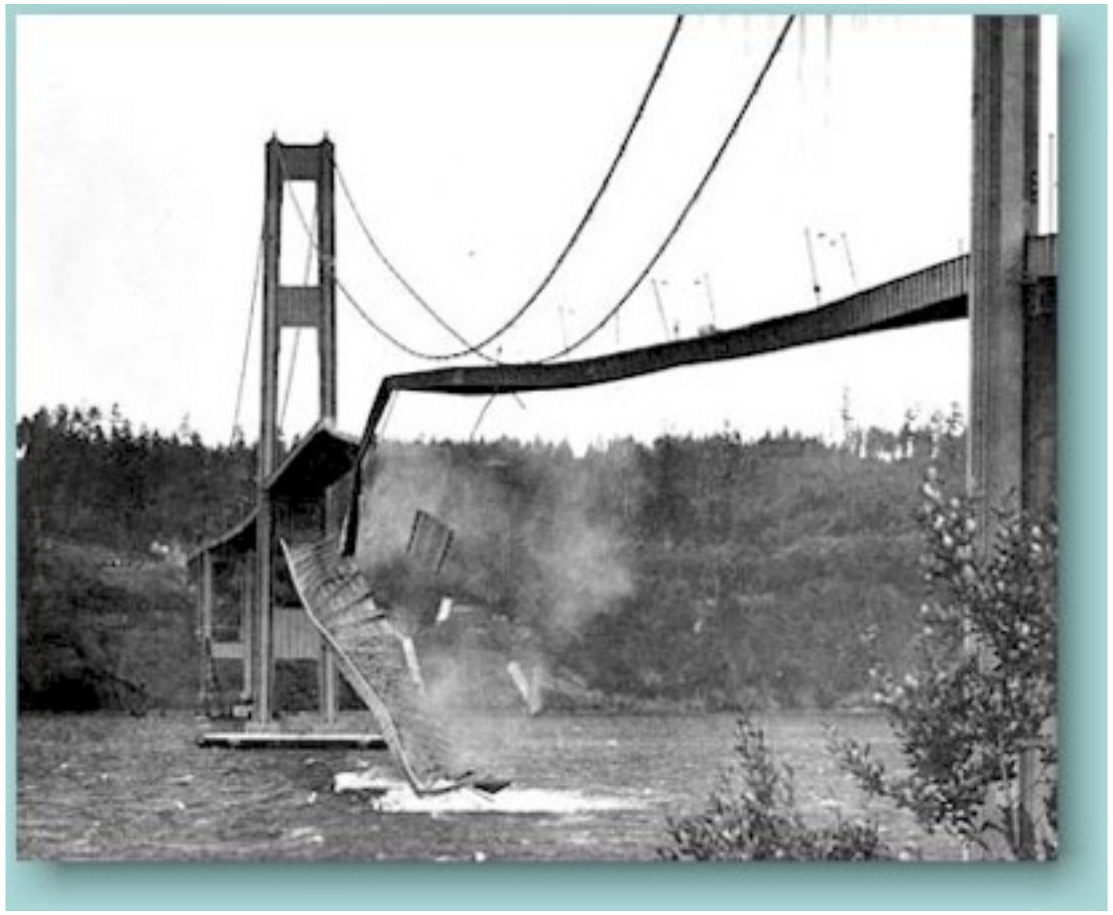


Model

Tacoma Bridge

The original, 5,939-foot-long Tacoma Narrows Bridge, popularly known as "Galloping Gertie," opened to traffic on July 1, 1940 after two years of construction, linking Tacoma and Gig Harbor. It collapsed just four months later during a 42-mile-per-hour wind storm on Nov. 7, 1940.

The bridge earned the nickname "Galloping Gertie" from its rolling, undulating behavior. Motorists crossing the 2,800-foot center span sometimes felt as though they were traveling on a giant roller coaster, watching the cars ahead disappear completely for a few moments as if they had been dropped into the trough of a large wave.



Oscillatory motion

The motion of an object can be predicted if the external forces acting upon it are known.

A special type of motion occurs when the force on the object is proportional to the displacement of the object from equilibrium.

If this force always acts towards the equilibrium position a back and forth motion will results about the equilibrium position.

This is known as periodic or oscillatory motion.

Familiar examples of periodic motion

1. Pendulum

2. Vibrations of a stringed instrument

3. Mass on a spring

Other examples include

1. Molecules in a solid

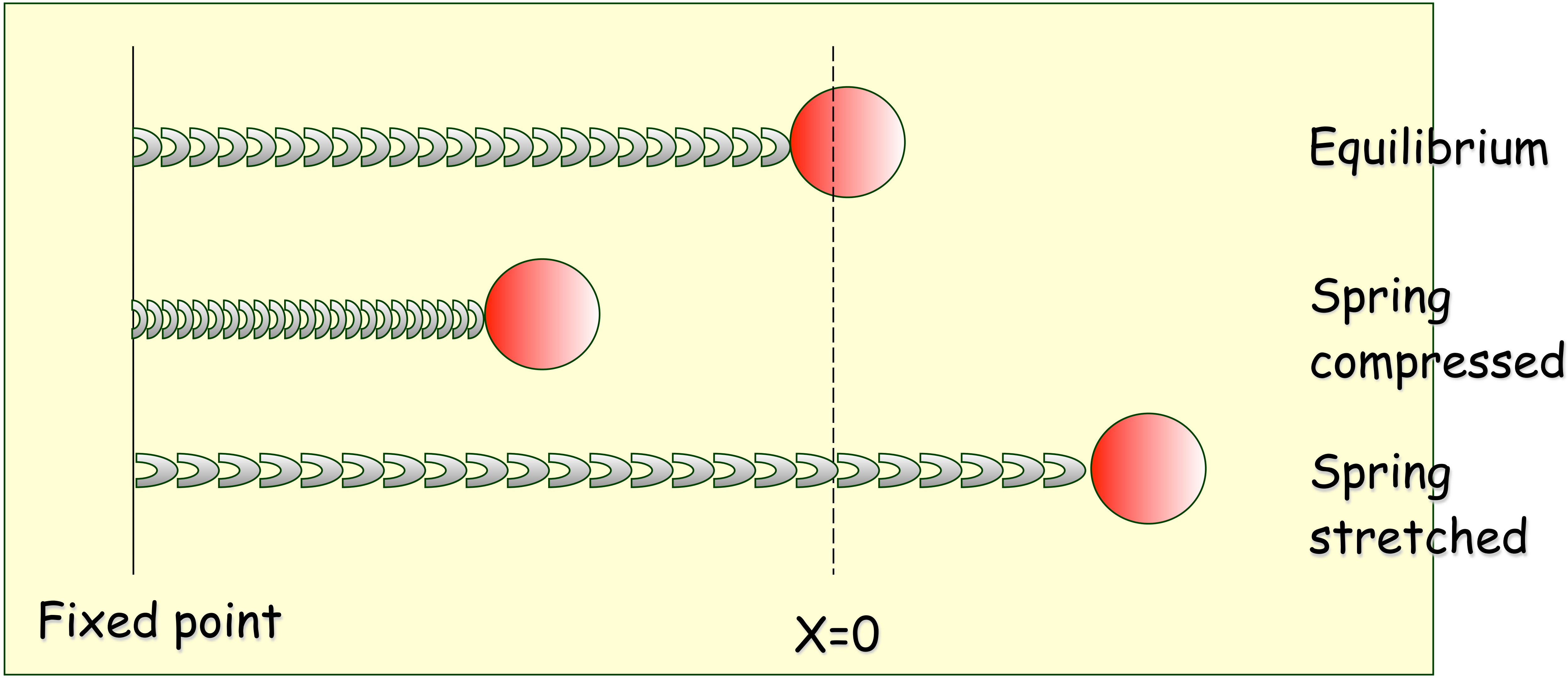
2. Air molecules in a sound wave

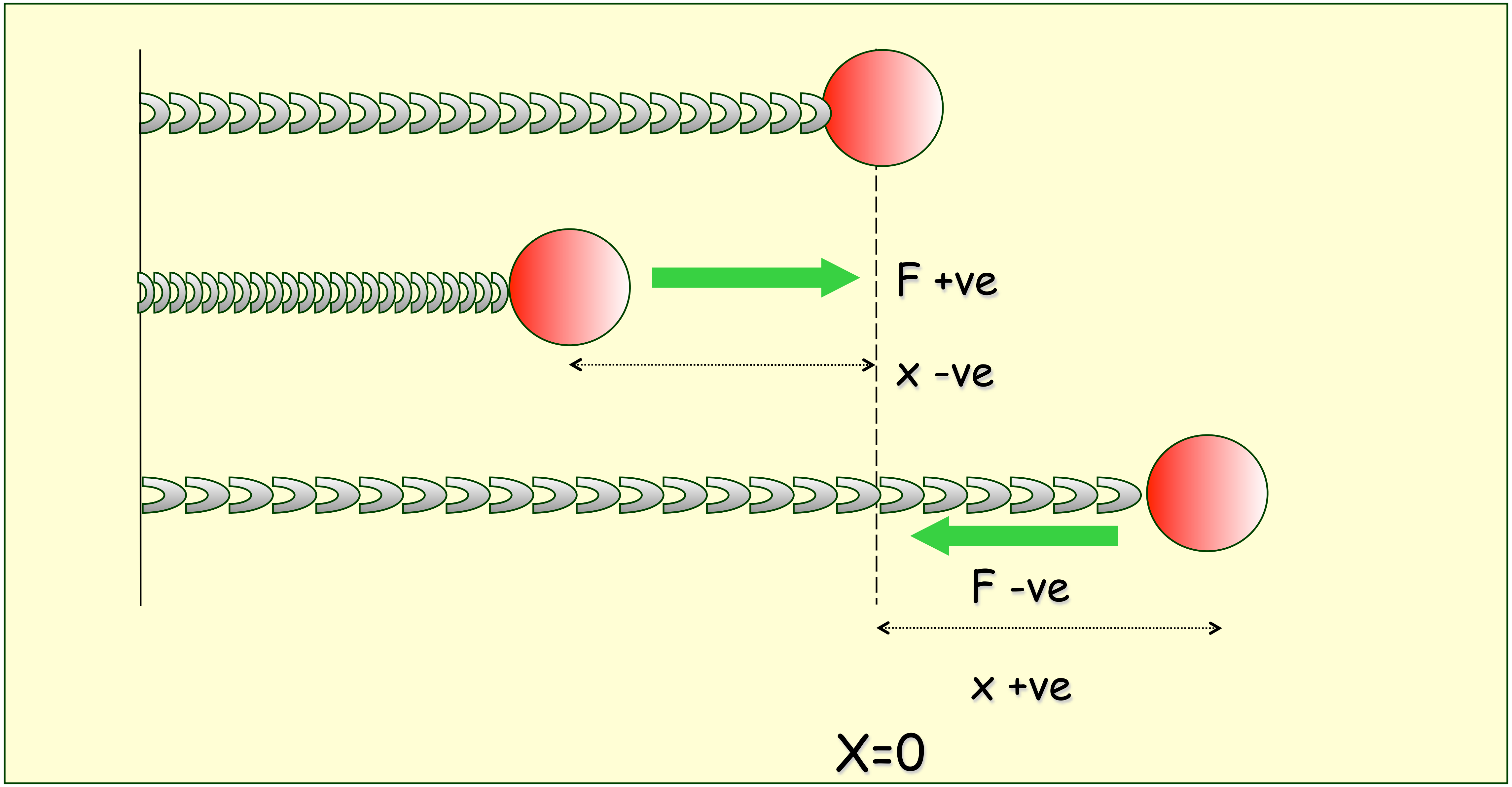
3. Alternating electric current

Simple Harmonic Motion

If an object oscillates between two positions for an indefinite length of time with no loss of mechanical energy the motion is said to be **simple harmonic motion**.

Example: Mass on a spring.





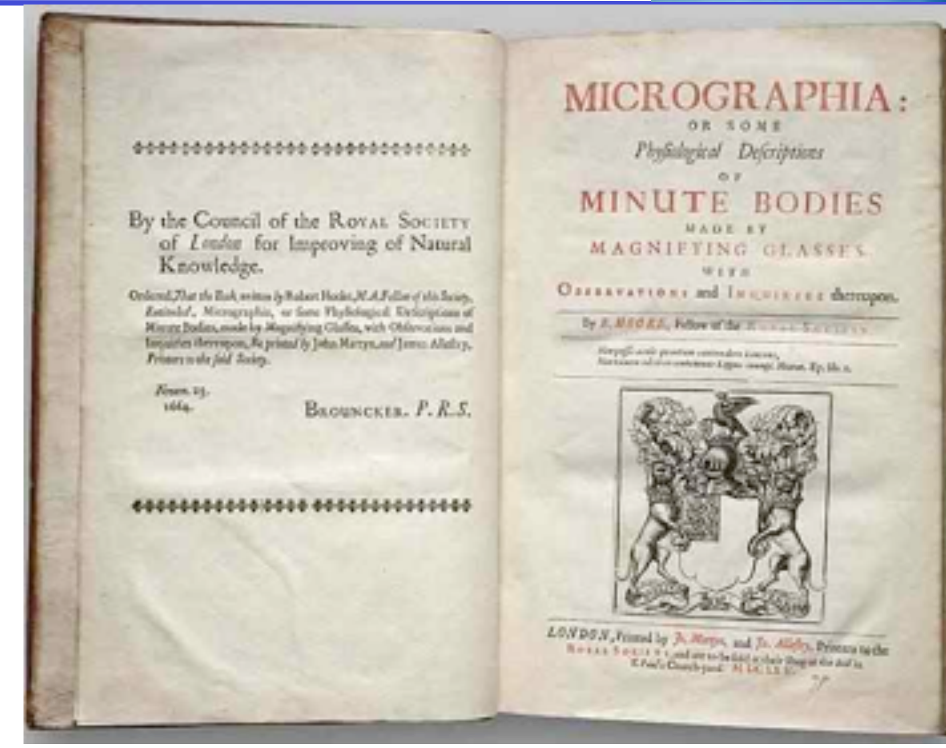
Spring exerts a force on the mass to restore it to its original position.

$$F \propto -x \quad \text{or} \quad F = -kx \qquad \text{(Hooke's Law)}$$

where k is a +ve constant, the **spring constant**

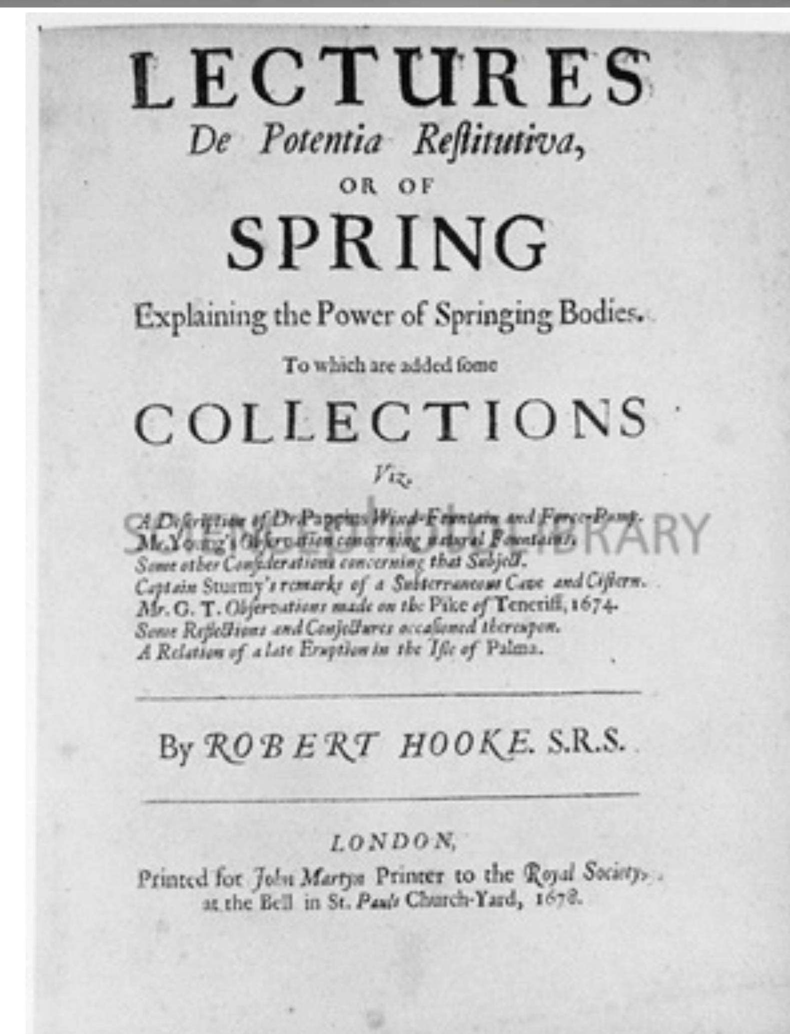
Hooke's law

Although **Robert Hooke**'s name is now usually associated with elasticity and springs, he was interested in many aspects of science and technology. His most famous written work is probably the Micrographia, a compendium of drawings he made of objects viewed under a magnifying glass.



ceiiinosssttuu

It's an anagram. In the time before patents and other intellectual property rights, publishing an anagram was a way to announce a discovery, establish priority, and still keep the details secret long enough to develop it fully. Hooke was hoping to apply his new theory to the design of timekeeping devices and didn't want the competition profiting off his discovery.



LECTURES
De Potentia Restitutiva,
OR OF
SPRING
Explaining the Power of Springing Bodies.
To which are added some
COLLECTIONS
Viz.
A Description of Dr Pappins Wind-Fountain and Force-Pump.
Mr Young's Observation concerning natural Fountains.
Some other Considerations concerning that Subject.
Captain Sturmy's *remarks of a Subterraneous Cave and Cistern.*
Mr. G. T. Observations made on the Pike *of* Teneriff, 1674.
Some Reflections and Conjectures occasioned thereupon.
A Relation of a late Eruption in the Isle of Palma.

By *ROBERT HOOKE* S.R.S.

LONDON,
Printed for *John Martyn* Printer to the *Royal Society*, at the Bell in St. *Pauls* Church-Yard, 1678.

1678: "About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, viz. ceiinossssttuu, that is **Ut tensio sic vis**."

A mass under a restoring force

From Newton's 2nd Law $F = ma$

where $a = \frac{d^2x}{dt^2}$

therefore $-kx = m\frac{d^2x}{dt^2}$

or $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This is the condition for simple harmonic motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

An object moves with simple harmonic motion (SHM) when the acceleration of the object is proportional to its displacement and in the opposite direction.

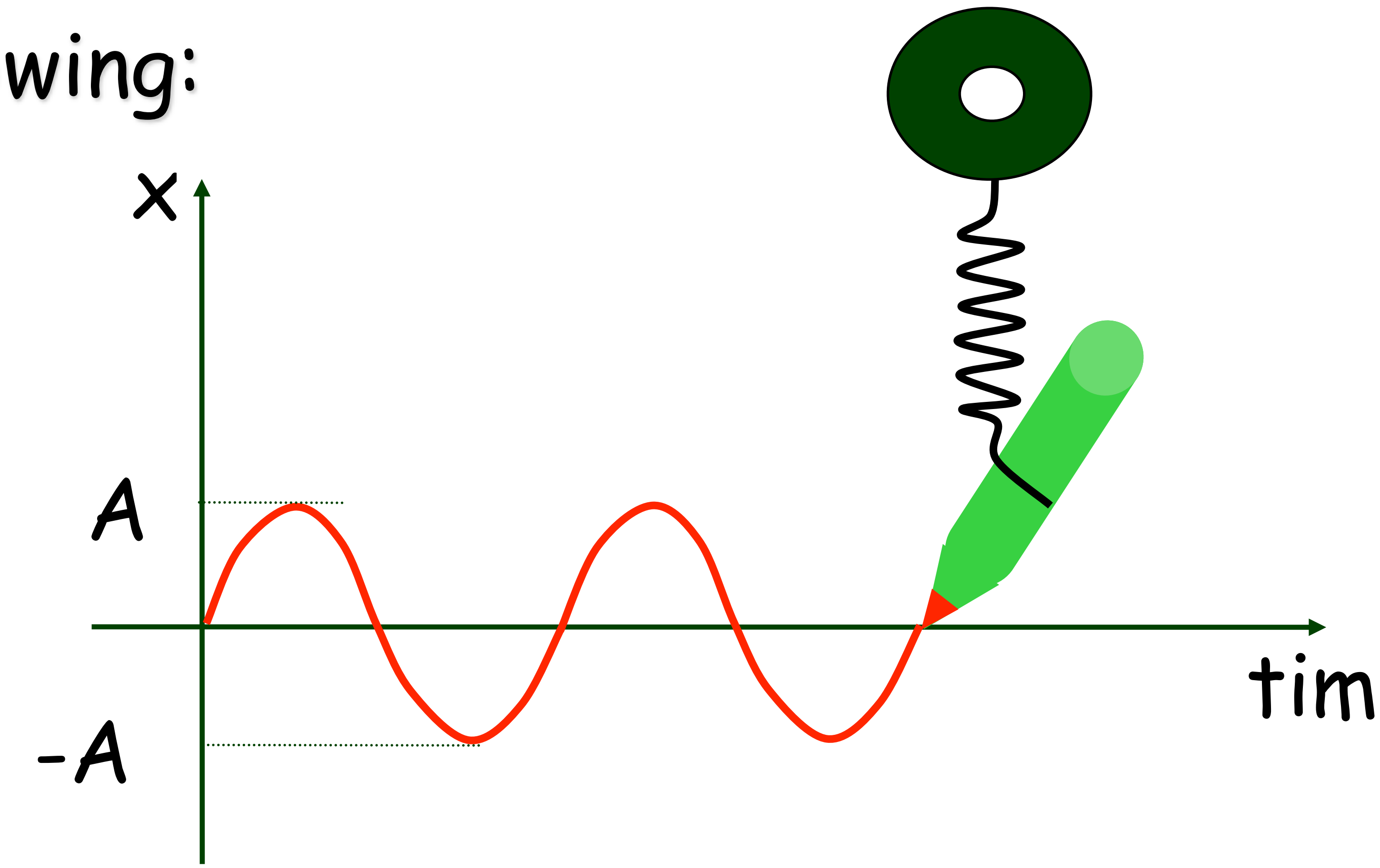
Some definitions:

The time taken to make one complete oscillation is the **period**, **T**.

The **frequency** of oscillation, **f** = 1/T in s^{-1} or Hertz

The distance from equilibrium to maximum displacement is the **amplitude** of oscillation, ***A***.

Consider the following:



The general equation for the curve traced out by the pen is $x = A \cos(\omega t + \delta)$

where $(\omega t + \delta)$ is the **phase** of the motion

and δ is the **phase constant**

We can show that the expression $x = A\cos(\omega t + \delta)$

is a solution of $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ by differentiating wrt time

$$x = A\cos(\omega t + \delta)$$

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \delta)$$

$$a = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \delta)$$

or $$a = -\omega^2 x$$

Compare this to $a = -(k/m)x$

$x = A\cos(\omega t + \delta)$ is a solution if $\omega = \sqrt{\frac{k}{m}}$

We can determine the amplitude of the oscillation (A) and the phase constant (δ) from the initial position x_o and the initial velocity v_o

$$\text{if} \quad x = A\cos(\omega t + \delta) \text{ then } \quad x_o = A\cos(\delta)$$

$$\text{if } v = -A\omega\sin(\omega t + \delta) \quad \text{then} \quad v_o = -A\omega\sin(\delta)$$

The system repeats the oscillation every T seconds

therefore $$x(t) = x(t+T)$$

and $$A\cos(\omega t + \delta) = A\cos(\omega(t + T) + \delta)$$
$$= A\cos(\omega t + \delta + \omega T)$$

The function will repeat when $\omega T = 2\pi$

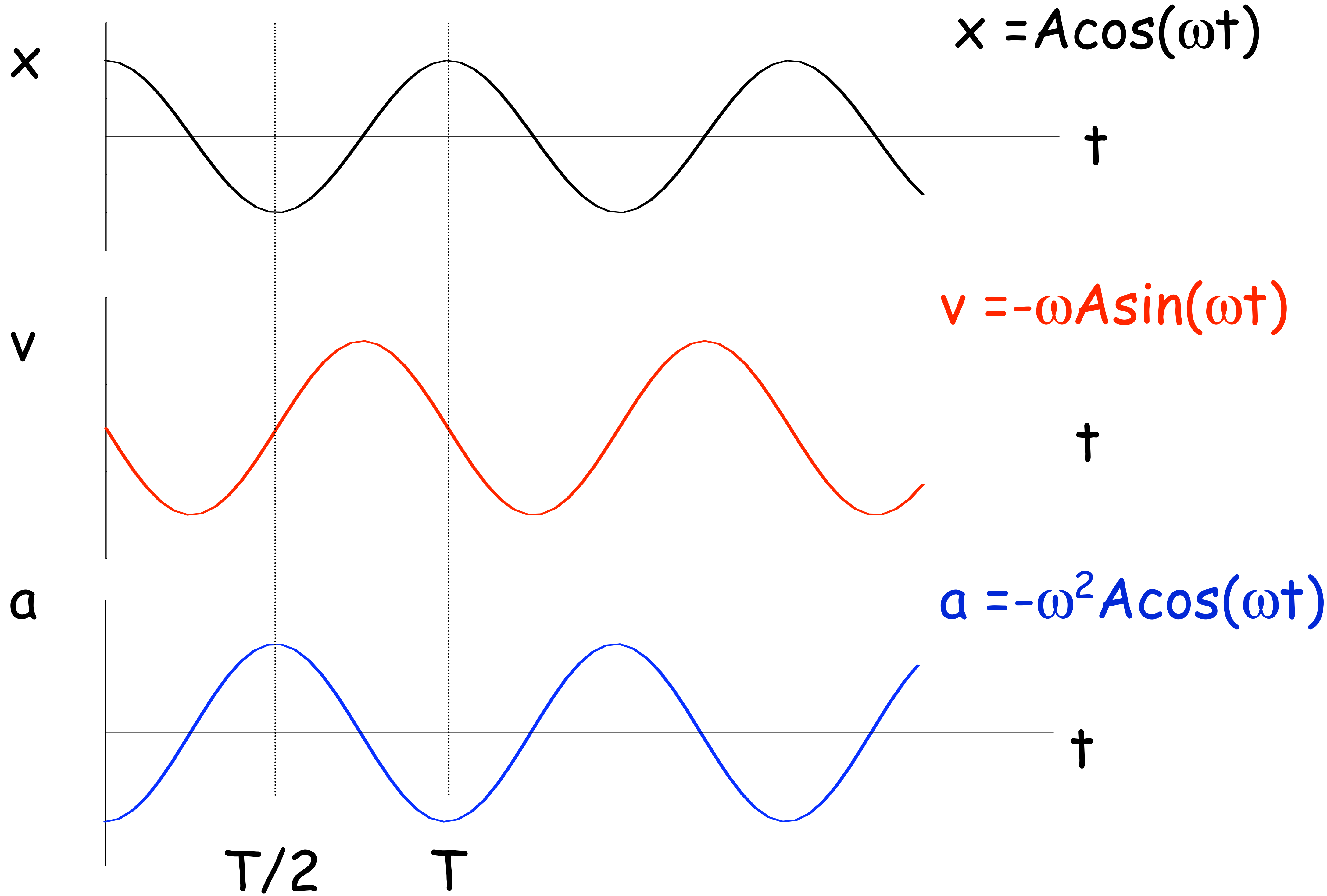
We can relate ω, f and the spring constant k using the following expressions.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

ω is known as the **angular frequency** and has units of rad·s^{-1}

x, v, a time dependence in SHM



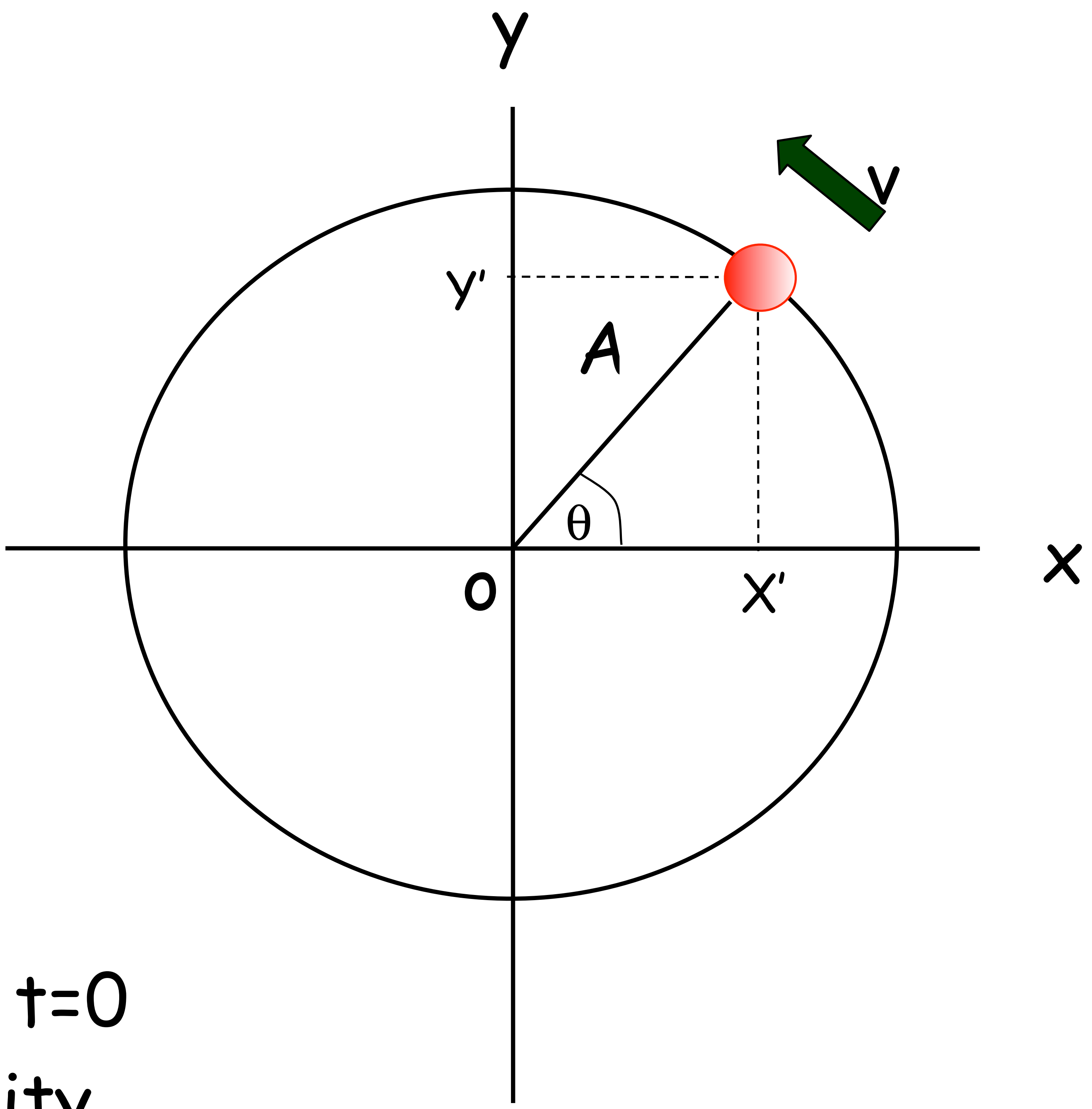
SHM and circular motion

Imagine a particle moving with constant speed v in a circle of radius A

The angular displacement of the particle relative to the x axis is given by

$$\theta = \omega t + \delta$$

Where δ = displacement at t=0
and $\omega = v/A$ = angular velocity

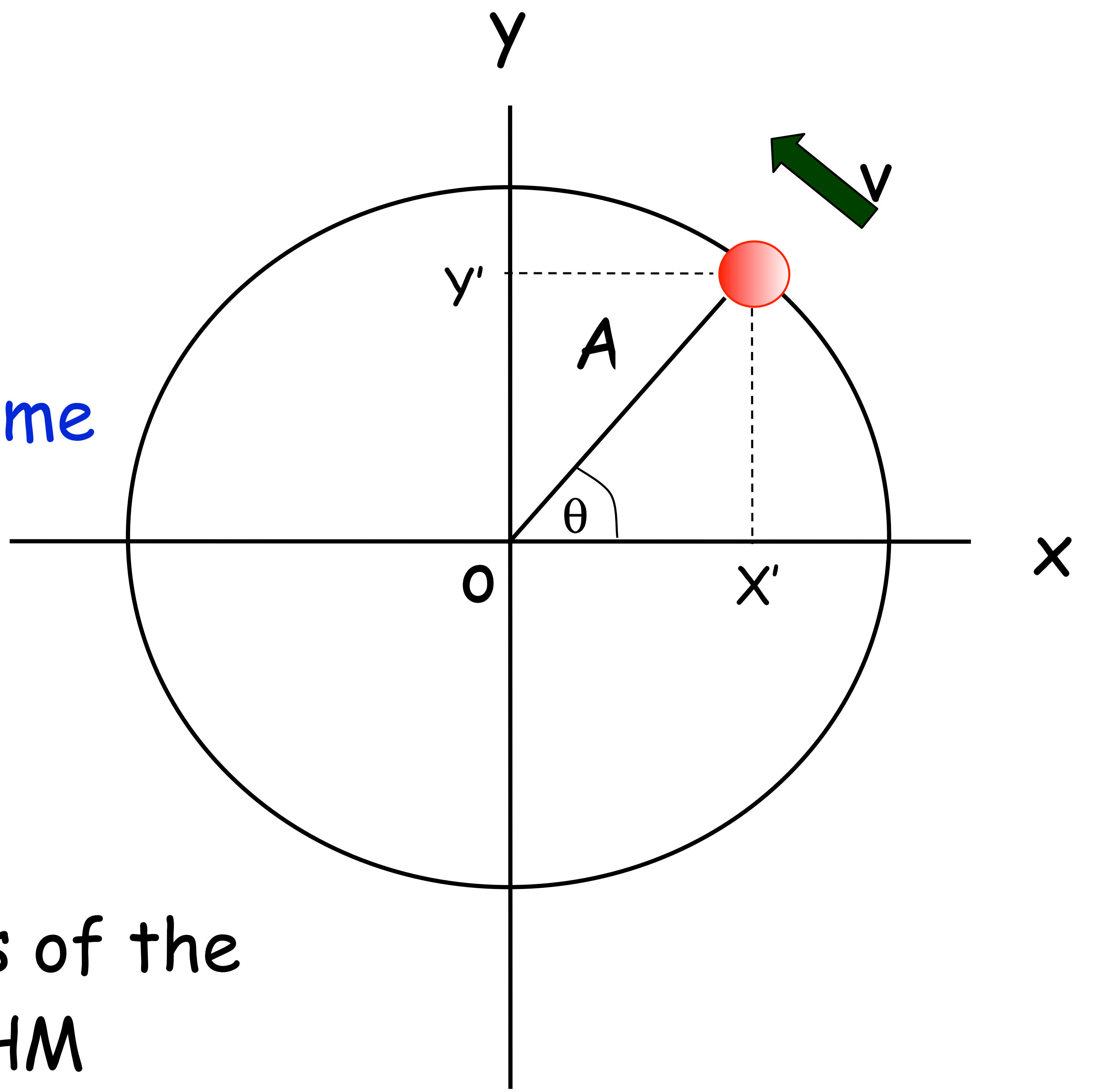


x position as a function of time

$$x' = A\cos(\theta)$$
$$= A\cos(\omega t + \delta)$$

y position as a function of time

$$y' = A\sin(\theta)$$
$$= A\sin(\omega t + \delta)$$
$$= A\cos(\omega t + \delta - \pi/2)$$



Both the x and y components of the particles motion describe SHM

But they differ in phase by $\pi/2$

When a particle moves with constant speed in a circle its projection on the diameter of the circle moves with simple harmonic motion.

This is true for both x and y

Circular motion is therefore the combination of two perpendicular simple harmonic motions with the same amplitude and frequency but with a relative phase difference of $\pi/2$

Problem

A 0.1 kg mass is suspended from a spring of negligible mass with a spring constant of 40N.m^{-1}. The mass oscillates vertically with an amplitude of 0.06m.

(a) Determine the angular frequency of the motion

(b) Express the height y of the mass above the equilibrium position as a function of time if at t=0 the mass is at its highest point.

(c) Express the height y of the mass above the equilibrium position as a function of time if at t=0 the mass is 0.03m above the equilibrium position and moving downwards .

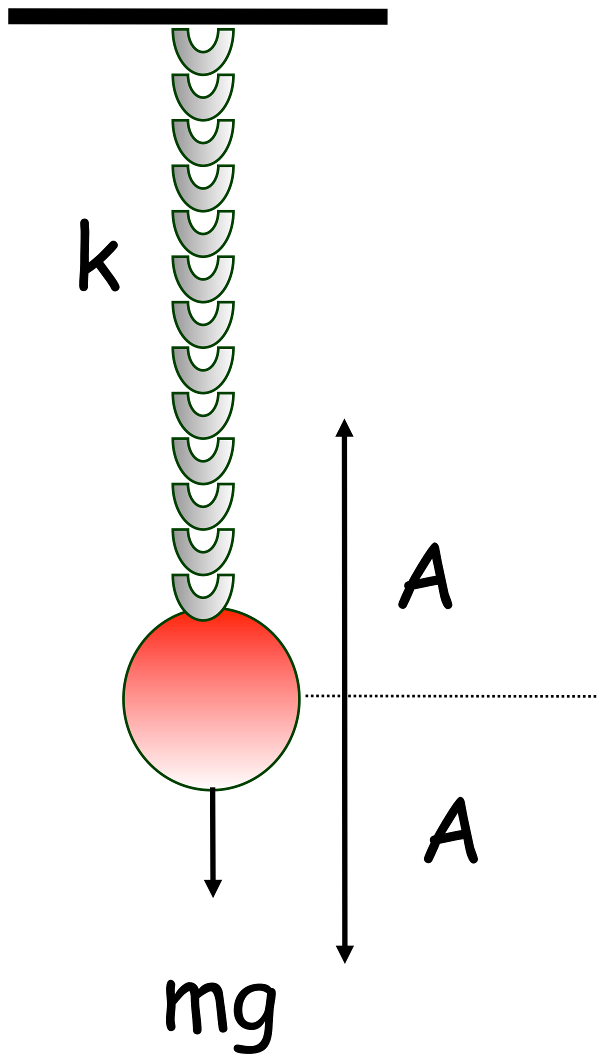
Answer

(a) Determine the angular frequency of the motion

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{40}{0.10}}$$

$$= 20\,\text{rad.s}^{-1}$$



$A = 0.06\text{m}$

$k = 40\text{N.m}^{-1}$

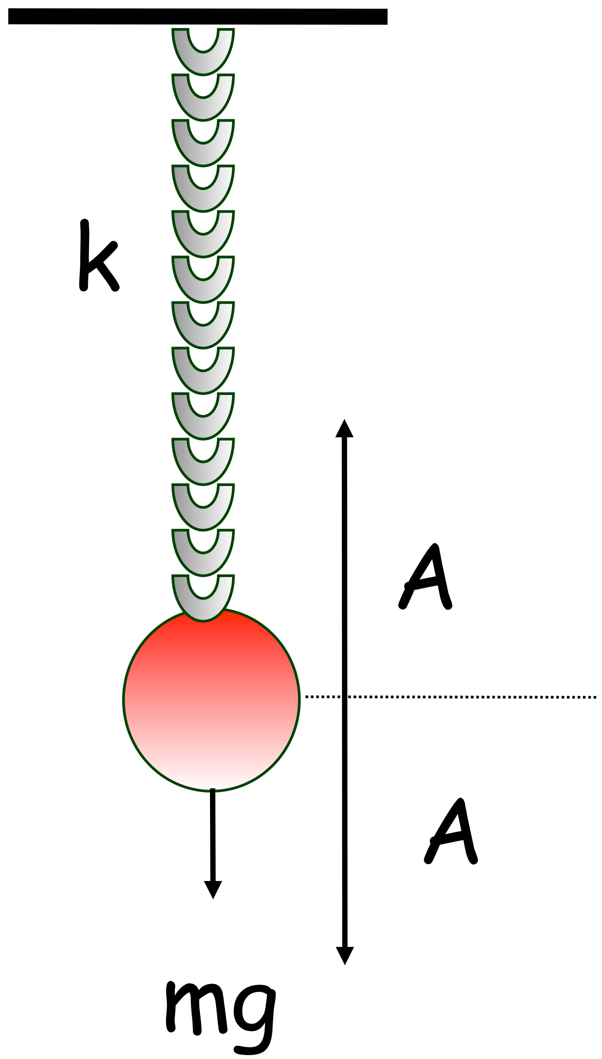
$m = 0.10\text{kg}$

(b) Determine y(t) if y(0)=A

In SHM

position $y(t) = A\cos(\omega t + \delta)$

velocity $v(t) = -A\omega\sin(\omega t + \delta)$



Initial conditions: t=0   y(0)=A   v=0

$$0.06 = 0.06\cos(0 + \delta)$$

$$0 = -20 \times 0.06\sin(0 + \delta)$$

ie $\cos(\delta) = 1$ and $\sin(\delta) = 0$

This is only true if $\delta = 0, \pm 2\pi, \pm 4\pi$.....

If $\delta = 0$

$$y(0) = 0.06\cos(20t)$$

$A = 0.06$m

$k = 40$N.m^{-1}

$m = 0.10$kg

(c) Determine y(t) if y(0)=0.03

position $y(t) = A\cos(\omega t + \delta)$

velocity $v(t) = -A\omega\sin(\omega t + \delta)$

Initial conditions: t=0 y(0)=0.03 v -ve

$$0.03 = 0.06\cos(\delta)$$

$$0 > -20 \times 0.06\sin(\delta)$$

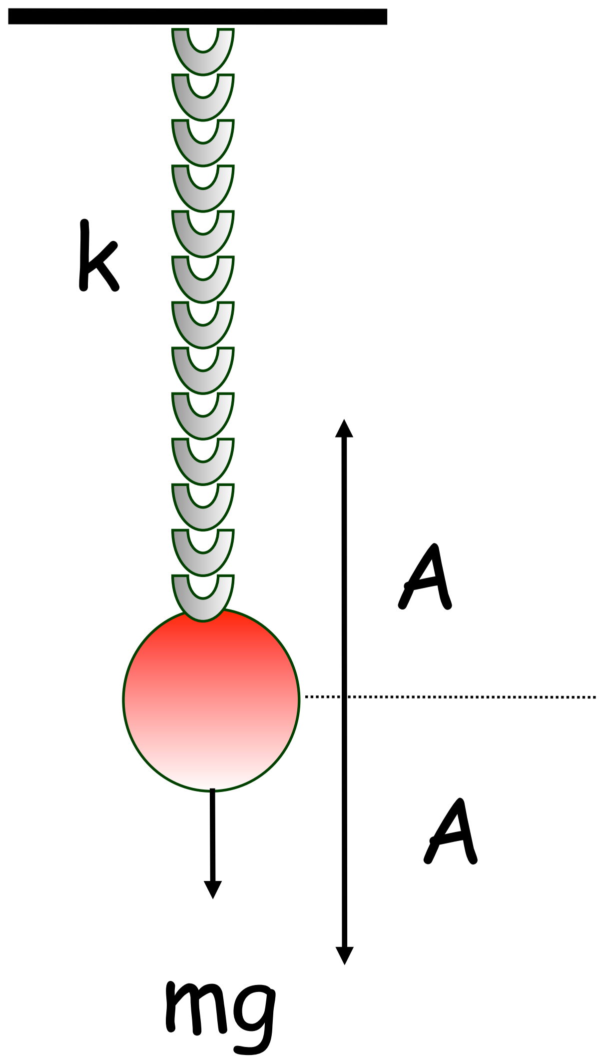
ie $\cos(\delta) = 0.5$ and $\sin(\delta) > 0$

This is only true if $\delta = \pi/3,\ 5\pi/3$ ...

or if $0 < \delta < \pi$

$\delta = \pi/3$

$$y(t) = 0.06\cos(20t + \pi/3)$$



A = 0.06m

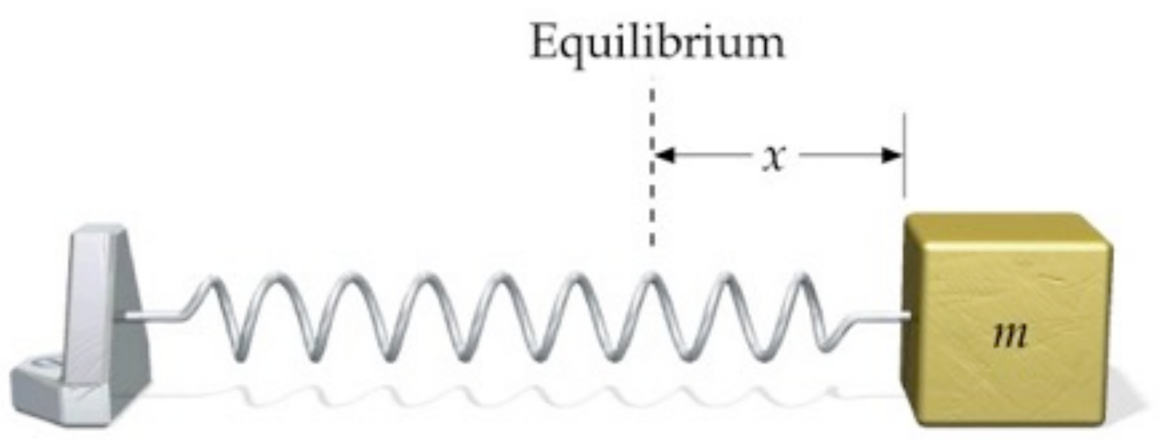
k = 40N.m^{-1}

m = 0.10kg

Energy in Simple Harmonic Motion

In SHM the total energy (E) of a system is **constant** but the kinetic energy (K) and the potential energy (U) vary wrt.

Consider a mass a distance x from equilibrium and acted upon by a restoring force



Kinetic Energy

$$K = \tfrac{1}{2} m v^2$$

$$v = -A\omega \sin(\omega t + \delta)$$

$$K = \tfrac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \delta)$$

Substitute $\omega^2 = k/m$

$$K = \tfrac{1}{2} k A^2 \sin^2(\omega t + \delta)$$

Potential Energy

$$U = \tfrac{1}{2} k x^2$$

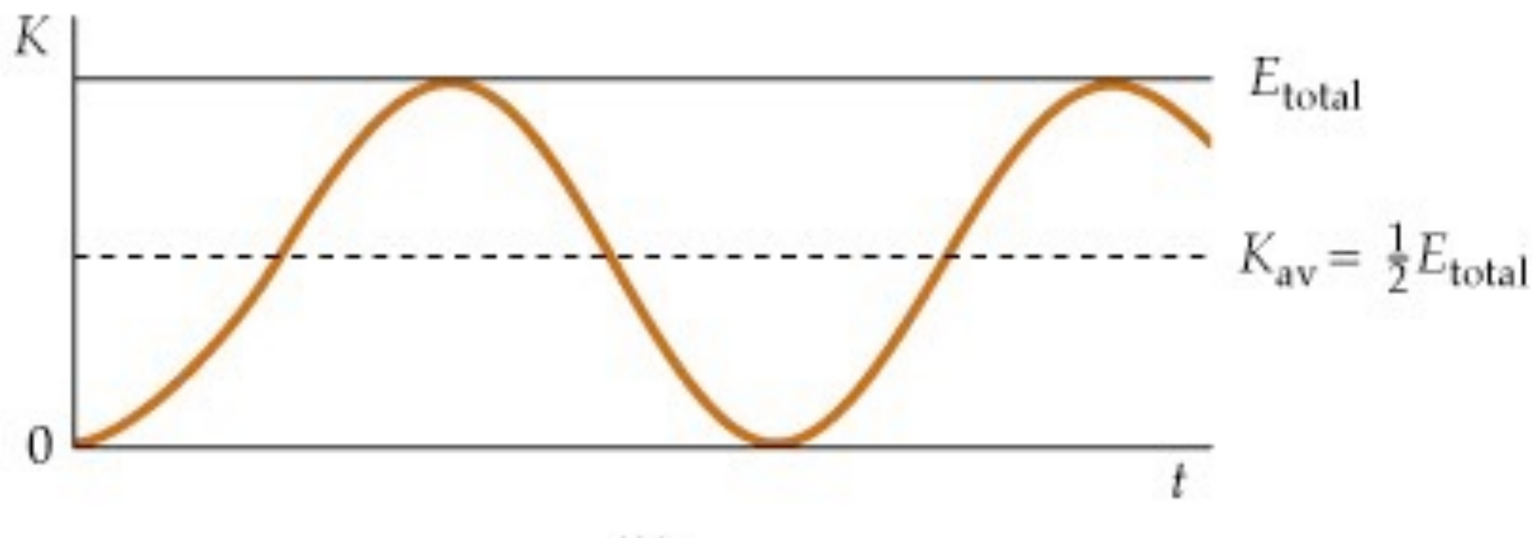
$$x = A\cos(\omega t + \delta)$$

$$U = \tfrac{1}{2} k A^2 \cos^2(\omega t + \delta)$$

Graphical representation

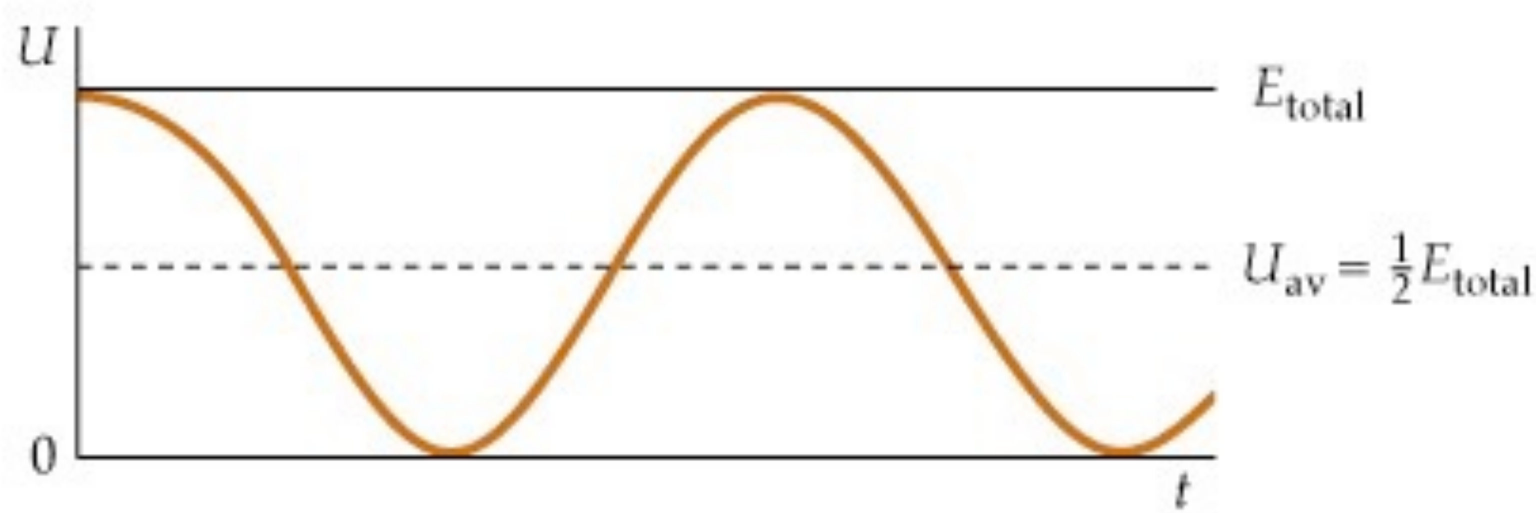
Kinetic

Energy



Potential

Energy



Total Energy

Total energy E = K + U

$$= \tfrac{1}{2}kA^2\sin^2(\omega t+\delta) + \tfrac{1}{2}kA^2\cos^2(\omega t+\delta)$$

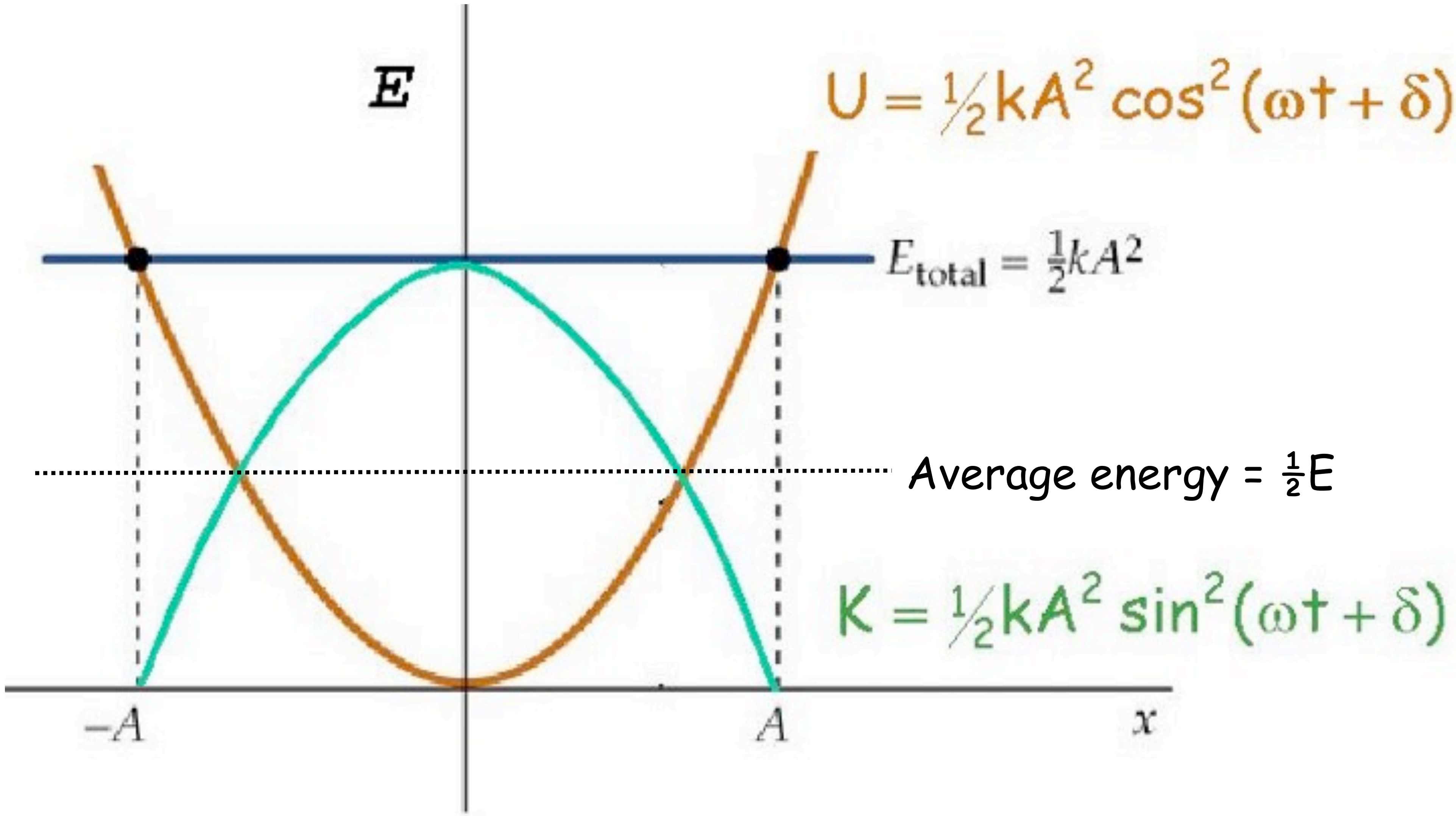
$$= \tfrac{1}{2}kA^2\left(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right)$$

but $\left(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right) = 1$

$$\therefore \quad E = \tfrac{1}{2}kA^2$$

In SHM the total energy of the system is proportional to the square of the amplitude of the motion

Graphical representation



At maximum displacement K=0 $\therefore$ E = U

At equilibrium U=0 and $v=v_{max}$ $\therefore$ E = K

At all times E = K + U is constant

Example: mass on a vertical spring

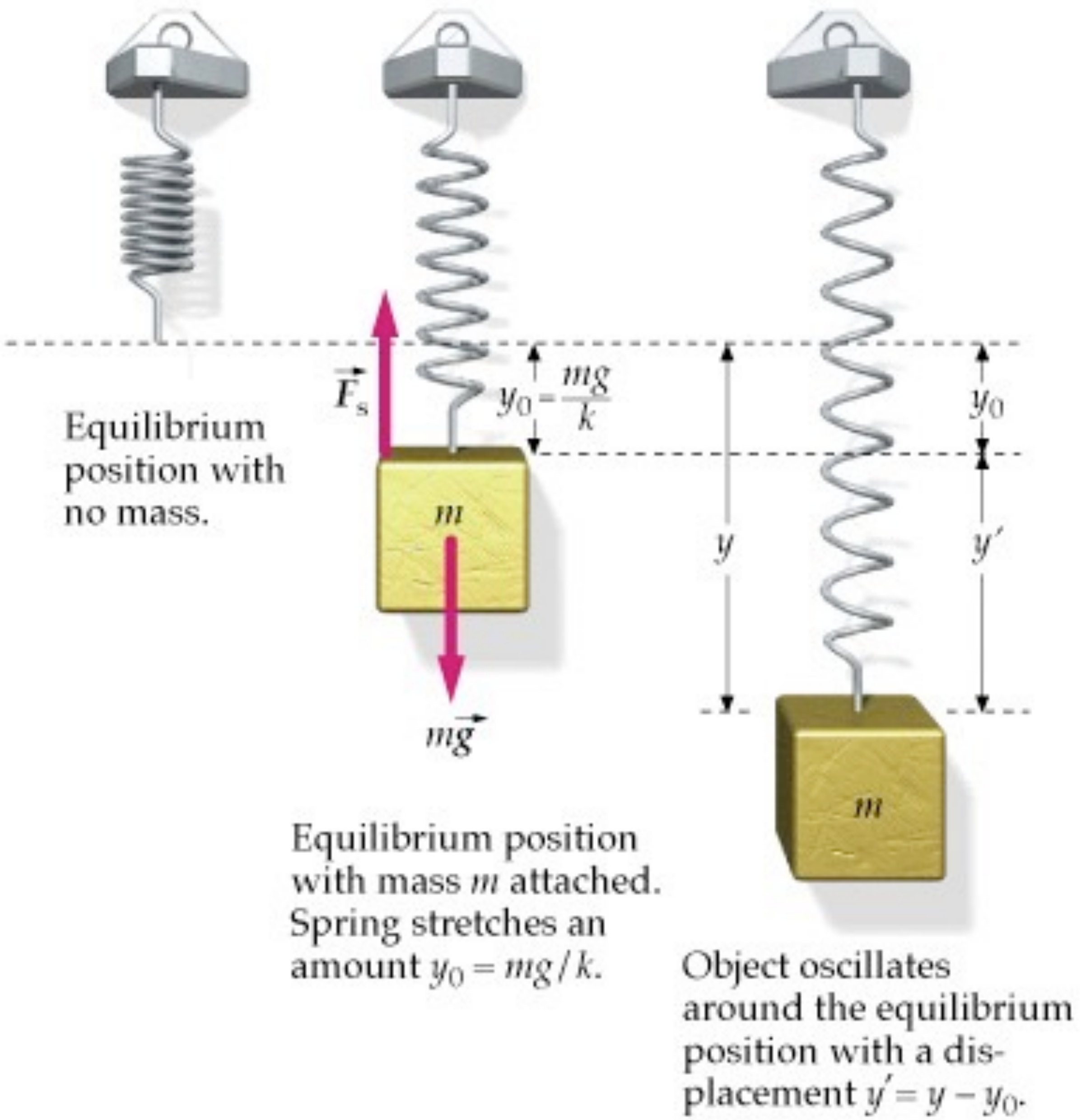
Show that, after an initial displacement, a mass on a spring oscillates with SHM and hence determine the energy of the system.

Consider:

(a) Change in equilibrium position when mass is added

(b) Oscillation after displacement

Define down to be positive



Define zero displacement as the end of unstretched spring

Attach mass, additional displacement = y_0

Displace mass to a point y from end of unstretched spring

The mass experiences a force downwards due to gravity and a force upwards due to the spring

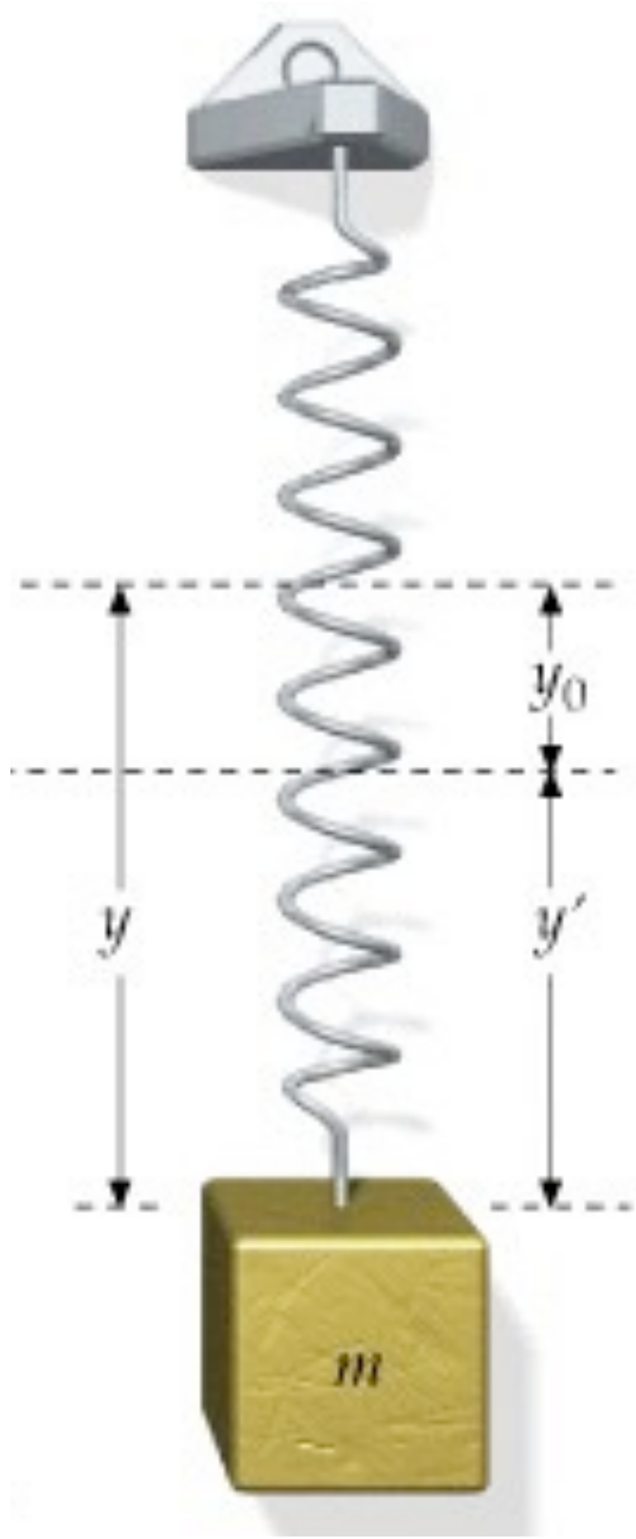
$$F = mg \quad \text{(down)} \qquad\qquad F = -ky \quad \text{(up)}$$

$\therefore$ from Newton's 2nd Law $\quad F_{TOTAL} = m\dfrac{d^2y}{dt^2} = -ky + mg$

$$F = m\frac{d^2y}{dt^2} = -ky + mg$$

This differs from the usual equation for SHM by mg

$$F = m\frac{d^2y}{dt^2} = -ky$$



$$F = m\frac{d^2y}{dt^2} = -ky + mg$$

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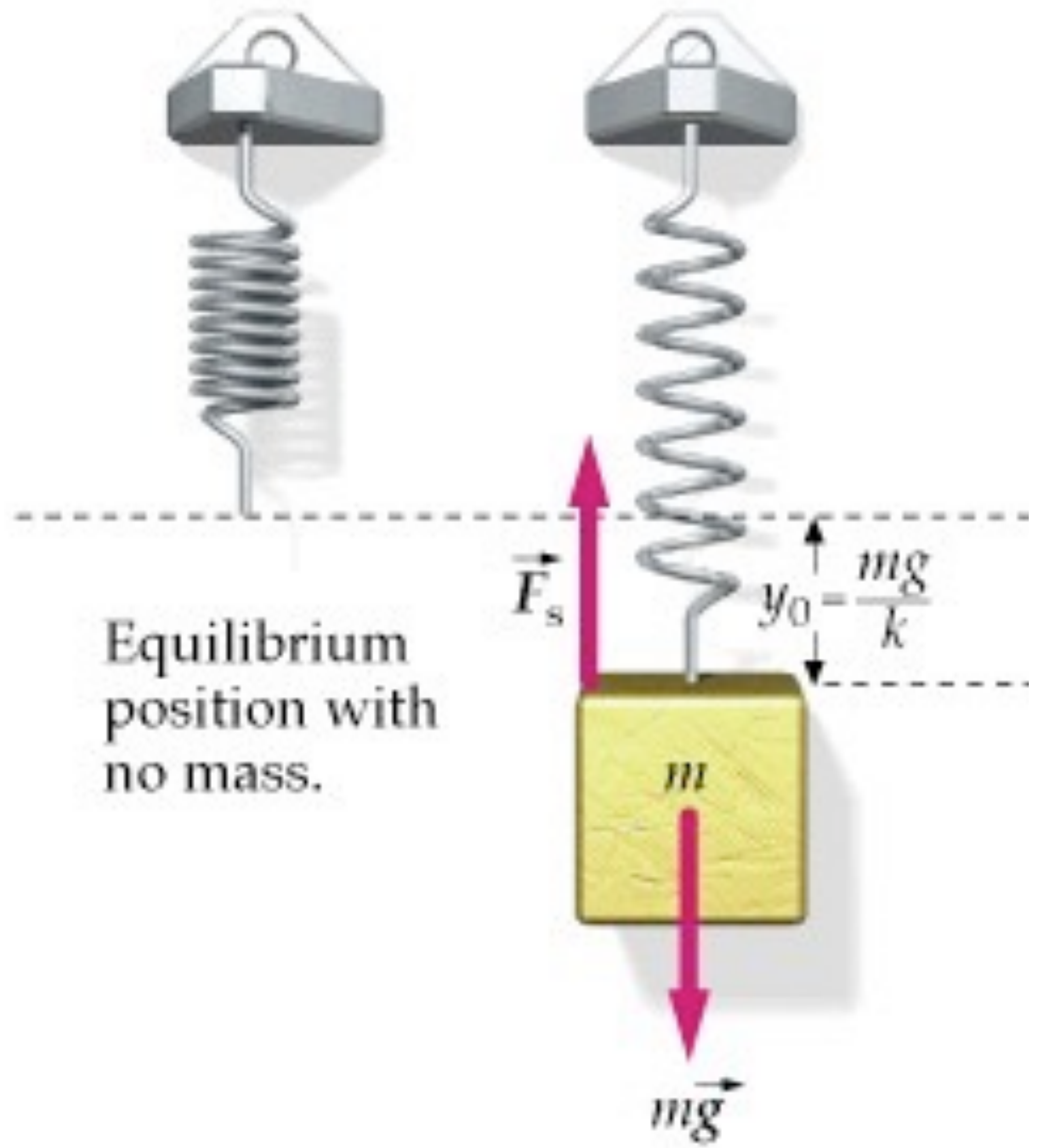
$$F = m\frac{d^2y}{dt^2} = -ky$$



Define $y = y_o + y^l$

where y^l = displacement

and $y_o = mg/k$ (from equilibrium)

$$m\frac{d^2y}{dt^2} = -ky + mg$$

Substitute $y = y_o + y^{l}$

$$m\frac{d^2(y_o + y')}{dt^2} = -k(y_o + y') + mg$$

$$= -ky_o - ky' + mg$$

But $y_o = mg/k$ $\therefore ky_o = mg$ and $dy_o/dt = 0$

$$m\frac{d^2(y')}{dt^2} = -ky' \qquad \text{or} \qquad \frac{d^2(y')}{dt^2} = -\frac{k}{m}y'$$

$$\frac{d^2(y')}{dt^2} = -\frac{k}{m}y'$$

Description of the motion of a mass on a vertical spring

This is SHM with the solution

$$y' = A\cos(\omega t + \delta)$$

With $\omega = \sqrt{\frac{k}{m}}$ (as we found for a horizontal system)

The only effect of the gravitational force mg is to shift the equilibrium position from y to y'

Potential energy of a mass on vertical spring

Total potential energy U = Spring potential energy U_s + Gravitational potential energy U_g

Generally, for some displacement Y

Spring potential energy $U_s = \frac{1}{2}kY^2$

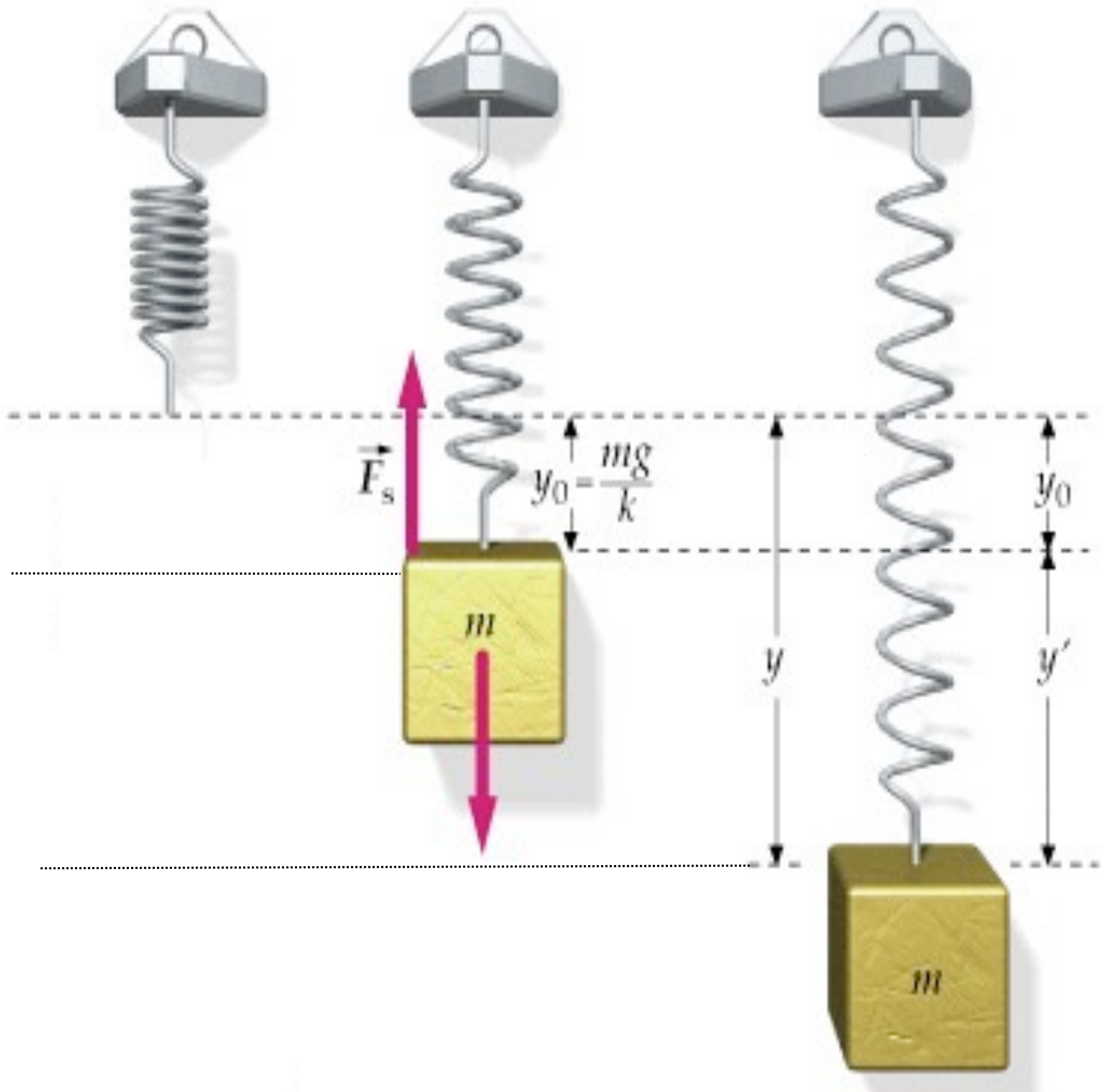
Gravitational potential energy $U_g = mgY$

U_s | U_g

define $U_s=0$ | define $U_g=0$

$\frac{1}{2}ky_o^{\ 2}$ | $-mgy_o$

$\frac{1}{2}ky^{\ 2}$ | $-mgy$



$$U = \left(\tfrac{1}{2}ky^2 - mgy\right) - \left(\tfrac{1}{2}ky_o{}^2 - mgy_o\right)$$

Substitute $y = y_o + y^l$

$$U = \tfrac{1}{2}k\,(y_o{}^2 + y^l)^2 - \tfrac{1}{2}ky_o{}^2 + mgy_o - mg(y_o + y^l)$$

$$= \tfrac{1}{2}ky_o{}^2 + ky_oy^l + \tfrac{1}{2}ky^{l\,2} - \tfrac{1}{2}ky_o{}^2 + mgy_o - mgy_o - mgy^l$$

$$= ky_oy^l - mgy^l + \tfrac{1}{2}ky^{l\,2}$$

$$= y^l\,(ky_o - mg) + \tfrac{1}{2}ky^{l\,2}$$

But $ky_o = mg$ (from forces when in equilibrium)

$$U = \tfrac{1}{2}ky^{l\,2}$$

Example

A 3kg mass stretches a spring 16cm when it hangs vertically in equilibrium. The spring is then stretched 5cm from its equilibrium position and the mass released.

Find the total energy and the potential energy of the spring when the mass is at its maximum displacement .

Answer

m = 3kg

y_o = 16cm = 0.16m

y^l = 5cm = 0.05m = A

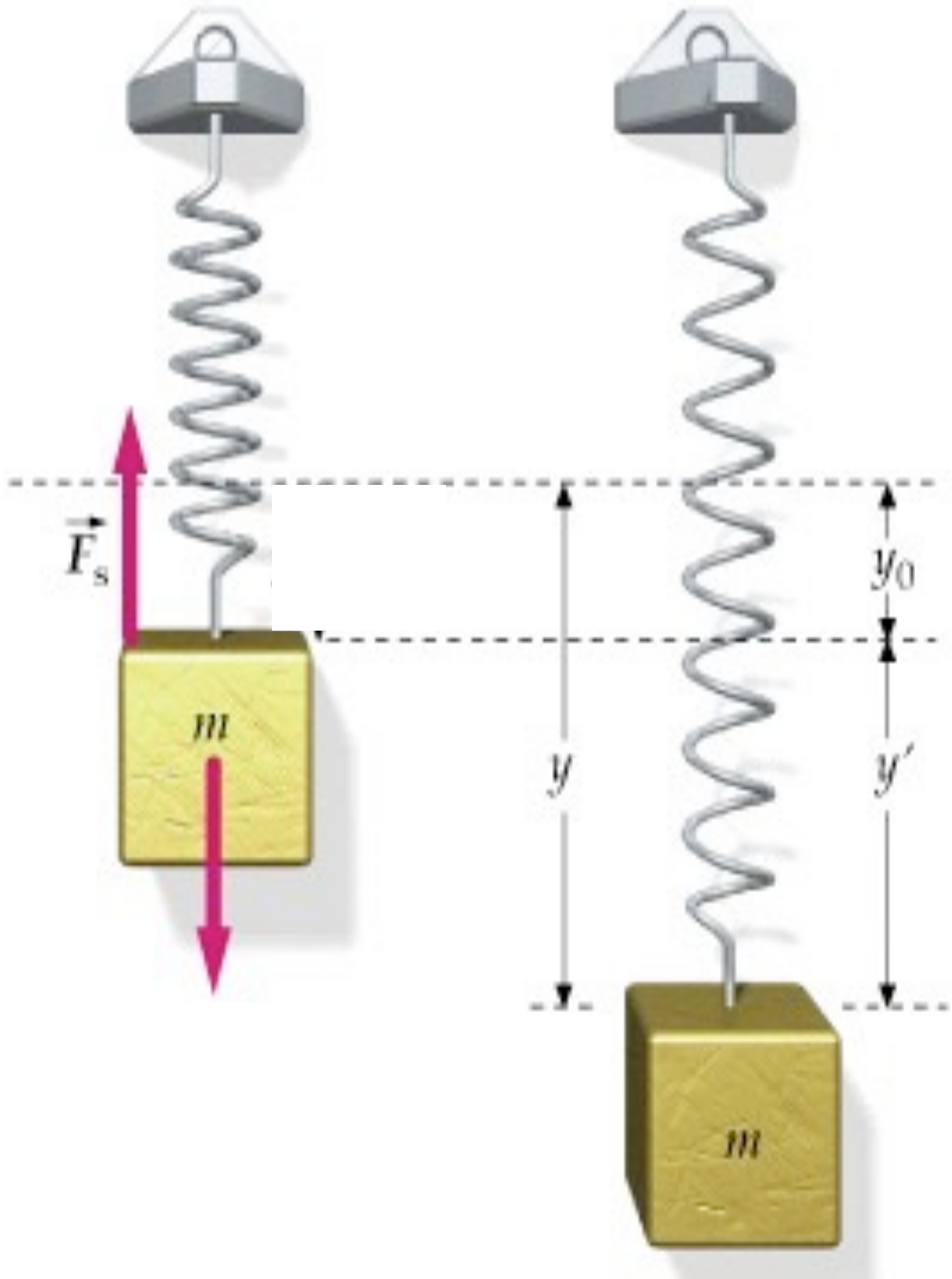
Total energy $E = \frac{1}{2}kA^2$

$ky_o = mg$

k = (3 x 9.81) / 0.16 = 184 N.m^{-1}

E = 0.5 x 184 x 0.05^2

= 0.23 J



$U = U_s + U_g$

At maximum displacement all the energy is potential energy

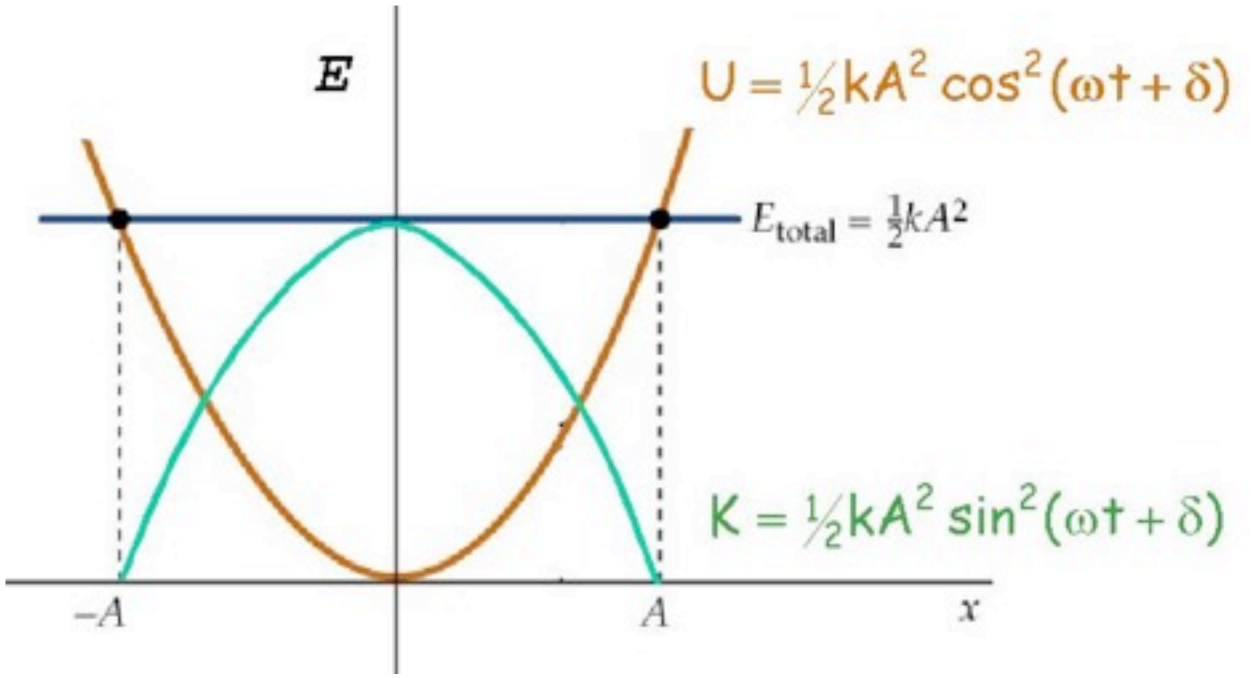
$E = U_s + U_g$

$U_s = E - U_g$

$= E - (-mgA)$

$= 0.23 + (3 \times 9.81 \times 0.05)$

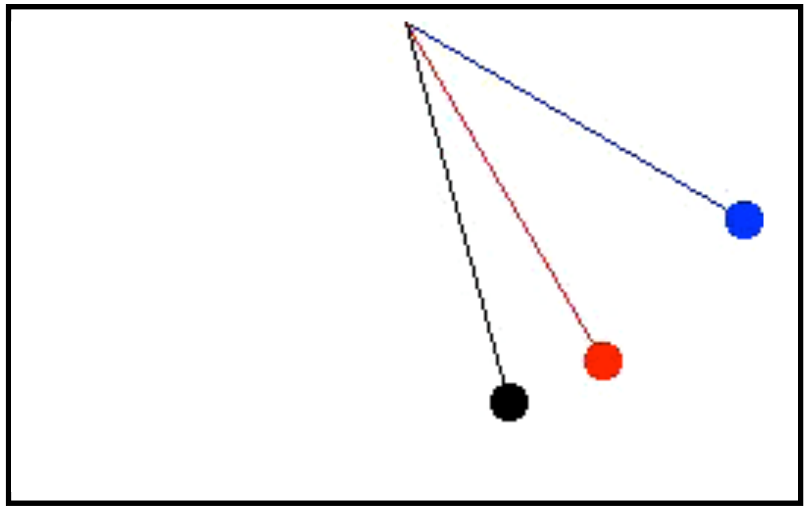
$= 1.70J$



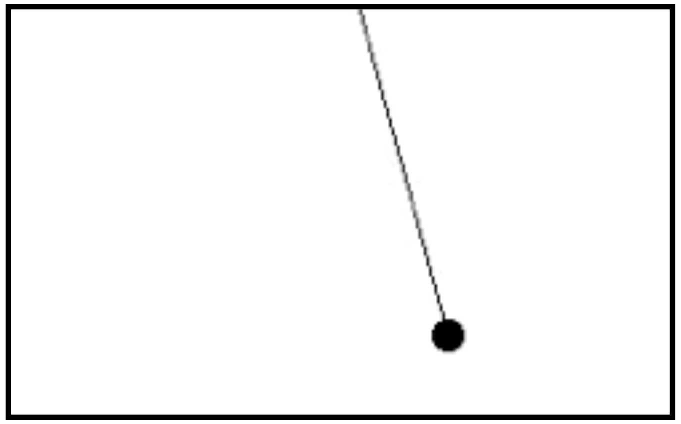
total energy is 0.23J and the potential energy is 1.70J

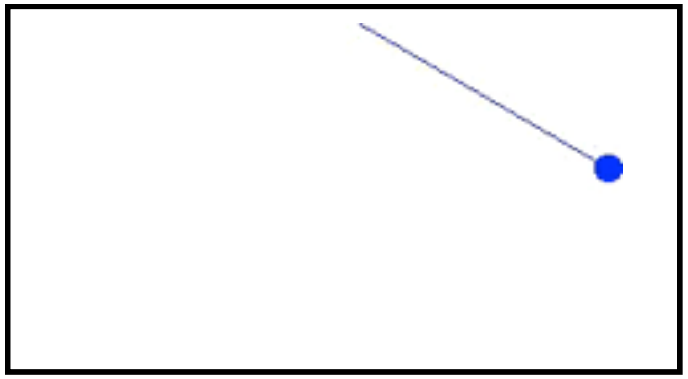
The Simple Pendulum

A simple pendulum consists of a string of length L and a bob of mass m.



When the mass is displaced and released from an initial angle ϕ with the vertical it will swing back and forth with a period T.





We are going to derive an expression for T.

Forces on mass:

mg (downwards)

tension (upwards)

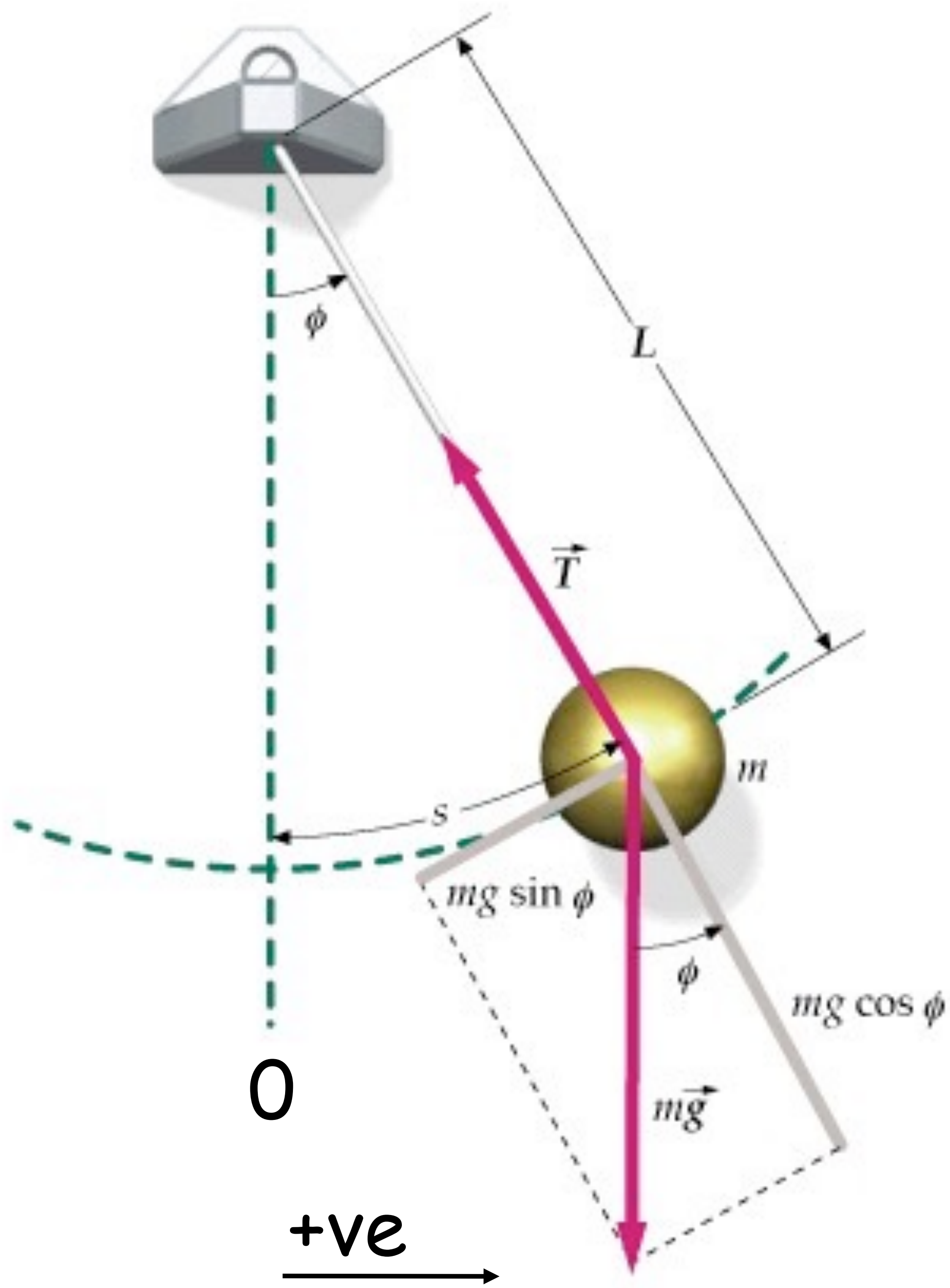
When mass is at an angle ϕ to the vertical these forces have to be resolved.

Tangentially:

weight = mg sin ϕ (towards 0)

tension = T cos 90 = 0

$$\sum F_{tang} = -mg\sin\phi$$



Using $\frac{\phi(\text{rads})}{2\pi} = \frac{s}{2\pi L}$

we find $s = L\phi$

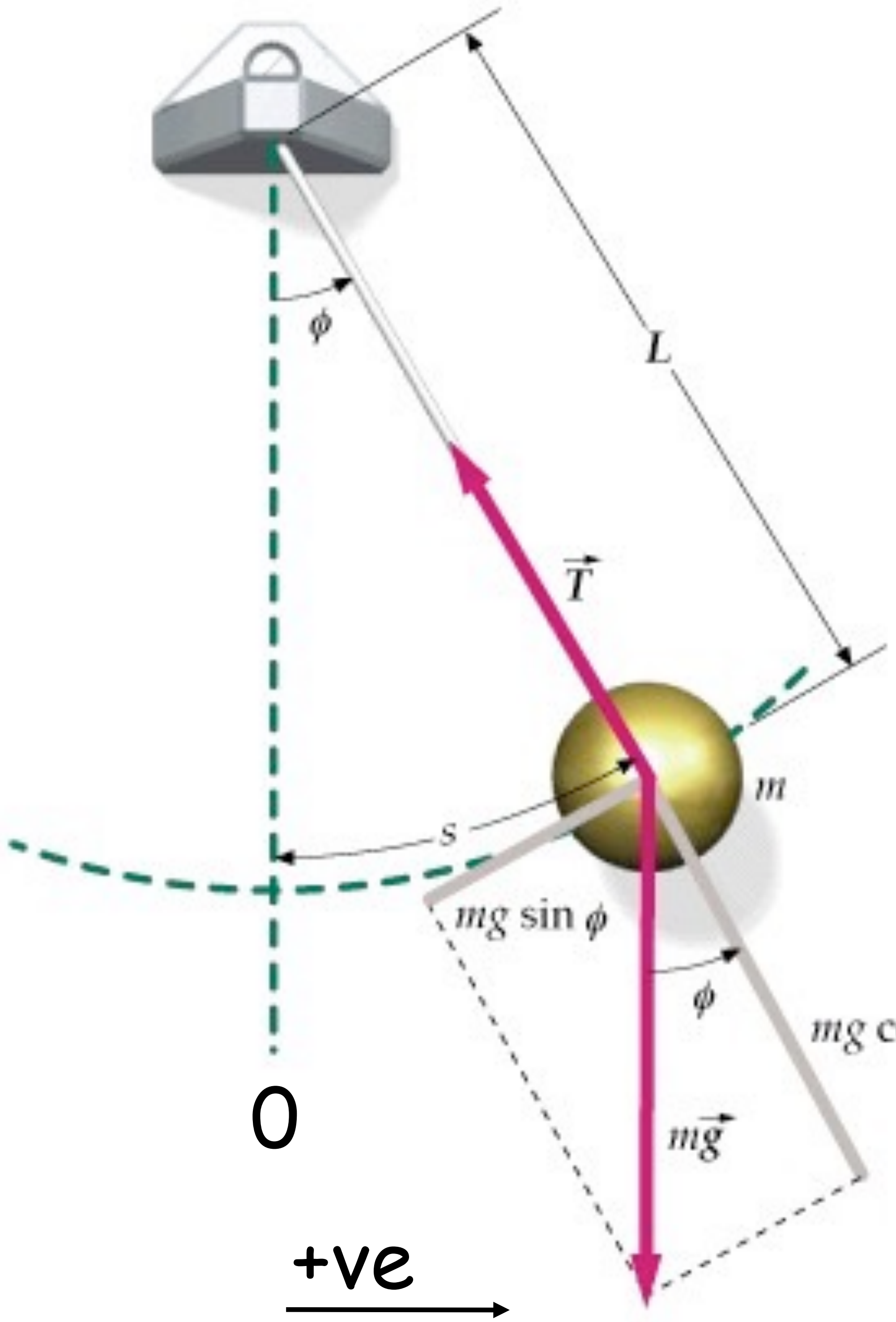
From Newton's 2nd Law (N2)

$$\sum F_{tang} = -mg\sin\phi$$

$$= ma$$

$$= m\frac{d^2s}{dt^2}$$

$$= mL\frac{d^2\phi}{dt^2}$$



$$-mg\sin\phi = mL\frac{d^2\phi}{dt^2}$$

or $$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin\phi$$

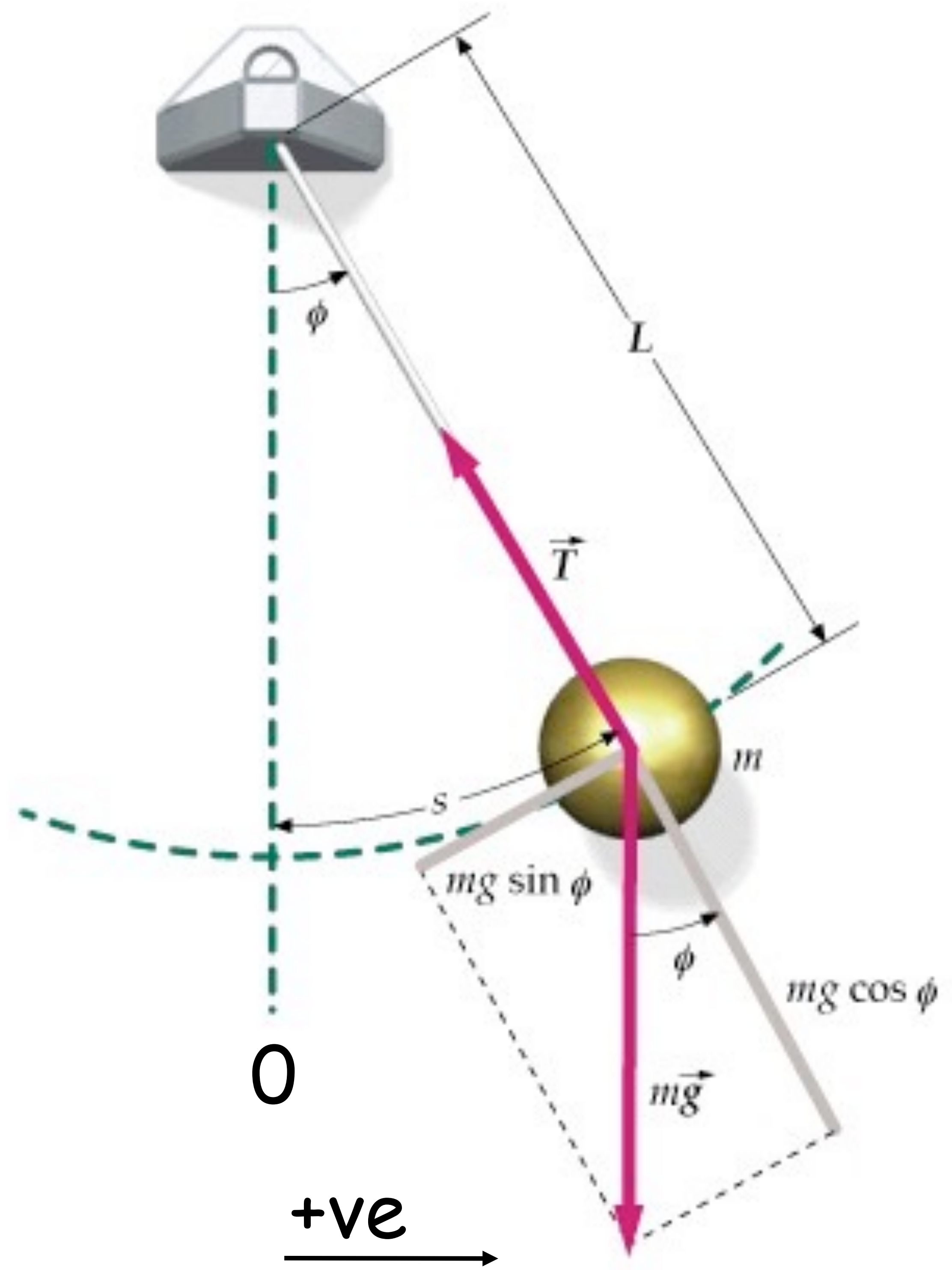
For small ϕ $\sin\phi \sim \phi$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\phi$$

ie SHM with $\omega^2 = \frac{g}{L}$

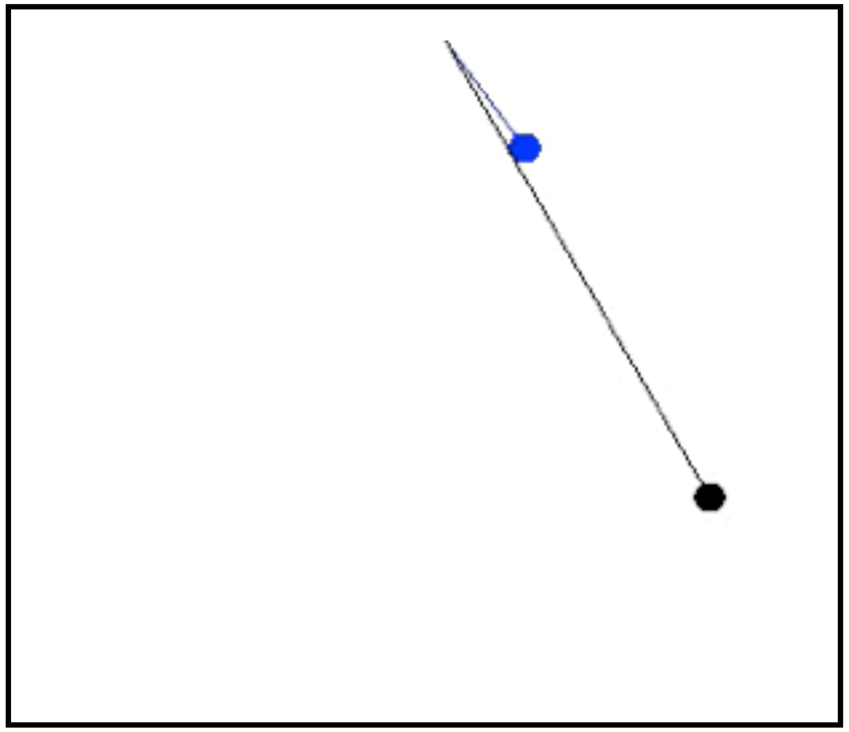
This has the solution

$$\phi = \phi_o \cos(\omega t + \delta)$$



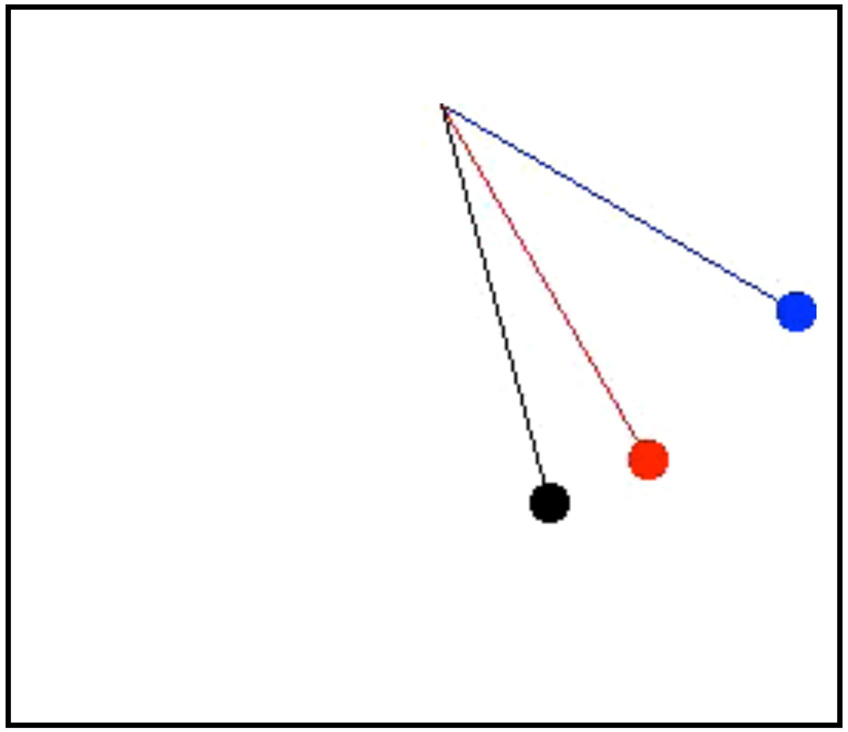
Period of the motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



ie the longer the pendulum the greater the period

Note: T does not depend upon amplitude of oscillation



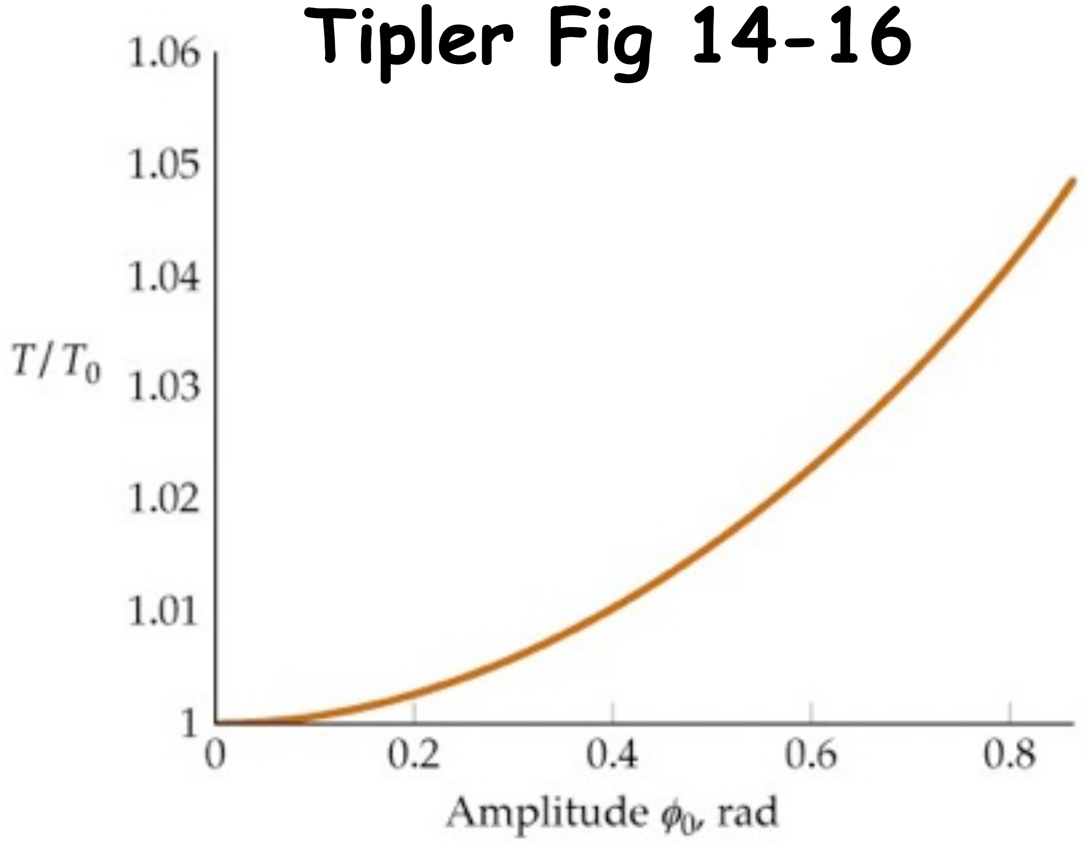
even if a clock pendulum changes amplitude it will still keep time

Period of the motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

This is only true for $\phi < 10^o$

Tipler Fig 14-16



Generally

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots\ldots\right)$$

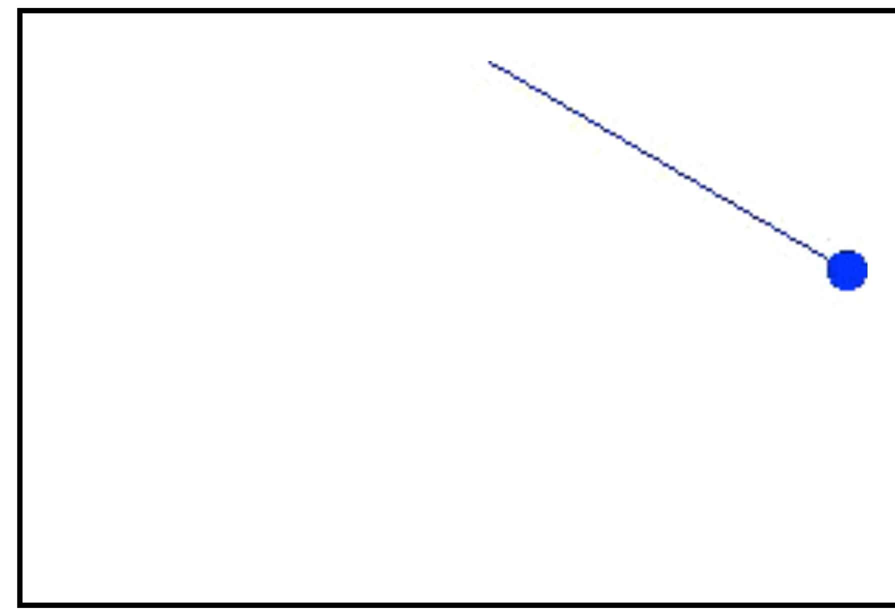
$$T = T_o\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots\ldots\right)$$

If the initial angular displacement is significantly large the small angle approximation is no longer valid

The error between the simple harmonic solution and the actual solution becomes apparent almost immediately, and grows as time progresses.

Dark blue pendulum is the simple approximation,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



light blue pendulum shows the numerical solution of the nonlinear differential equation of motion.

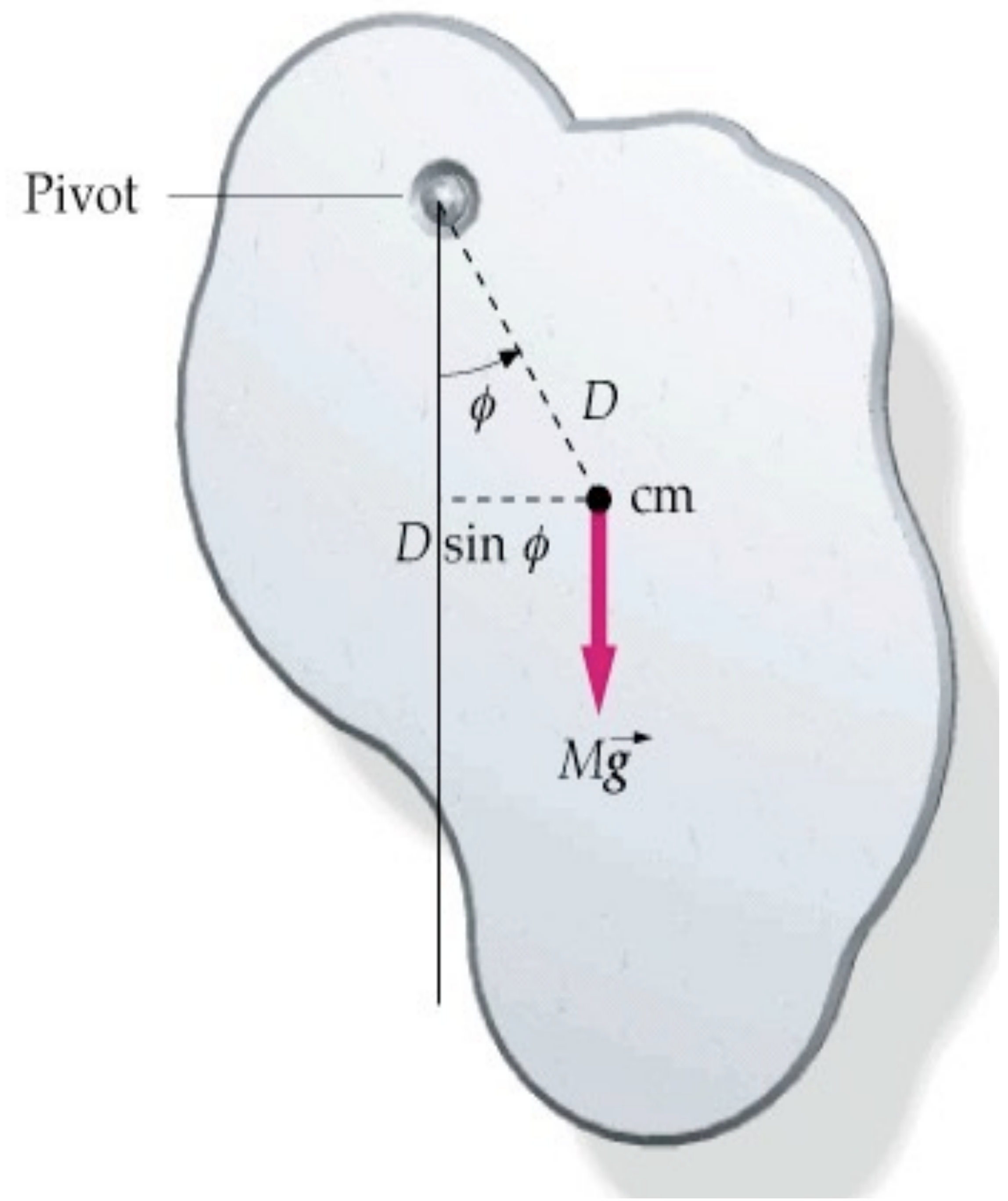
$$T = T_0 \left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots..\right)$$

The Physical Pendulum

When a rigid object (of any shape) is pivoted about a point other than its centre of mass (CoM) it will oscillate when displaced from equilibrium

Such a system is called a physical pendulum
Consider the mass opposite

M = mass of object
D = pivot-CoM distance
ϕ = angle of displacement



Torque about pivot = $MgD \sin\phi$

(This will tend to restore equilibrium)

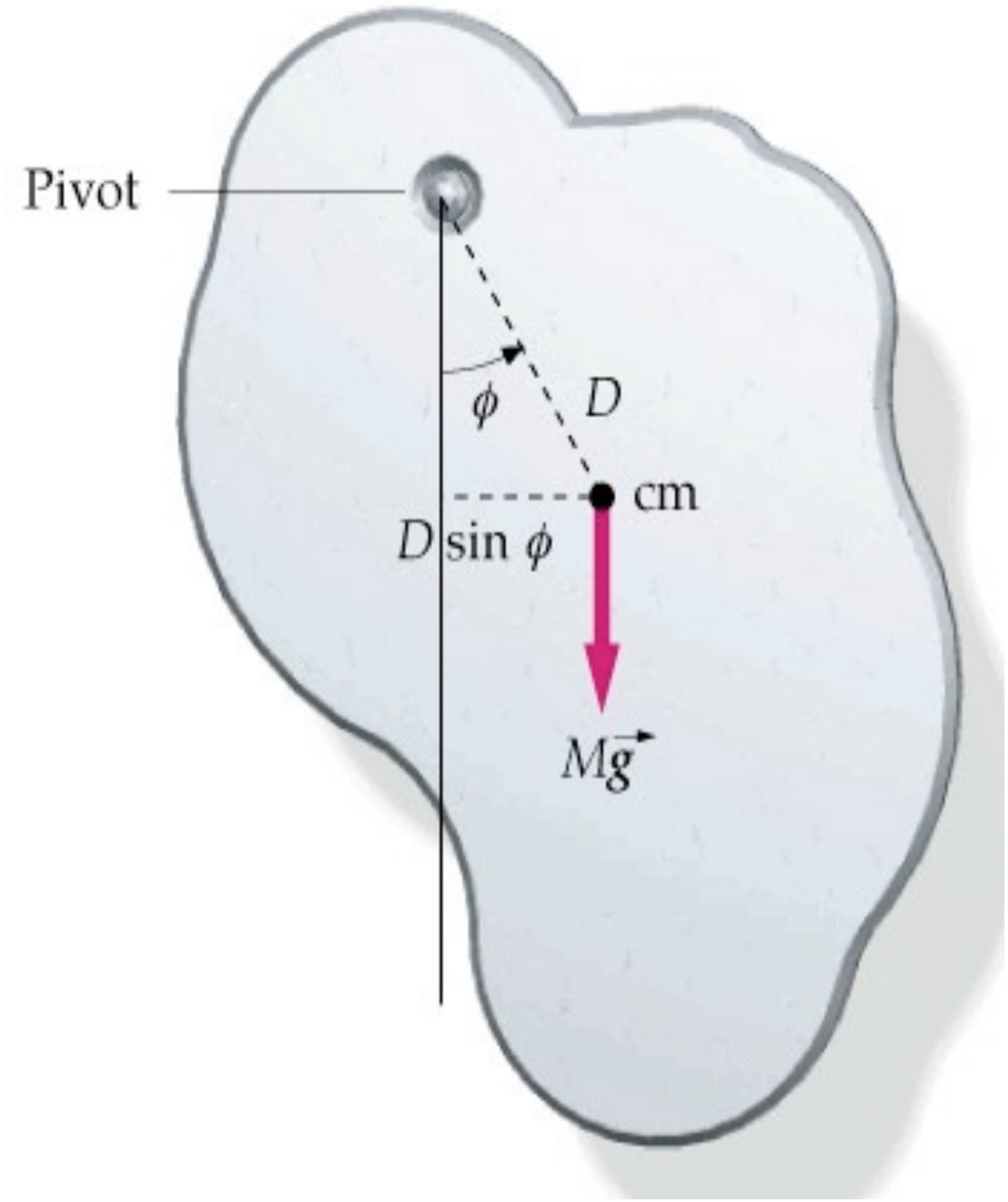
From N2

$$\tau = I\alpha = I\frac{d^2\phi}{dt^2}$$

$$-MgD\sin\phi = I\frac{d^2\phi}{dt^2}$$

$$\frac{d^2\phi}{dt^2} = -\frac{MgD}{I}\sin\phi$$

$$\frac{d^2\phi}{dt^2} = -\frac{MgD}{I}\phi \quad \text{for small } \phi$$



$$\frac{d^2\phi}{dt^2} = -\frac{MgD}{I}\phi$$

or $$\frac{d^2\phi}{dt^2} = -\omega^2\phi$$ where $\omega^2 = MgD/I$

for small angles the motion is SHM with

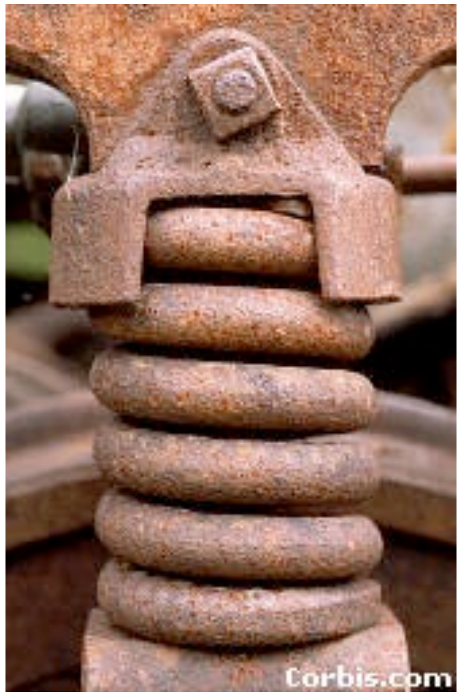
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MgD}}$$

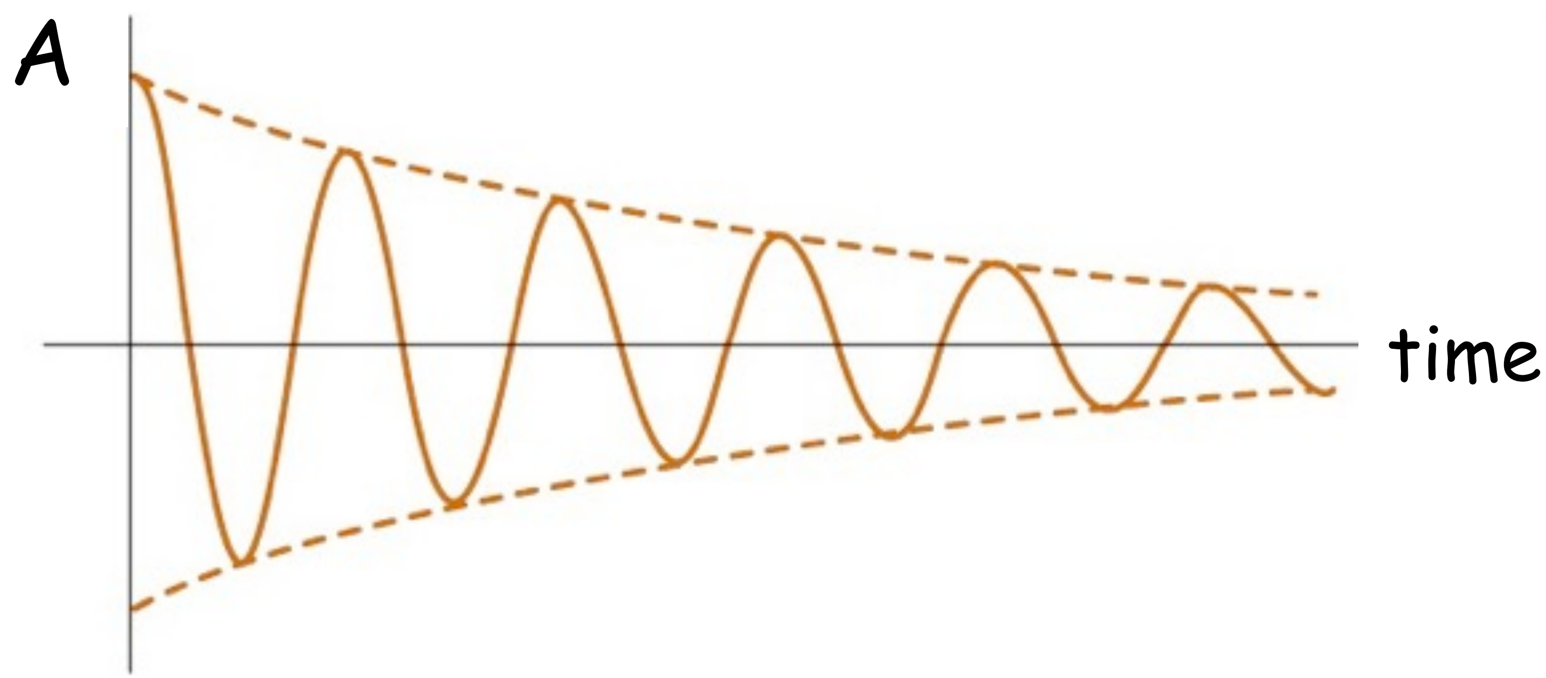
for large angles

$$T = T_o \left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + ...\right)$$

All real oscillations are subject to frictional or dissipative forces.

These forces remove energy from the oscillating system and reduce A.





Consider mass m on the end of a spring with a spring constant k

Restoring force = kx when mass is a distance x from equilibrium

drag force $\propto$ dx/dt

$$F = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

where $\gamma = b/m$ and $\omega^2 = k/m$

Auxiliary equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega_0 = (k/m)^{1/2}$

In order to find the auxiliary eq. one tries: $x(t) = e^{-\beta t}$

$$\beta^2 - \gamma\beta + \omega_o^2 = 0 \qquad \beta_{1/2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

1) $\gamma > 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t}$

2) $\gamma = 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

3) $\gamma < 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t}$

Initial conditions

and the constants can be determined applying the initial conditions, e.g. $x(0)=x_0$ and $v(0)=0$.

1) $$x(t) = e^{-\frac{\gamma}{2}t}\left[\left(\frac{x_0}{2} - \frac{\gamma x_0}{4\omega}\right)e^{-\omega t} + \left(\frac{x_0}{2} + \frac{\gamma x_0}{4\omega}\right)e^{+\omega t}\right]$$ overdamped

2) $$x(t) = e^{-\frac{\gamma}{2}t}\left[x_0 + \frac{\gamma x_0}{2}t\right]$$ critically damped

3) $$x(t) = e^{-\frac{\gamma}{2}t}\left[\left(x_0\right)\cos\omega t + \left(\frac{\gamma x_0}{2\omega}\right)\sin\omega t\right]$$ underdamped

with $$\omega = \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

Weak damping

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega^2 = k/m$

Weak damping: dissipative force is small compared to the restoring force

Oscillations continue, but gradually decrease in amplitude

Guess a solution to the differential equation above - exponential function will ensure the oscillations die at long t

first guess: $x(t) = e^{-\beta t} f(t)$

where β is a +ve constant and f(t) is to be determined

$$x = e^{-\beta t} f$$

$$\frac{dx}{dt} = -\beta e^{-\beta t} f + e^{-\beta t}\frac{df}{dt} \qquad = e^{-\beta t}\left(-\beta f + \frac{df}{dt}\right)$$

$$\frac{d^2x}{dt^2} = \beta^2 e^{-\beta t} f - \beta e^{-\beta t}\frac{df}{dt} - \beta e^{-\beta t}\frac{df}{dt} + e^{-\beta t}\frac{d^2f}{dt^2}$$

$$= e^{-\beta t}\left(\beta^2 f - 2\beta\frac{df}{dt} + \frac{d^2f}{dt^2}\right)$$

substitute these expressions into $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$

$$e^{-\beta t}\left(\beta^2 f - 2\beta\frac{df}{dt} + \frac{d^2f}{dt^2}\right) + \gamma\, e^{-\beta t}\left(-\beta f + \frac{df}{dt}\right) + \omega^2 e^{-\beta t} f = 0$$

After some tidying up we get

$$\frac{d^2f}{dt^2} + (\gamma - 2\beta)\frac{df}{dt} + (\beta^2 - \beta\gamma + \omega_o^2)f = 0$$

If $\gamma = 2\beta$ (or $\beta = \gamma/2$) we get an equation for SHM

$$\frac{d^2f}{dt^2} + (\omega_o^2 - \frac{\gamma^2}{4})\, f = 0 \qquad \frac{d^2x}{dt^2} + \omega^2 x = 0$$

ie $f = x\cos(\omega t + \delta)$ and $\omega^2 = (\omega_o^2 - \frac{\gamma^2}{4})$

when the dissipative force is small $\omega_o^2 >> \frac{\gamma^2}{4}$

and $\omega = \left[(\omega_o^2 - \frac{\gamma^2}{4}) \right]^{1/2} \approx \omega_o$

choosing f to have its maximum value x_o at t=0

we can write $f(t) = x_o \cos \omega t$

Therefore the displacement at any time t is given by

$$x(t) = x_o \, e^{\frac{-\gamma t}{2}} \cos(\omega t)$$

Strong damping

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega^2 = k/m$

Strong damping: $\gamma > \frac{\omega_o}{20}$ oscillations rapidly cease

if $\omega_o^2 < \frac{\gamma^2}{4}$ no oscillations will occur

Our solution becomes $\frac{d^2f}{dt^2} - \alpha^2 f = 0$ with $\alpha^2 = \frac{\gamma^2}{4} - \omega_o^2$

exp($-\alpha t$) and exp($+\alpha t$) both satisfy this equation giving

$f = Ae^{-\alpha t} + Be^{+\alpha t}$ and displacement $x = e^{\frac{-\gamma t}{2}}\left(Ae^{-\alpha t} + Be^{+\alpha t}\right)$

Critical damping

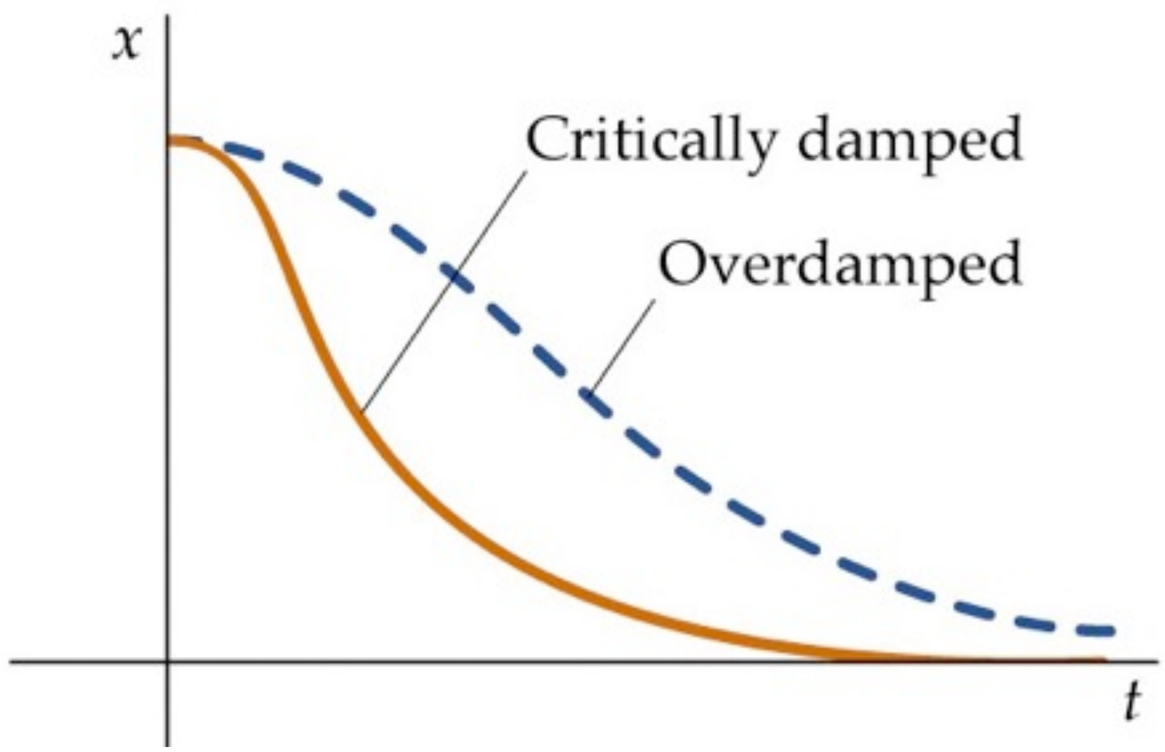
$$\frac{d^2f}{dt^2} + (\omega_o^2 - \frac{\gamma^2}{4})\, f = 0$$

If $\gamma = 2\omega_o$ the mass returns to equilibrium most quickly

and $$\frac{d^2f}{dt^2} = 0$$

eg: shock absorbers, CD platform

$$\therefore \quad f = A + Bt$$



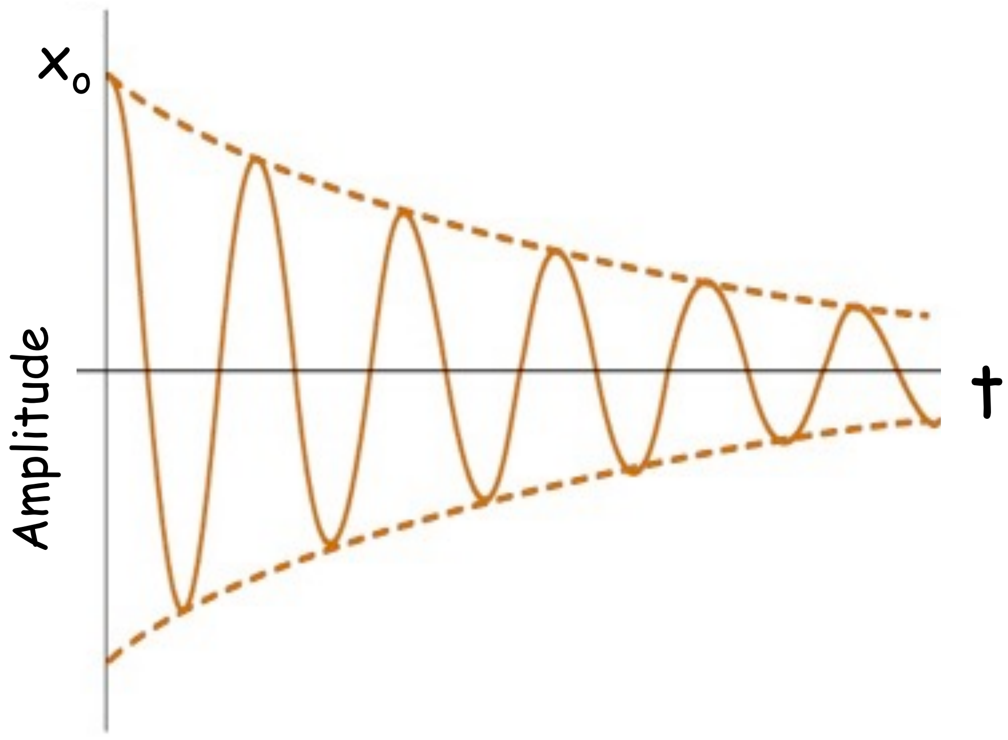
$df/dt=B \qquad d^2f/dt^2=0$

and $$x = e^{\frac{-\gamma t}{2}}(A + Bt)$$

Energy of a damped oscillator

Generally $E = \frac{1}{2} m\omega^2 A^2$ energy $E \propto$ amplitude A^2

if amplitude is decreasing exponentially then energy will also decrease exponentially



$$x(t) = x_o\, e^{\frac{-\gamma t}{2}} \cos(\omega t)$$

max displacement when cos=1

$$x(t) = x_o\, e^{\frac{-\gamma t}{2}}$$

$$\therefore \quad E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

Quality factor - Q

A damped oscillator is often described by its quality-factor or Q-factor

$$Q = \frac{\omega_o m}{b} = \frac{\omega_o}{\gamma}$$

this can be related to the fractional energy lost per cycle

$$E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

$$= E_o\, e^{-\gamma t}$$

$$dE = -\gamma\, E_o\, e^{-\gamma t} dt$$

$$= -\gamma\, E\, dt$$

In a weakly damped system the energy lost / cycle is small

$$dE = \Delta E \quad \text{and} \quad dt = T$$

$$\Delta E = -\gamma E T$$

$$\frac{|\Delta E|}{E} = \gamma T$$

$$\frac{|\Delta E|}{E} = \frac{\gamma 2\pi}{\omega_o}$$

but $Q = \dfrac{\omega_o}{\gamma}$ ie $\gamma = \dfrac{\omega_o}{Q}$

$$\boxed{\frac{|\Delta E|}{E} = \frac{2\pi}{Q}}$$

Example

When middle C on the piano (f=262Hz) is struck, it loses half its energy after 4s.

(a) what is the decay time ?

(b) What is the Q factor for this piano wire ?

(c) What is the fractional energy loss per cycle ?

(a) what is the decay time ?

$$E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

$$= E_o e^{-\gamma t}$$

decay time $\tau = 1/\gamma$

$$\therefore \quad E = E_o e^{-t/\tau}$$

When t=4s, $E = \frac{1}{2} E_o$

$$\frac{1}{2} E_o = E_o e^{-4/\tau}$$

$$2 = e^{4/\tau}$$

take natural logs of each side

$$\ln 2 = \frac{4}{\tau}$$

$$\tau = 5.77$$

The decay time is 5.77s

(b) What is the Q factor for this piano wire ?

$$Q = \frac{\omega_o}{\gamma} \quad \text{and} \quad \tau = 1/\gamma$$

$$Q = \omega_o \tau$$

$$= 2\pi f \tau$$

$$= 2\,\pi\, 262 \times 5.77$$

$$= 9.5 \times 10^3$$

Q factor for the piano wire = 9.5×10^3 (unitless)

Q is quite large, but this is to be expected for piano wire. Large Qs are also found for crystal and glass.

(c) What is the fractional energy loss per cycle ?

$$\frac{|\Delta E|}{E} = \gamma\, T$$

$$\frac{|\Delta E|}{E} = \frac{T}{\tau} = \frac{1}{f\tau}$$

$$= \frac{1}{262 \times 5.77}$$

$$= 6.61 \times 10^{-4}$$

$$\frac{|\Delta E|}{E} = \frac{2\pi}{Q}$$

$$= \frac{2\pi}{9.5 \times 10^{3}}$$

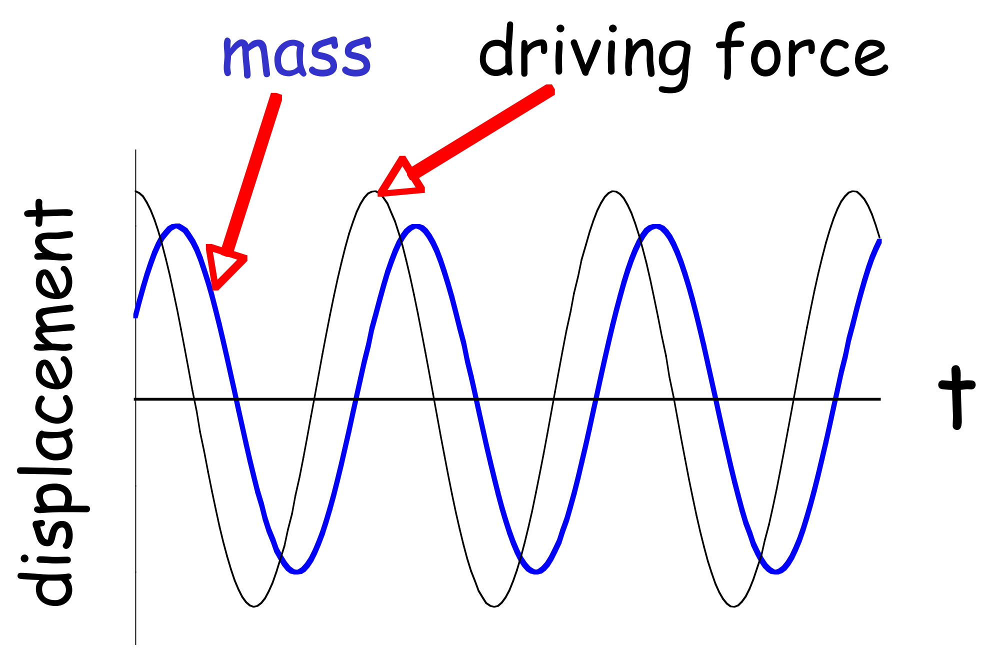
$$= 6.61 \times 10^{-4}$$

The fractional energy loss per cycle = 6.61×10^{-4}

Driven oscillations

Consider the steady state behaviour of a mass oscillating on a spring under the influence of a driving force.

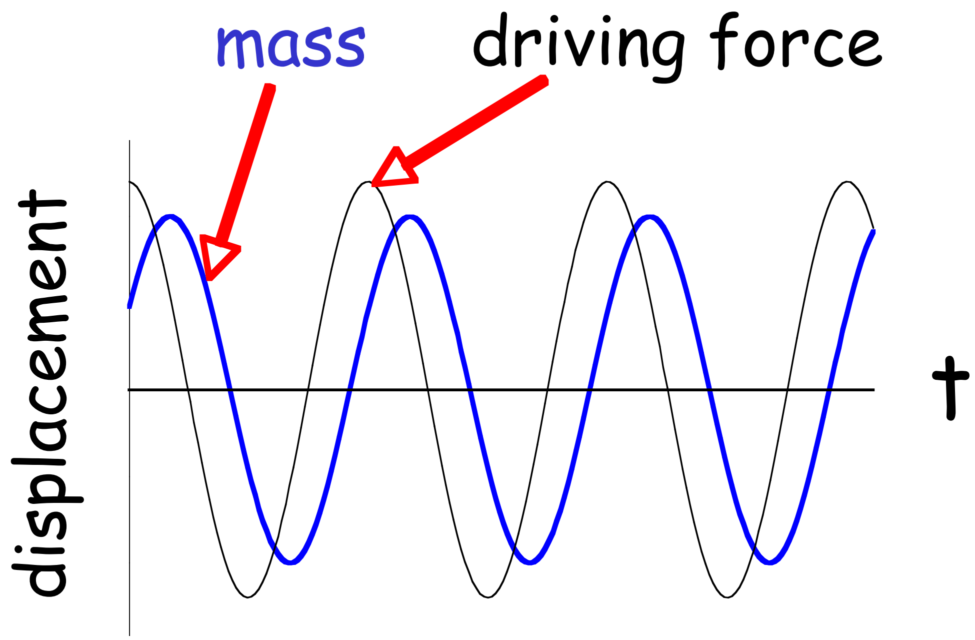
The mass oscillates at the same frequency of the driving force with a constant amplitude x_o.



The oscillations are out of phase, ie the displacement lags behind the driving force.

Force = $F_o\cos(\omega t)$ has +ve peaks at $t = 0, 2\pi/\omega, 4\pi/\omega$..........

+ve peaks of the displacement occur at $t = \Delta t, (2\pi/\omega)+\Delta t, (4\pi/\omega)+\Delta t$ ....



$$\therefore \text{the displacement } x = x_o \cos(\omega t - \phi) \quad \text{where} \quad \phi = \omega \Delta t = \frac{2\pi \Delta t}{T}$$

This describes a displacement with the same frequency as the driving force, has constant amplitude and a phase lag ϕ with respect to the driving force.

Equation of motion for a driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t)$$ where $\gamma = b/m$ and $\omega^2 = k/m$

Solution of this equation is $x = x_o\cos(\omega t - \phi)$

To determine the x_o and ϕ we need to substitute the solution into the equation of motion.

We need $$\frac{dx}{dt} = -\omega x_o \sin(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 x_o \cos(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t)$$

$$-\omega^2 x_o\cos(\omega t-\phi) - \gamma\omega x_o\sin(\omega t-\phi) + \omega_o^2 x_o\cos(\omega t-\phi) = \frac{F_o}{m}\cos(\omega t)$$

$$(\omega_o^2-\omega^2)x_o\cos(\omega t-\phi) - \gamma\omega x_o\sin(\omega t-\phi) = \frac{F_o}{m}\cos(\omega t)$$

This equation must be true at all times.

To solve for x_o and ϕ we need to consider two situations.

1. $(\omega t - \phi) = 0$ $\quad\therefore \sin(\omega t - \phi) = 0$ and $\cos(\omega t)=\cos\phi$

2. $(\omega t - \phi) = \pi/2$ $\quad\therefore \cos(\omega t - \phi) = 0$ and $\cos(\omega t)=\cos(\pi/2 + \phi)$

This leaves us with two simultaneous equations:

$$(\omega_o^2 - \omega^2)x_o = \frac{F_o}{m}\cos(\phi)$$

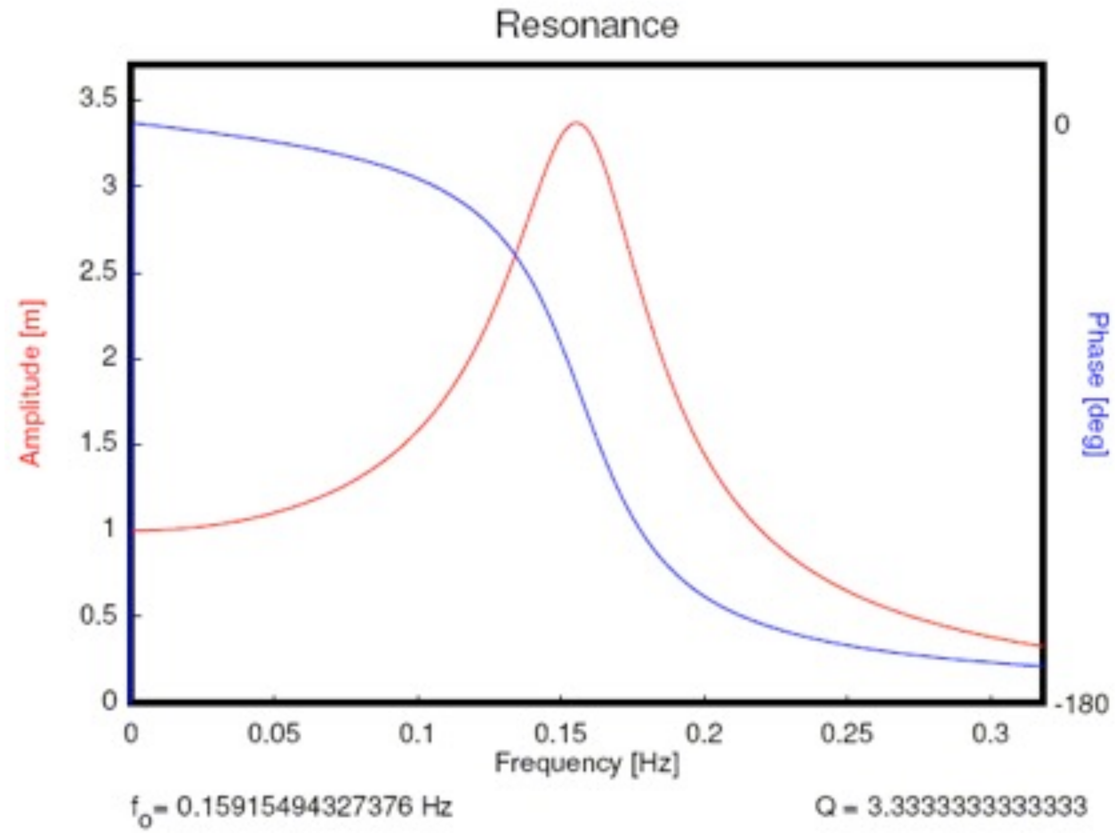
$$-\gamma\omega x_o = \frac{F_o}{m}\cos(\frac{\pi}{2} + \phi)$$

Remember $\cos(\frac{\pi}{2} + \phi) = -\sin\phi$ and $\cos^2 A + \sin^2 A = 1$

The solutions are

$$x_o = \frac{F_o / m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\tan\phi = \frac{\omega\gamma}{(\omega_o^2 - \omega^2)}$$



Resonance

The amplitude and energy of a system in the steady state depends on the amplitude and the frequency of the driver.

With no driving force the system will oscillate at its **natural frequency** ω_o

If the driving frequency $\sim \omega_o$ the energy absorbed by the oscillator is maximum and large amplitude oscillations occur

This is known as **resonance** and the natural frequency of the system is therefore called the **resonance frequency**

Resonance occurs in many systems - washing machines, breaking a glass with sound, child on a swing...........

Absorbed power

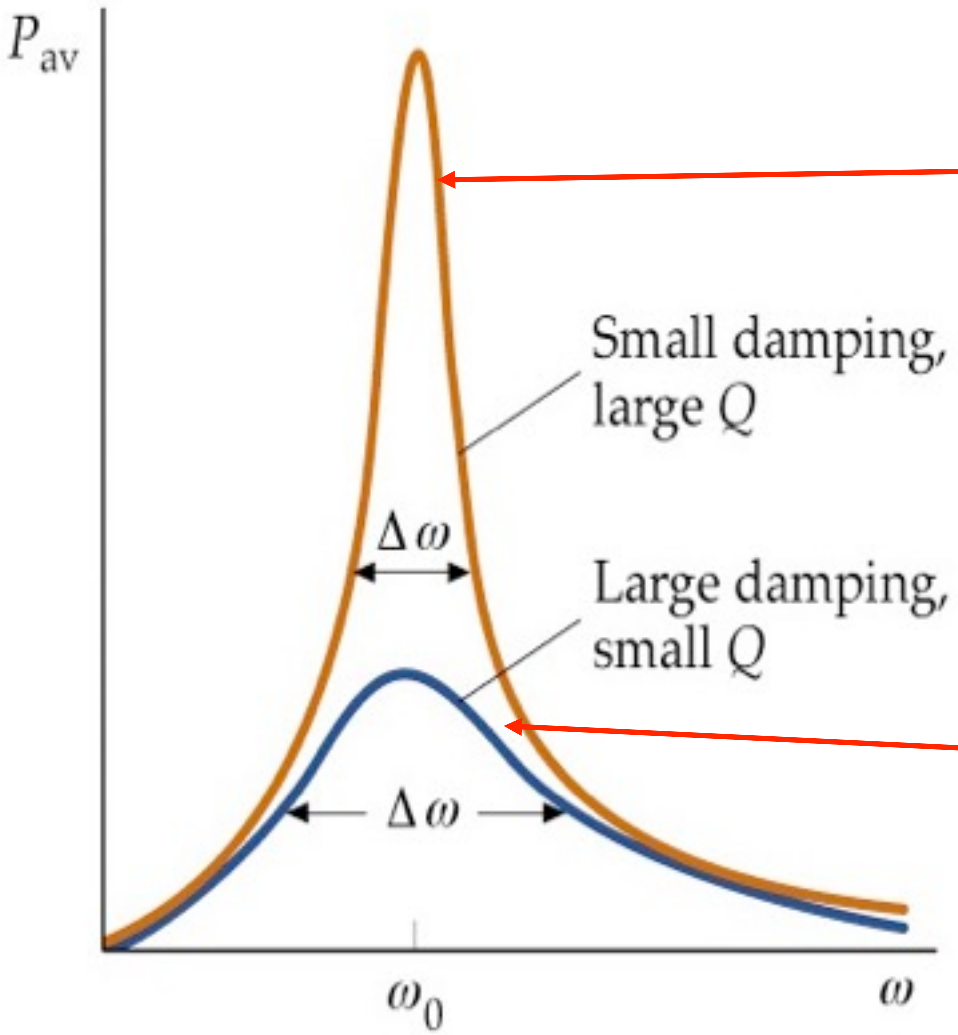
The average rate at which power is absorbed equals the average power delivered by the driving force, to replace the energy dissipated by the drag force.

Over a period it is:

$$\langle P\rangle = \left\langle \frac{Fdx}{dt} \right\rangle = \left\langle -b\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)\right\rangle = b\left\langle \left(\frac{dx}{dt}\right)^2 \right\rangle$$

$$\langle P\rangle = \frac{F_0^2}{2m\gamma}\left[\frac{\gamma^2\omega^2}{\left(\omega_0^2-\omega^2\right)+\gamma^2\omega^2}\right]$$

$$\langle P \rangle = \frac{F_0^2}{2m\gamma}\left[\frac{\gamma^2\omega^2}{\left(\omega_0^2-\omega^2\right)+\gamma^2\omega^2}\right]$$



When damping is small oscillator absorbs much more energy from driving force.

Resonance peak is narrow

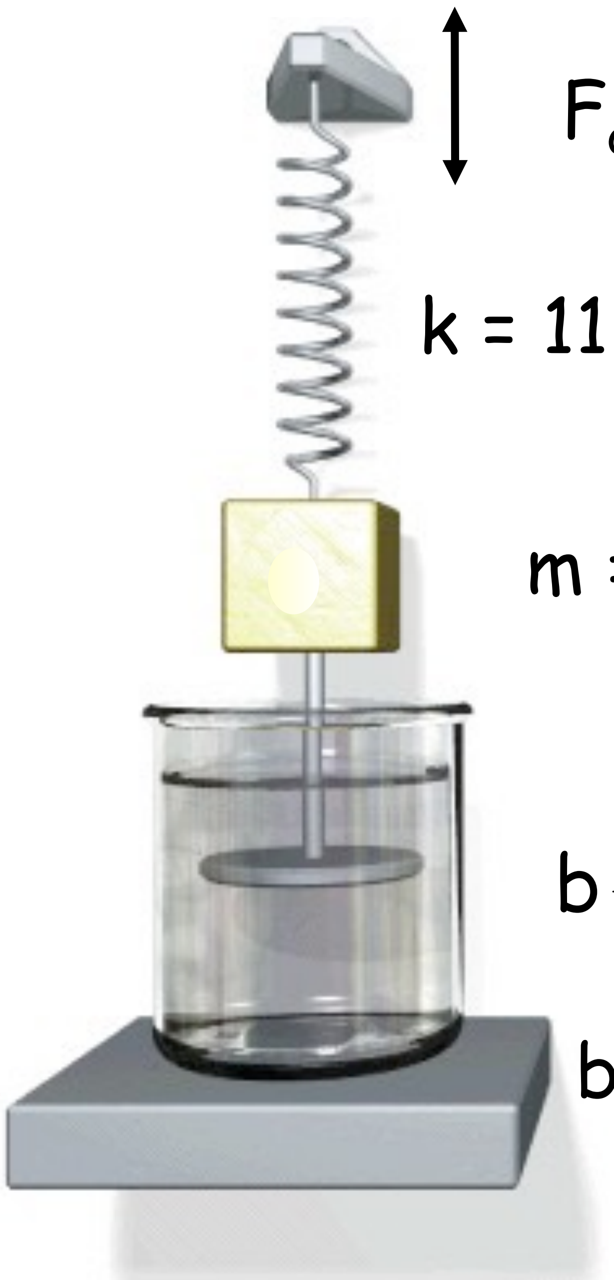
When damping is large oscillator resonance curve is broad

For small damping $\frac{\Delta\omega}{\omega_o} = \frac{\Delta f}{f_o} = \frac{1}{Q}$

Example

A mass of 25g on the end of a spring with force constant $11 Nm^{-1}$ is subject to a harmonic force of amplitude 2N. The viscous retarding force is given by b(dx/dt) with $b=0.3 kg\ s^{-1}$.

Determine the amplitude of the displacement and the phase difference between the force and the displacement in the steady state motion when the angular frequency is 5 rad s^{-1}.



$F_o = 2$ N

$k = 11$ Nm^{-1}

$m = 25$g

$= 0.025$kg

$b\dfrac{dx}{dt}$

$b = 0.3$ kg s^{-1}

$\omega = 5$ rad s^{-1}

Amplitude of displacement

$$x_o = \frac{F_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

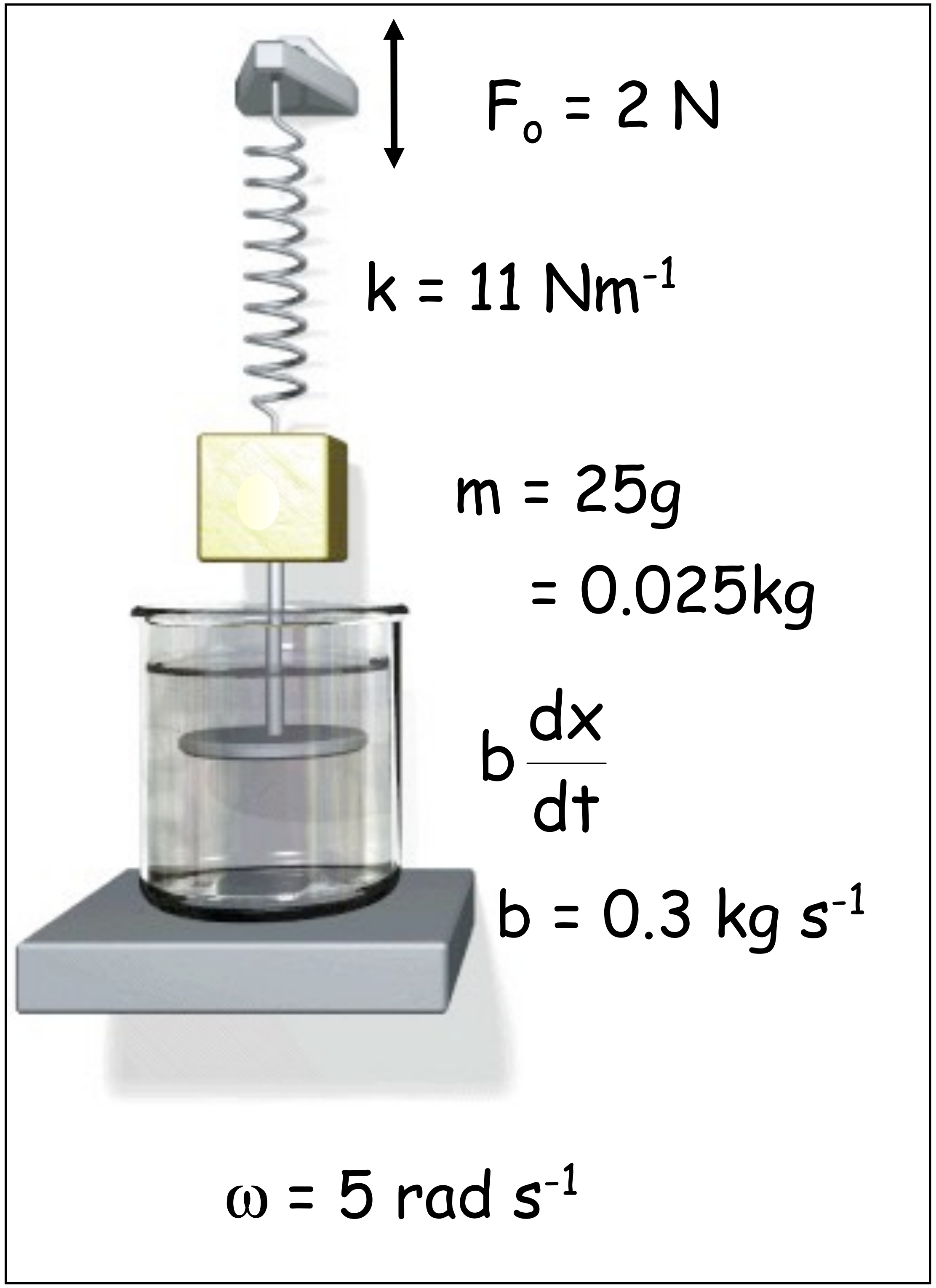
where $\omega_o^2 = k/m$ and $\gamma = b/m$

$$= \frac{2/0.025}{\sqrt{(11/0.025 - 5^2)^2 + 5^2(0.3/0.025)^2}}$$

$$= \frac{80}{419.3}$$

$$= 0.19$$

Amplitude of displacement = 0.19m



phase of displacement

$$\tan\phi = \frac{\omega\gamma}{(\omega_o^2 - \omega^2)}$$

where $\omega_o^2 = k/m$

$$\tan\phi = \frac{5(0.3/0.025)}{((11/0.025) - 5^2)}$$

$$\phi = \tan^{-1}\left(\frac{60}{415}\right)$$

$$= 8.2$$

phase of displacement lags phase of force by 8.2°