

Corso di Laurea in Fisica - UNITS  
ISTITUZIONI DI FISICA  
PER IL SISTEMA TERRA

# SURFACE WAVES

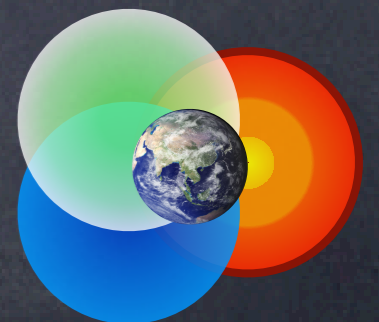
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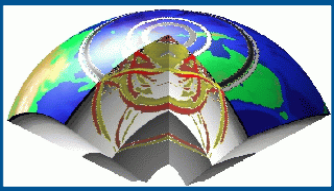
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# The Wave Equation: Potentials



*On Waves Propagated along the Plane Surface of an Elastic Solid.* By Lord RAYLEIGH, D.C.L., F.R.S.

[Read November 12th, 1885.]

It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity.\*

Denoting the displacements by  $\alpha$ ,  $\beta$ ,  $\gamma$ , and the dilatation by  $\theta$ , we have the usual equations

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

$\mathbf{u}$  displacement

$\Phi$  scalar potential

$\Psi_i$  vector potential

$$\partial_{tt}^2 \Phi = \alpha^2 \nabla^2 \Phi$$

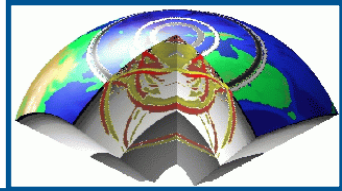
$$\partial_{tt}^2 \Psi_i = \beta^2 \nabla^2 \Psi_i$$

$\alpha$  P-wave speed

$\beta$  S-wave speed



# Rayleigh Waves

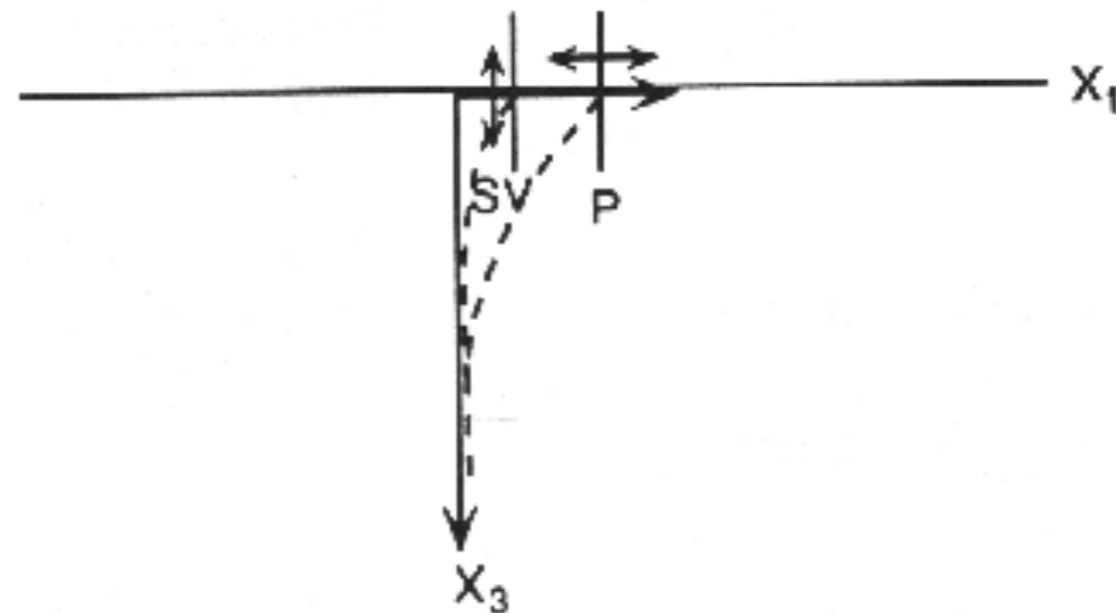
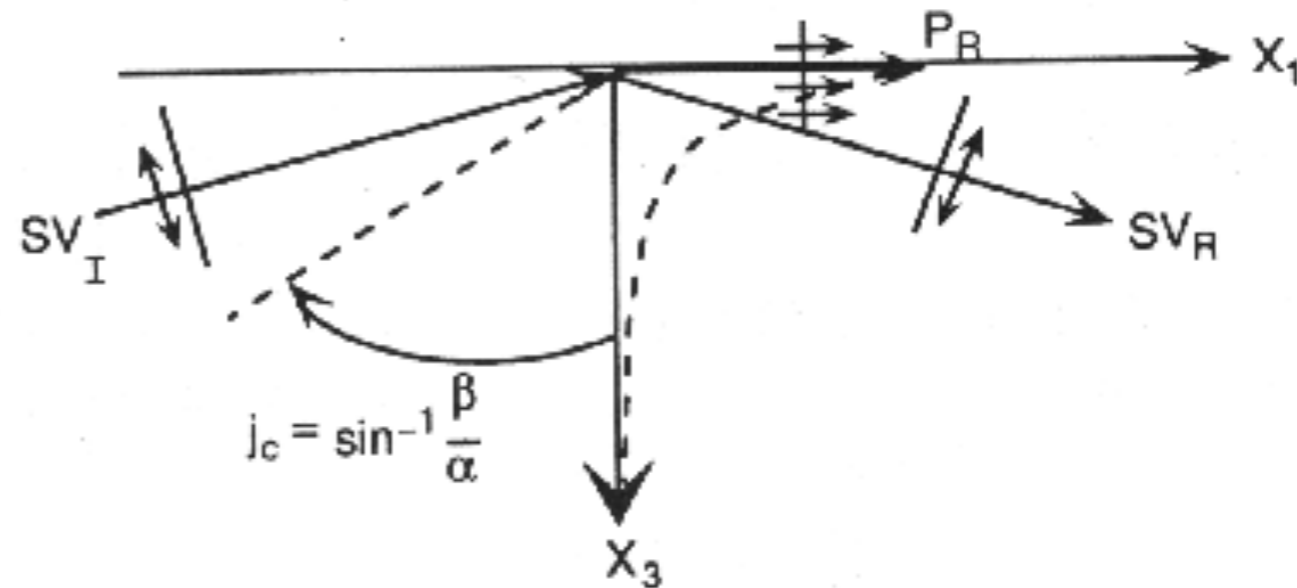


SV waves incident on a free surface: conversion and reflection

An **evanescent** P-wave propagates along the free surface decaying exponentially with depth.

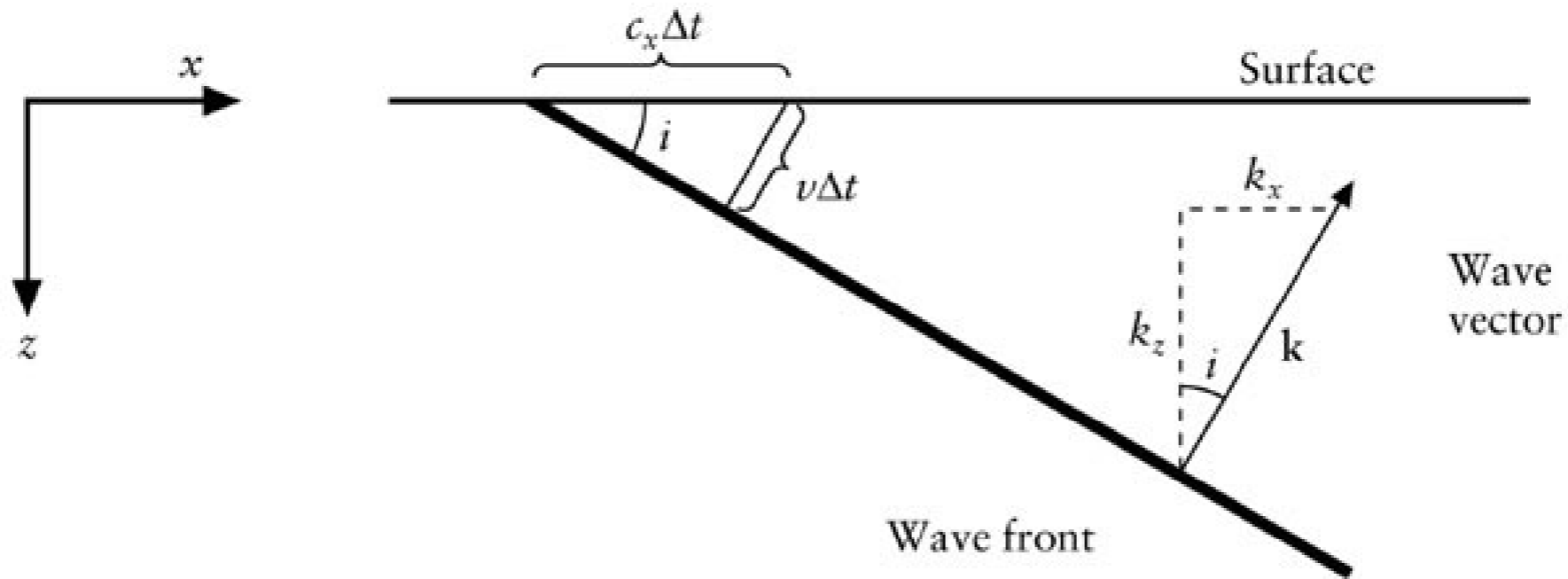
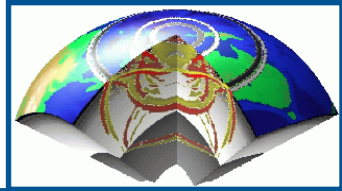
The reflected post-critically reflected SV wave is totally reflected and phase-shifted. These two wave types can only exist together, they both satisfy the free surface boundary condition:

-> Surface waves





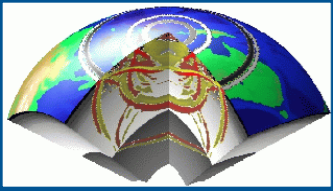
# Apparent horizontal velocity



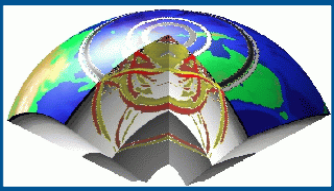
$$k_x = k \sin(i) = \omega \frac{\sin(i)}{\alpha} = \frac{\omega}{c}$$

$$k_z = k \cos(i) = \sqrt{k^2 - k_x^2} = \omega \sqrt{\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c} \sqrt{\left(\frac{c}{\alpha}\right)^2 - 1} = k_x r_\alpha$$

In current terminology,  $k_x$  is  $k$ !



# Surface waves: Geometry



We are looking for plane waves traveling along one horizontal coordinate axis, so we can - for example - set

$$\partial_y(\cdot) = 0$$

And consider only wave motion in the  $x, z$  plane. Then

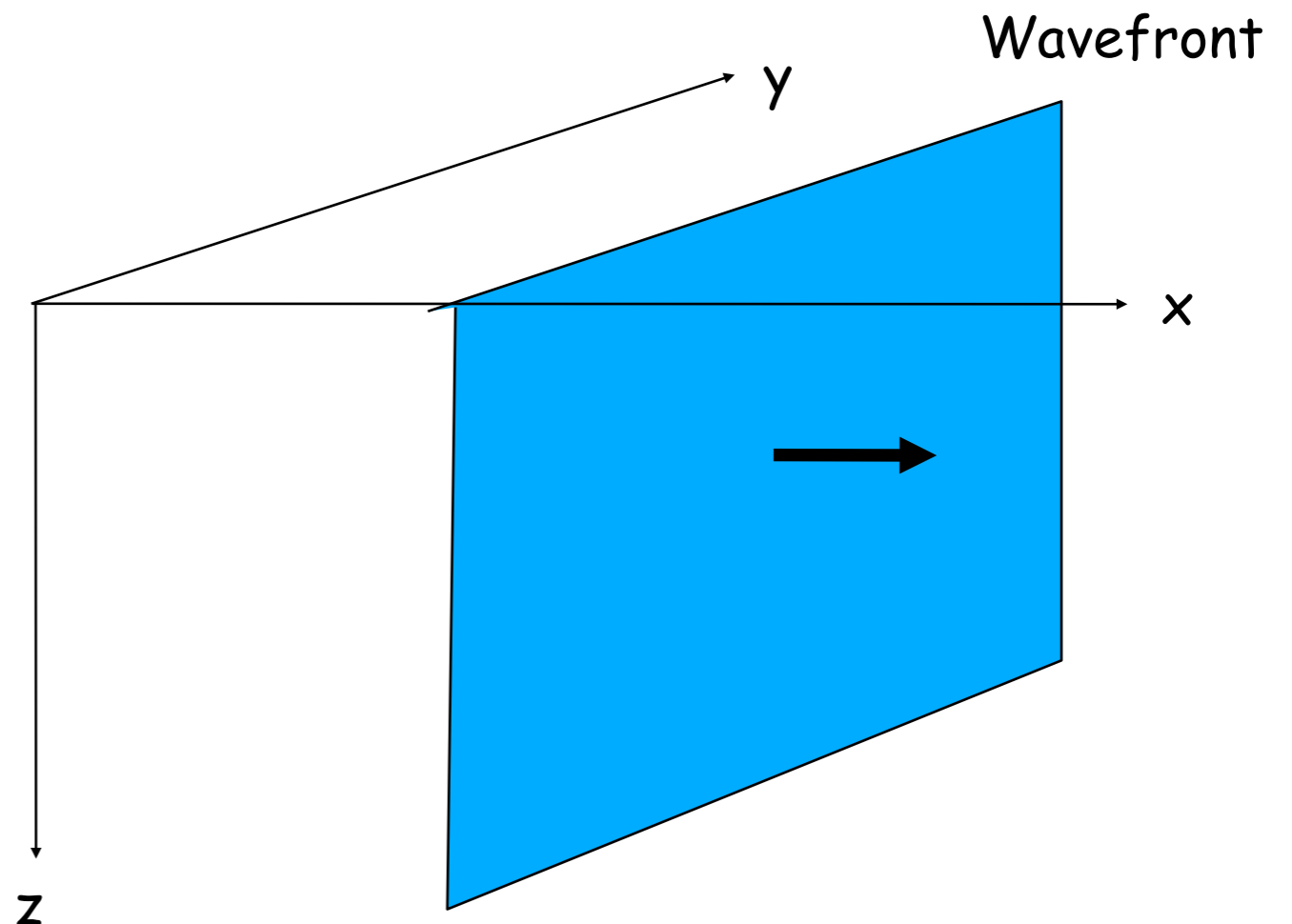
$$u_x = \partial_x \Phi - \partial_z \Psi$$

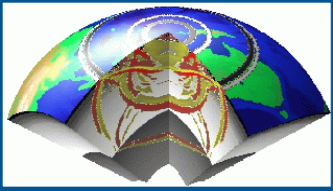
$$u_z = \partial_z \Phi + \partial_x \Psi$$

As we only require  $\Psi_y$  we set  $\Psi_y = \Psi$  from now on. Our trial solution is thus

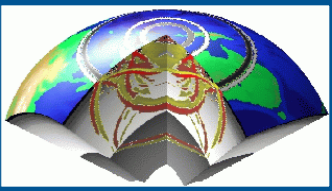
$$\Phi = A \exp[ik(x \pm r_\alpha z - ct)]$$

$$\Psi = B \exp[ik(x \pm r_\beta z - ct)]$$



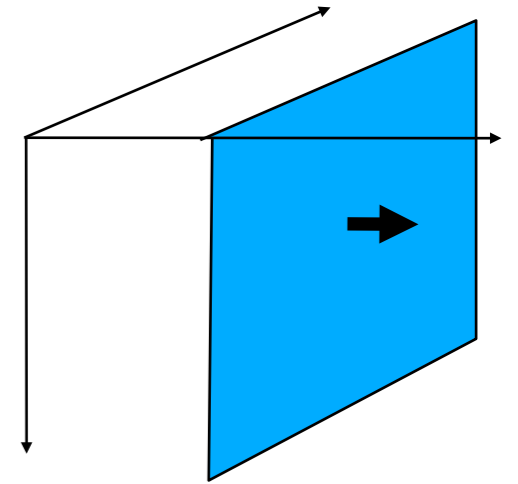


# Condition of existence



With that ansatz one has that, in order to desired solution exists, the coefficients

$$r_{\alpha} = \pm \sqrt{\frac{c^2}{\alpha^2} - 1} \quad r_{\beta} = \pm \sqrt{\frac{c^2}{\beta^2} - 1}$$



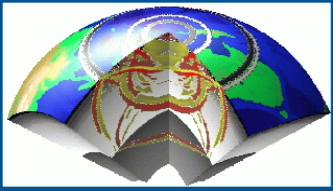
have to express a decay along  $z$ , i.e.

$$c < \beta < \alpha$$

to obtain

$$\Phi = A \exp \left[ i(kx - \omega t) - kz \sqrt{1 - \frac{c^2}{\alpha^2}} \right] = A \exp \left( -kz \sqrt{1 - \frac{c^2}{\alpha^2}} \right) \exp [i(kx - \omega t)]$$
$$\Psi = B \exp \left[ i(kx - \omega t) - kz \sqrt{1 - \frac{c^2}{\beta^2}} \right] = B \exp \left( -kz \sqrt{1 - \frac{c^2}{\beta^2}} \right) \exp [i(kx - \omega t)]$$

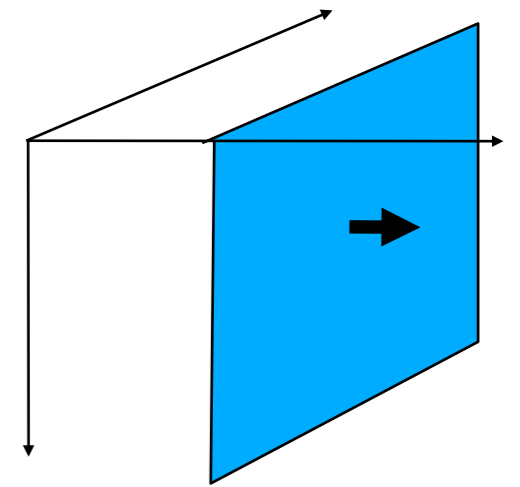




# Surface waves: Boundary Conditions



Analogous to the problem of finding the reflection-transmission coefficients we now have to satisfy the boundary conditions at the free surface (stress free)



$$\sigma_{zz} = 0 = \sigma_{zx}$$

In isotropic media we have

$$\sigma_{zz} = \lambda(\partial_x u_x + \partial_z u_z) + 2\mu \partial_z u_z$$

$$\sigma_{xz} = 2\mu \partial_x u_z$$

where

$$u_x = \partial_x \Phi - \partial_z \Psi$$

$$u_z = \partial_z \Phi + \partial_x \Psi$$

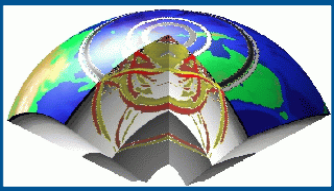
and

$$\Phi = A \exp[ik(x \pm r_\alpha z - ct)]$$

$$\Psi = B \exp[ik(x \pm r_\beta z - ct)]$$

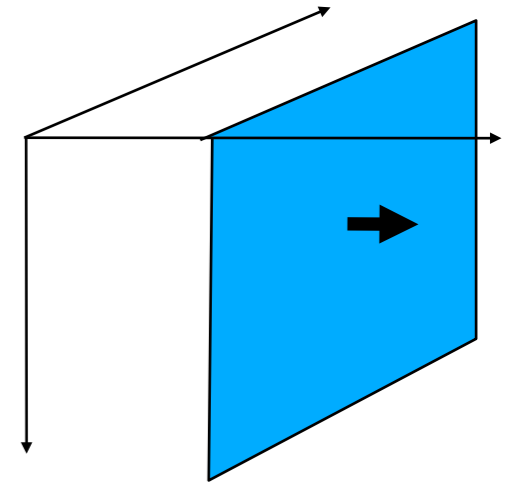


# Rayleigh waves: solutions



This leads to the following relationship for  $c$ , the phase velocity:

$$(2 - c^2 / \beta^2)^2 = 4(1 - c^2 / \alpha^2)^{1/2} (1 - c^2 / \beta^2)^{1/2}$$



For simplicity we take a fixed relationship between  $\alpha$  and shear-wave velocity (Poisson's medium):

$$\alpha = \sqrt{3} \beta$$

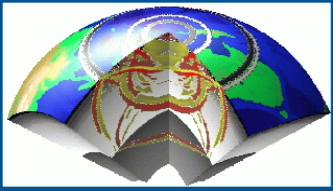
... to  
obtain

$$\frac{c^6}{\beta^6} - 8 \frac{c^4}{\beta^4} + \frac{56}{3} \frac{c^2}{\beta^2} - 16 = 0$$

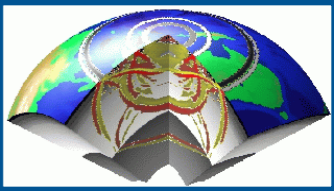
... and the only root which fulfills the condition  $c < \beta$  is

$$c \cong 0.92 \beta$$

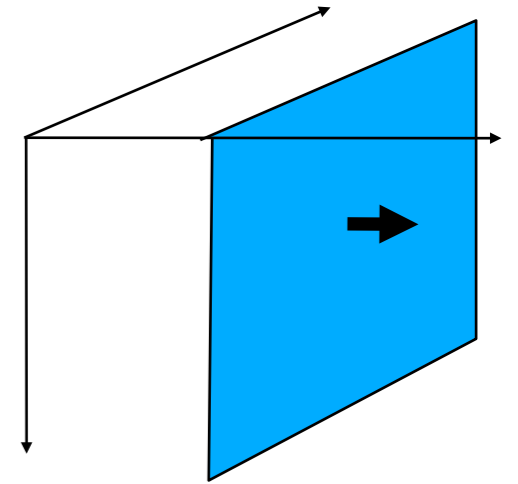




# Displacement



Putting this value back into our solutions we finally obtain the displacement in the x-z plane for a plane harmonic surface wave propagating along direction x



$$u_x = C(e^{-0.8475kz} - 0.57773e^{-0.3933kz})\sin k(x - ct)$$

$$u_z = C(-0.8475e^{-0.8475kz} + 1.4679e^{-0.3933kz})\cos k(x - ct)$$

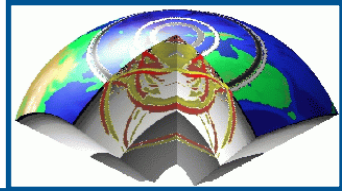
This development was first made by Lord Rayleigh in 1885.

It demonstrates that YES there are solutions to the wave equation propagating along a **free surface!**

Some remarkable facts can be drawn from this particular form:

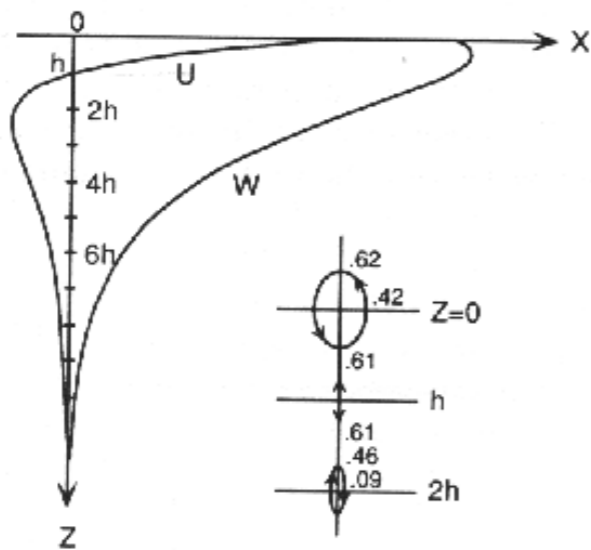
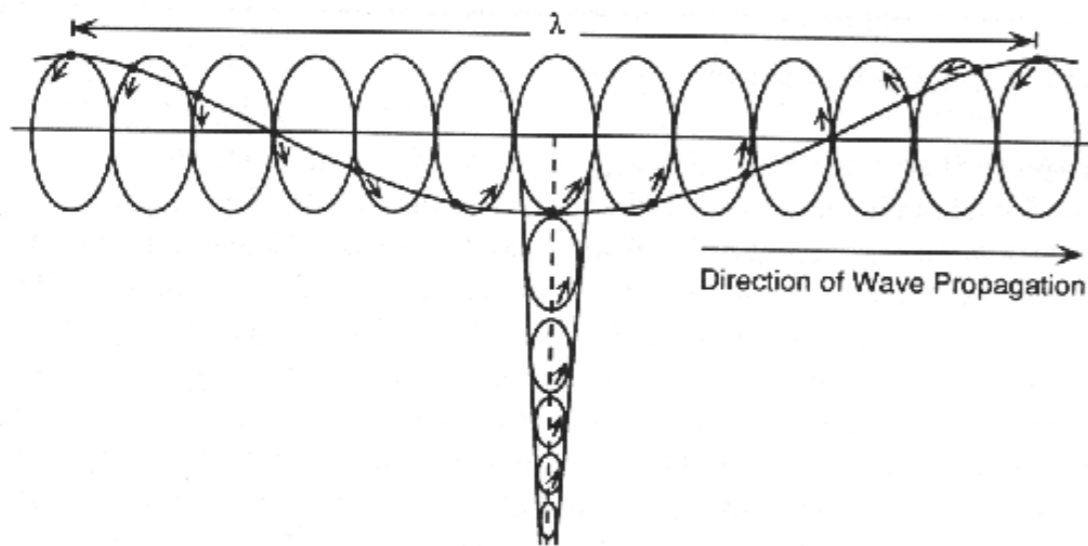


# Particle Motion (1)

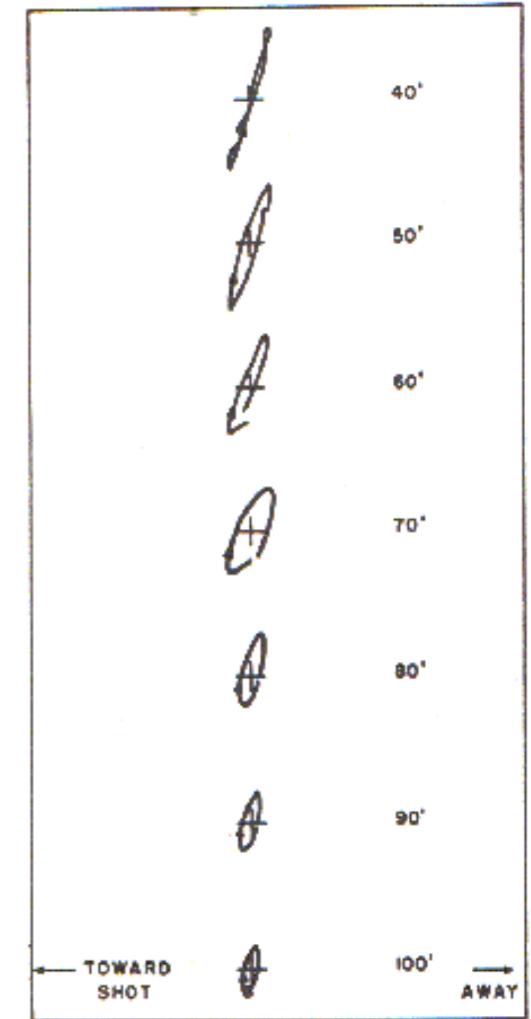
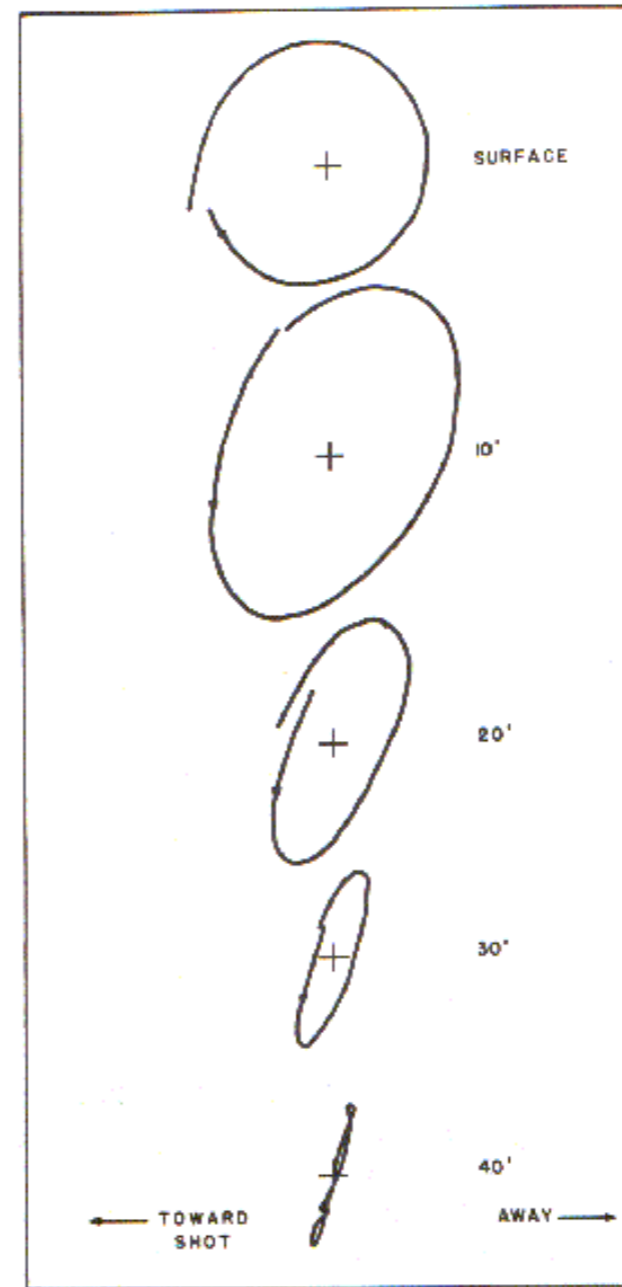


How does the particle motion look like?

theoretical

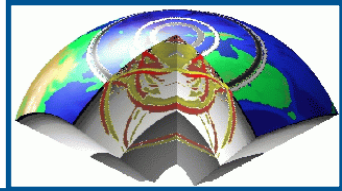


experimental



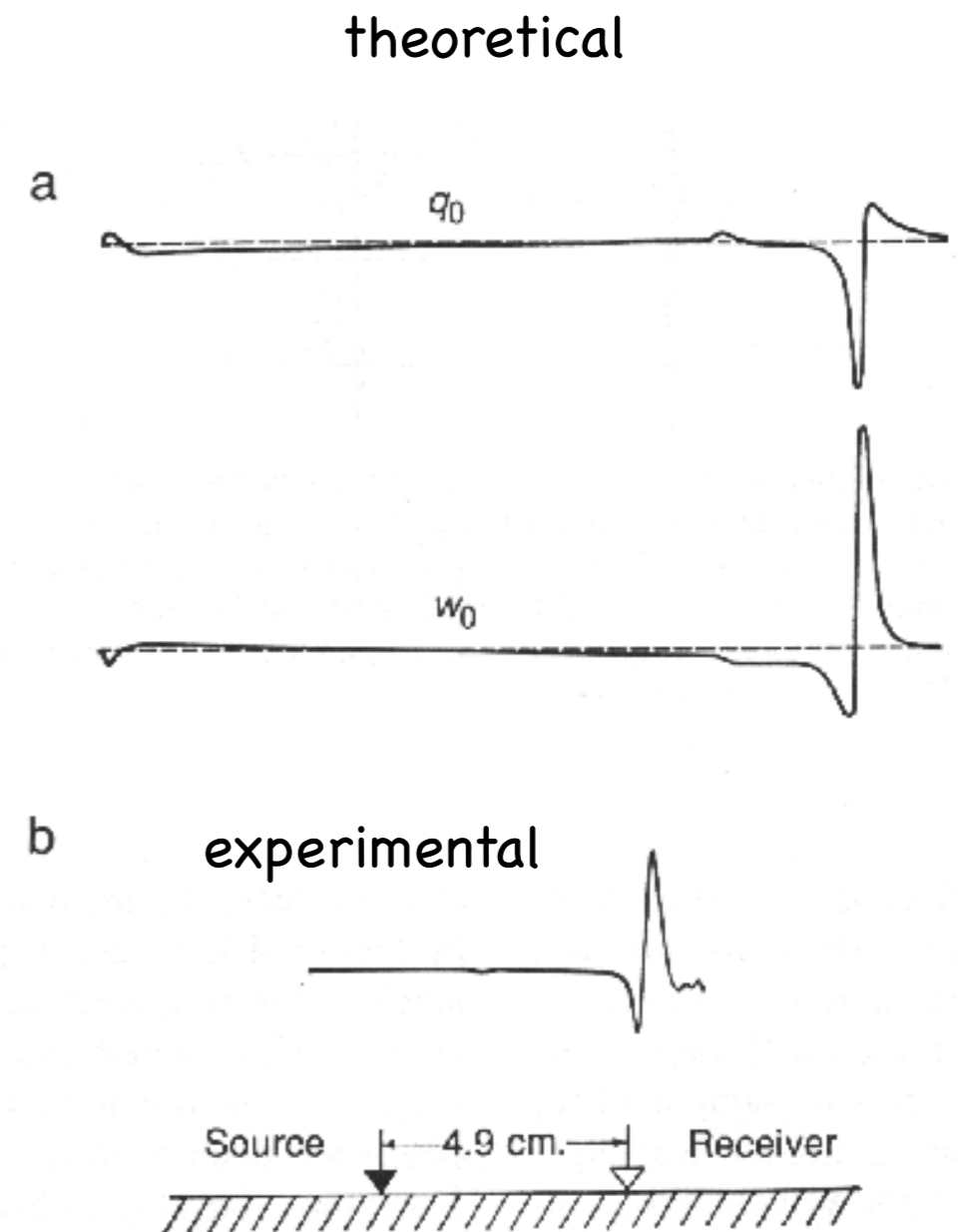


# Lamb's Problem and Rayleigh waves



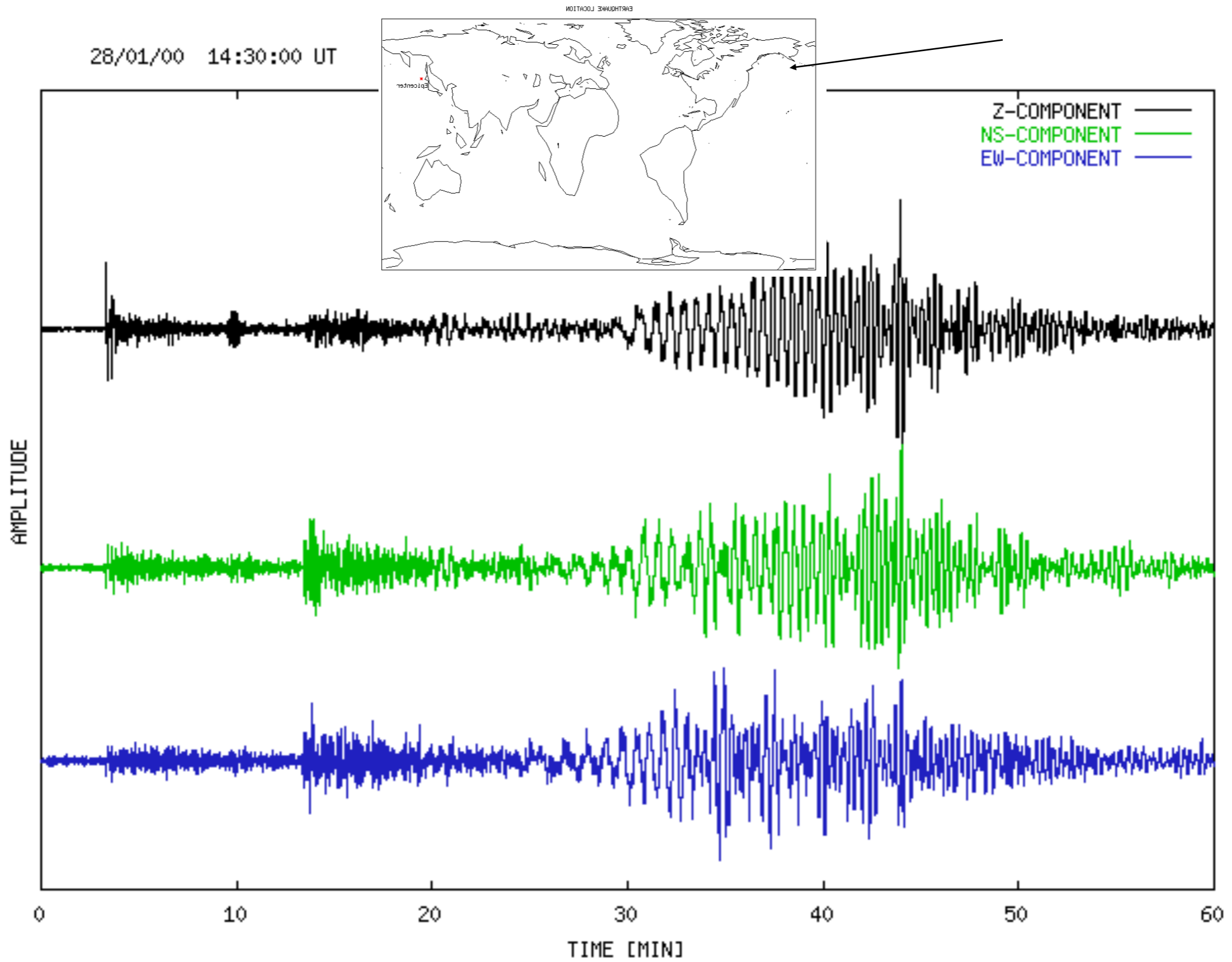
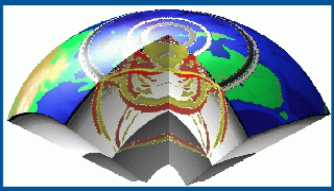
Transient solution to an impulsive vertical point force at the surface of a half space is called **Lamb's problem** (after Horace Lamb, 1904).

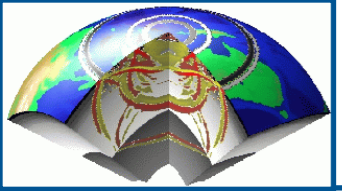
- the two components are out of phase by  $\pi/2$
- for small values of  $z$  a particle describes an ellipse and the motion is retrograde
- at some depth  $z$  the motion is linear in  $z$
- below that depth the motion is again elliptical but prograde
- the phase velocity is independent of  $k$ : **there is no dispersion** for a homogeneous half space
- Right Figure: radial and vertical motion for a source at the surface



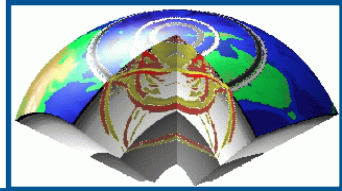


# Data Example





# Dispersion relation

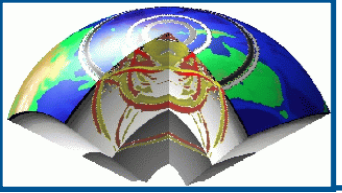


- In physics, the dispersion relation is the relation between the energy of a system and its corresponding momentum. For example, for massive particles in free space, the dispersion relation can easily be calculated from the definition of kinetic energy:

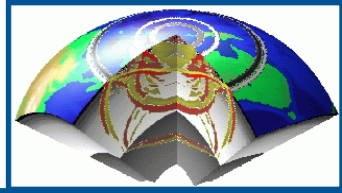
$$E = \frac{1}{2} mv^2 = \frac{p^2}{2m}$$

- For electromagnetic waves, the energy is proportional to the frequency of the wave and the momentum to the wavenumber. In this case, Maxwell's equations tell us that the dispersion relation for vacuum is linear:  $\omega = ck$ .
- The name "**dispersion relation**" originally comes from optics. It is possible to make the effective speed of light dependent on wavelength by making light pass through a material which has a non-constant index of refraction, or by using light in a non-uniform medium such as a waveguide. In this case, the waveform will spread over time, such that a narrow pulse will become an extended pulse, i.e. be dispersed.



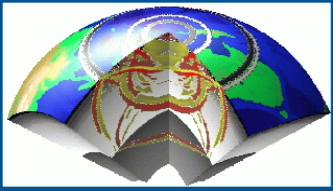


# Dispersion...



- In optics, dispersion is a phenomenon that causes the separation of a wave into spectral components with different wavelengths, due to a dependence of the wave's speed on its wavelength. It is most often described in light waves, but it may happen to any kind of wave that interacts with a medium or can be confined to a waveguide, such as sound waves. There are generally two sources of dispersion: **material dispersion**, which comes from a frequency-dependent response of a material to waves; and **waveguide dispersion**, which occurs when the speed of a wave in a waveguide depends on its frequency.
- In optics, the phase velocity of a wave  $v$  in a given uniform medium is given by:  $v=c/n$ , where  $c$  is the speed of light in a vacuum and  $n$  is the refractive index of the medium. In general, the refractive index is some function of the frequency  $f$  of the light, thus  $n = n(f)$ , or alternately, with respect to the wave's wavelength  $n = n(\lambda)$ . For visible light, most transparent materials (e.g. glasses) have a refractive index  $n$  decreases with increasing wavelength  $\lambda$  ( $dn/d\lambda < 0$ , i.e.  $dv/d\lambda > 0$ ). In this case, the medium is said to have **normal dispersion** and if the index increases with increasing wavelength the medium has **anomalous dispersion**.





# Effect of dispersion...



Demonstration: sum two harmonic waves with slightly different angular frequencies and wavenumbers:

$$u(x, t) = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$

$$\omega_1 = \omega + \delta\omega \quad \omega_2 = \omega - \delta\omega \quad \omega \gg \delta\omega$$

$$k_1 = k + \delta k \quad k_2 = k - \delta k \quad k \gg \delta k$$

Add the two cosines:

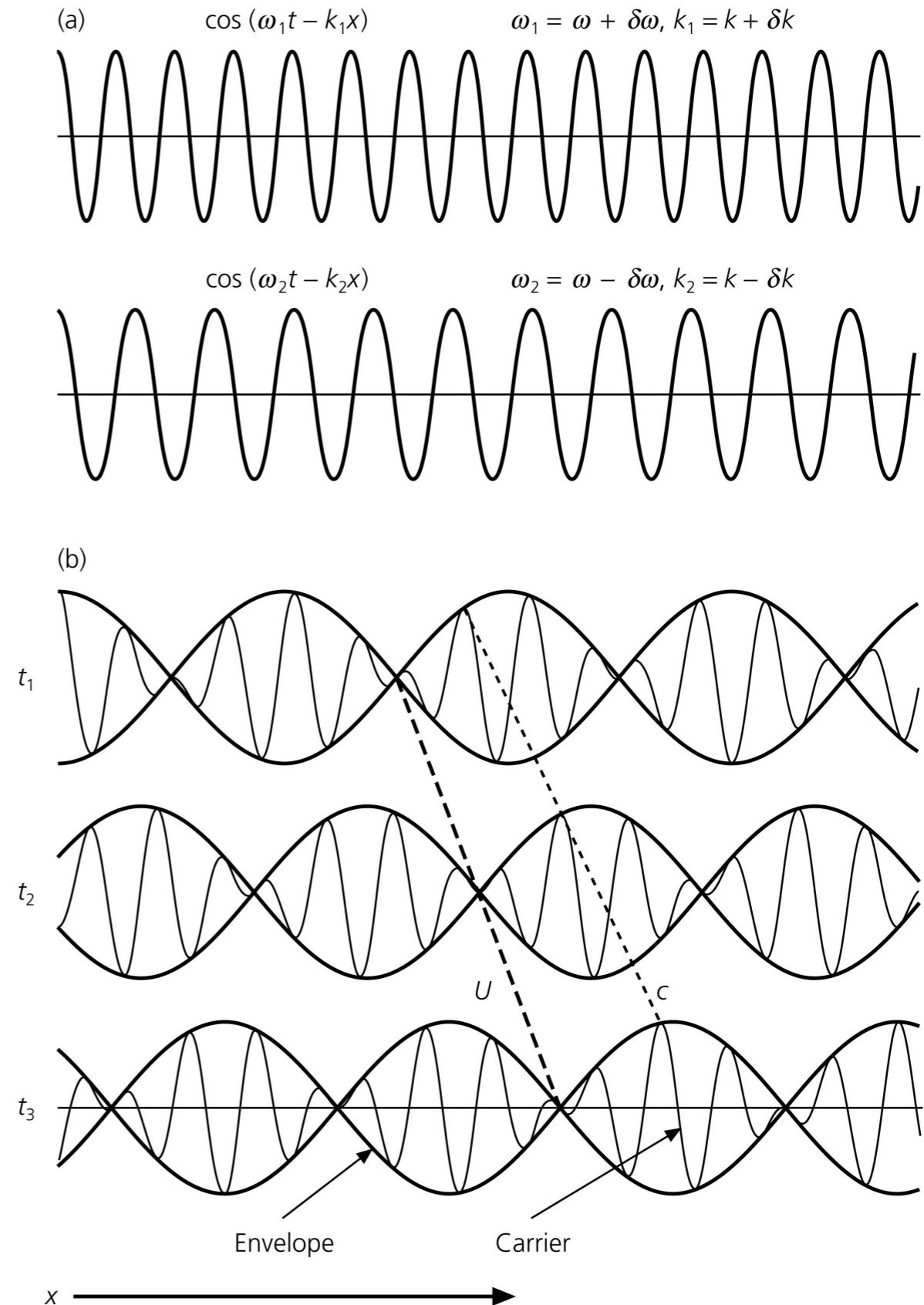
$$\begin{aligned} u(x, t) &= \cos(\omega t + \delta\omega t - kx - \delta kx) \\ &+ \cos(\omega t - \delta\omega t - kx + \delta kx) \\ &= 2 \cos(\omega t - kx) \cos(\delta\omega t - \delta kx) \end{aligned}$$

The envelope (beat) has a *group velocity*:

$$U = \delta\omega / \delta k$$

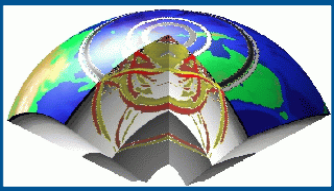
The individual peaks move with a *phase velocity*:

$$c = \omega / k$$





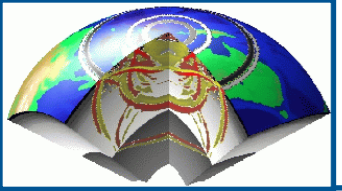
# Group velocity



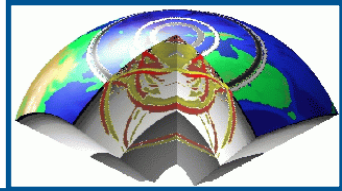
- Another consequence of dispersion manifests itself as a temporal effect. The phase velocity is the velocity at which the phase of any one frequency component of the wave will propagate. This is not the same as the **group velocity of the wave, which is the rate that changes in amplitude** (known as the envelope of the wave) will propagate. The group velocity  $v_g$  is related to the phase velocity  $v$  by, for a homogeneous medium (here  $\lambda$  is the wavelength in vacuum, not in the medium):

$$v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk} = v + k \frac{dv}{dk} = v - \lambda \frac{dv}{d\lambda}$$

and thus in the normal dispersion case  
 $v_g$  is always  $< v$  !



# Dispersion relation



- In classical mechanics, the Hamilton's principle the perturbation scheme applied to an averaged Lagrangian for an harmonic wave field gives a characteristic equation:  $\Delta(\omega, \mathbf{k}_i) = 0$

*Transverse wave in a string*

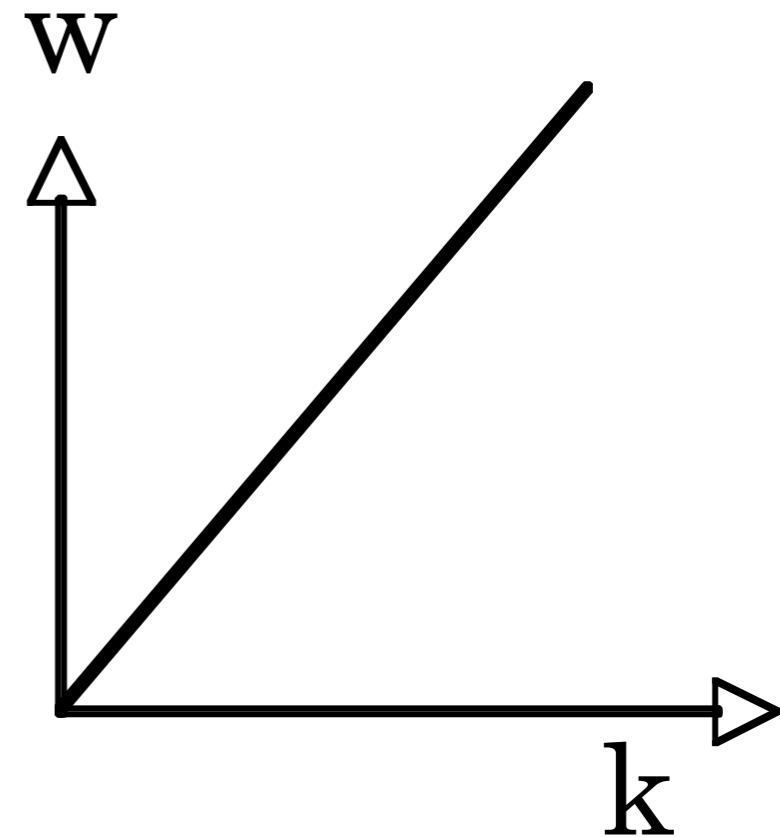
$$\left( \frac{\partial^2}{\partial x^2} - \frac{\mu}{F} \frac{\partial^2}{\partial t^2} \right) \phi = 0 \Rightarrow \omega = \pm kc$$

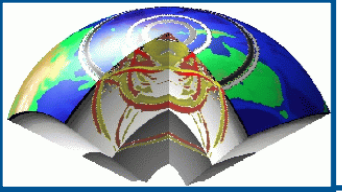
*Acoustic wave*

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\rho}{B} \frac{\partial^2}{\partial t^2} \right) \phi = 0 \Rightarrow \omega = \pm kc$$

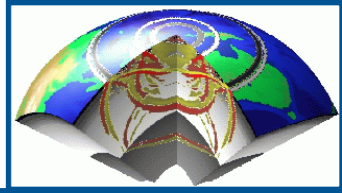
*Longitudinal wave in a rod*

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2}{\partial t^2} \right) \phi = 0 \Rightarrow \omega = \pm kc$$

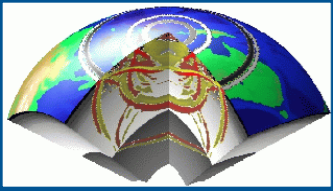




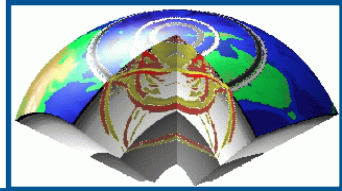
# Dispersion examples



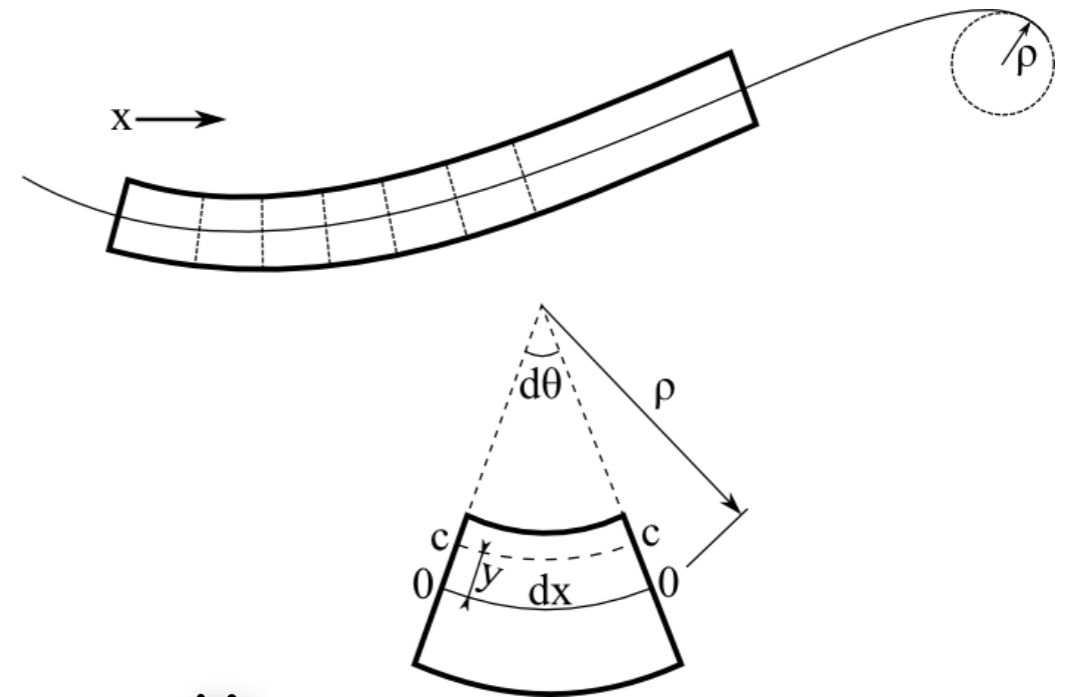
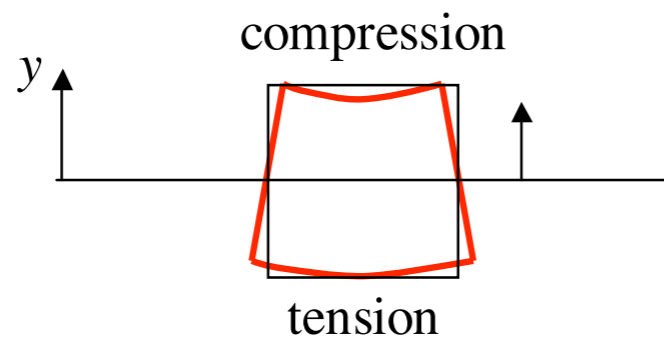
- Discrete systems: lattices
- Stiff systems: rods and thin plates
- Boundary waves: plates and rods  
Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!



# Stiffness...



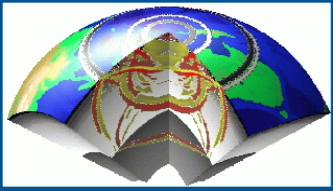
- How "**stiff**" or "flexible" is a material? It depends on whether we pull on it, twist it, bend it, or simply compress it. In the simplest case the material is characterized by two independent "stiffness constants" and that different combinations of these constants determine the response to a pull, twist, bend, or pressure.



## Euler-Bernoulli equation

$$\left( \frac{\partial^4}{\partial x^4} - \frac{\rho A}{EI} \frac{\partial^2}{\partial t^2} \right) y = 0 \Rightarrow \omega = \pm k^2 \sqrt{\frac{EI}{\rho A}}$$





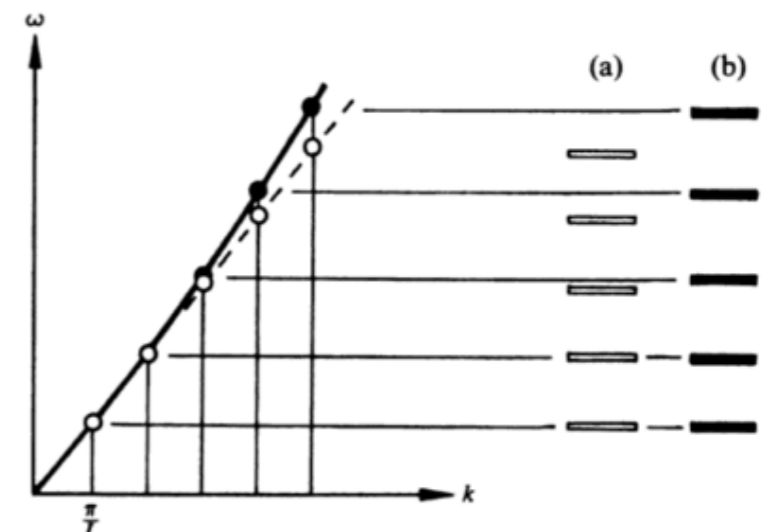
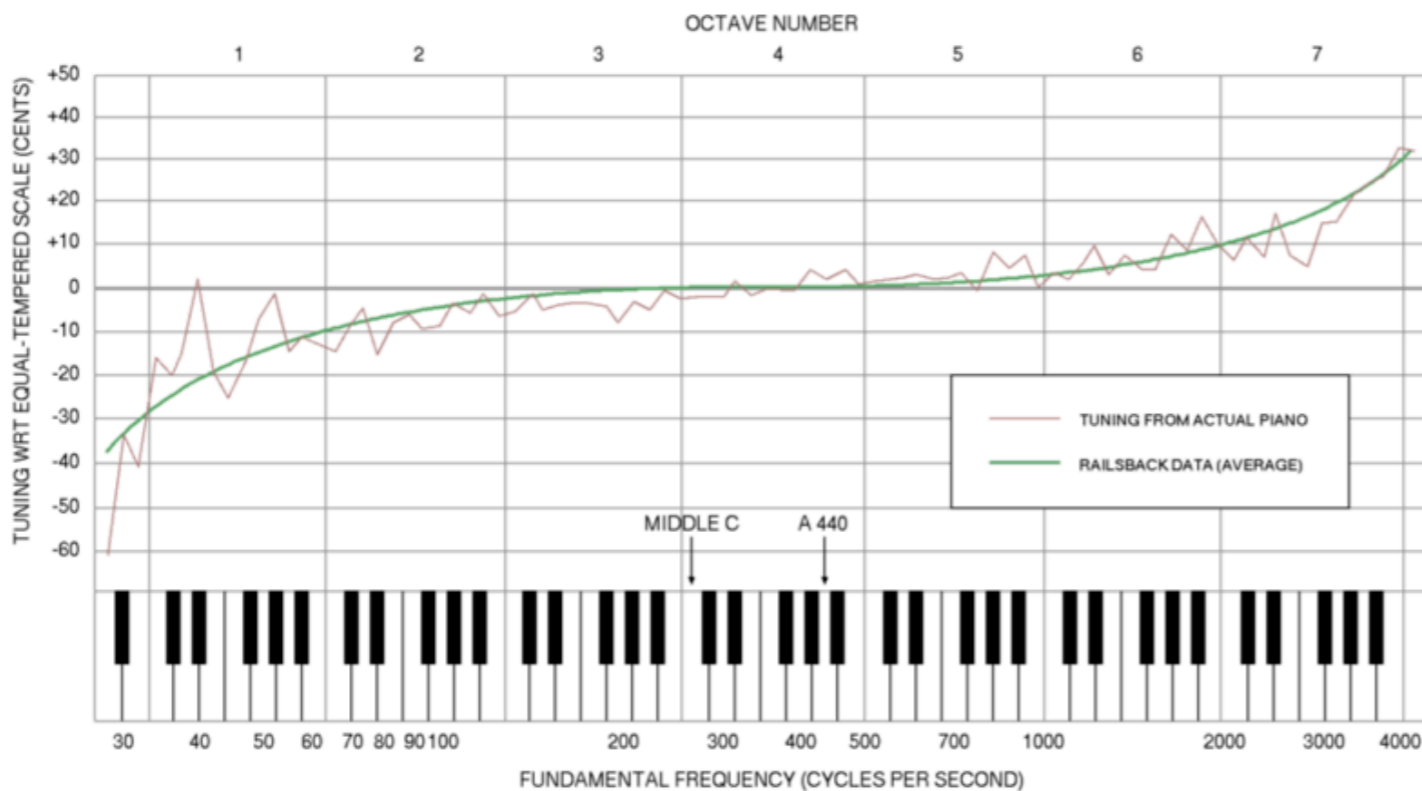
# Stiffness...



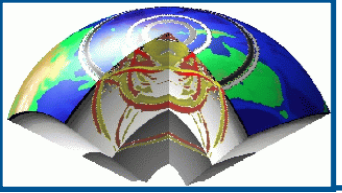
- Stiffness in a vibrating string introduces a restoring force proportional to the bending angle of the string and the usual stiffness term added to the wave equation for the ideal string. Stiff-string models are commonly used in piano synthesis and they have to be included in tuning of piano strings due to inharmonic effects.

$$\left( \frac{\partial^4}{\partial x^4} + \frac{E}{\rho} \frac{\partial^2}{\partial x^2} - \frac{\rho A}{EI} \frac{\partial^2}{\partial t^2} \right) w = 0 \Rightarrow \omega = \pm k \sqrt{\frac{E}{\rho}} \left( 1 + k^2 \sqrt{\frac{I}{A}} \right)^{1/2}$$

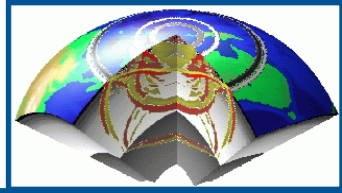
$$\Rightarrow \omega \approx \pm k \sqrt{\frac{E}{\rho}} \left( 1 + \frac{1}{2} k^2 \sqrt{\frac{I}{A}} \right)$$







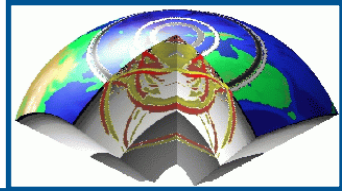
# Dispersion examples



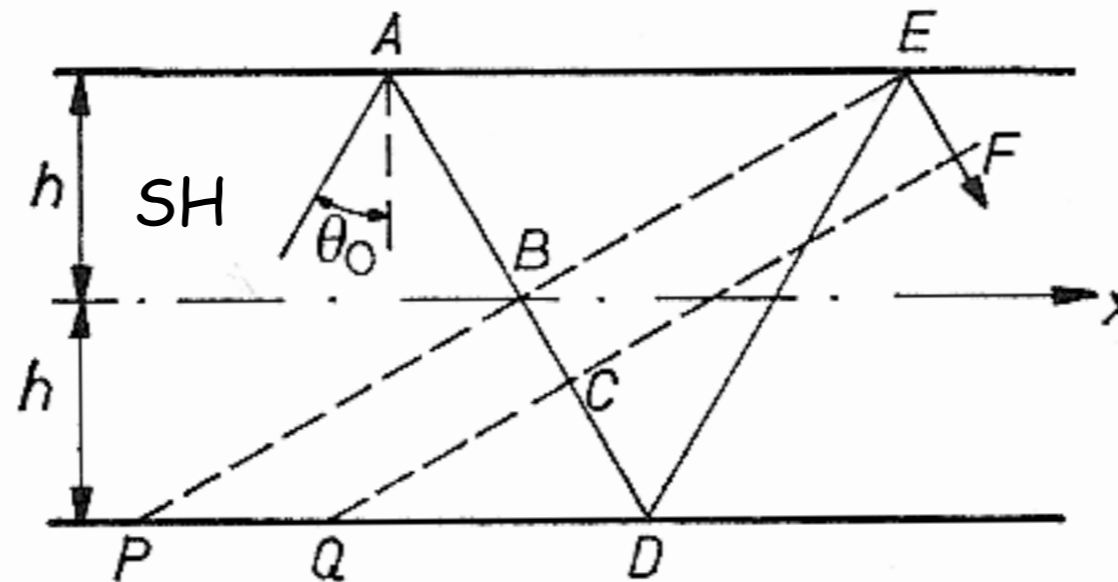
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- Stiff systems: rods and thin plates
- Boundary waves: plates and rods  
Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!



# SH Waves in plates: Geometry



In an elastic half-space no SH type surface waves exist. Why? Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer delimited by two free boundaries, i.e. a homogeneous plate?



The condition of interference of multiply reflected waves at the rigid boundaries is:

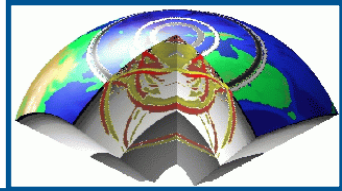
$$\frac{BDE}{\lambda} = \frac{CDEF}{\lambda} = n \quad BDE = 2(2h) \cos \theta_0 \quad \cos \theta_0 = n \frac{\lambda}{2(2h)} = n \frac{\pi}{(2h)k}$$

$$k \cos \theta_0 (2h) = k_z (2h) = k_x r_\beta (2h) = n\pi$$

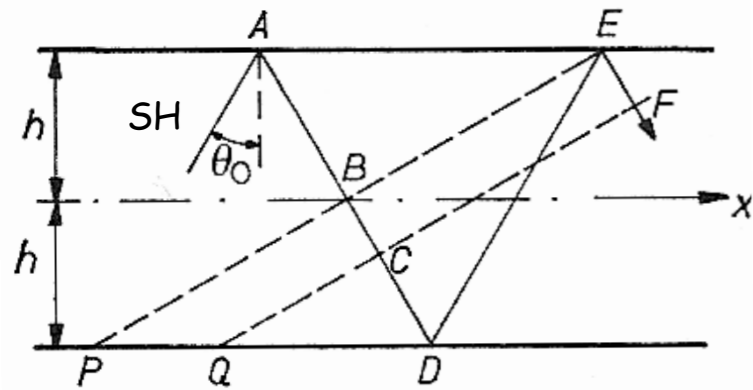
Repeated reflection in the layer allow interference between incident and reflected SH waves: SH reverberations can be totally trapped.



# SH waves: trapping



$$u_y = A \exp[i(\omega t + \omega \eta_\beta z - kx)] + B \exp[i(\omega t - \omega \eta_\beta z - kx)]$$



$$k = k_x = \frac{\omega}{c}; \quad \omega \eta_\beta = k_z = \frac{\omega}{c} \sqrt{\frac{c^2}{\beta^2} - 1} = k r_\beta$$

$$u_y = A \exp[i(\omega t + k r_\beta z - kx)] + B \exp[i(\omega t - k r_\beta z - kx)]$$

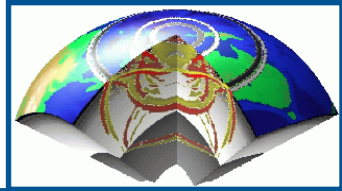
The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are: free surface conditions

$$\sigma_{zy}(0) = \mu \left. \frac{\partial u_y}{\partial z} \right|_0 = i k r_\beta \mu \left\{ A \exp[i(\omega t - kx)] - B \exp[i(\omega t - kx)] \right\} = 0$$

$$\sigma_{zy}(2h) = \mu \left. \frac{\partial u_y}{\partial z} \right|_{2h} = i k r_\beta \mu \left\{ A \exp[i(\omega t + k r_\beta 2h - kx)] - B \exp[i(\omega t - k r_\beta 2h - kx)] \right\} = 0$$

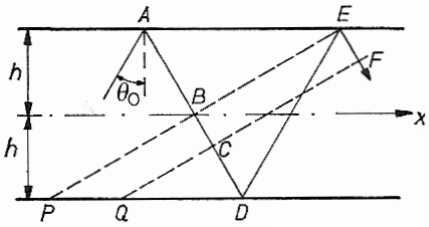


# SH waves: eigenvalues...



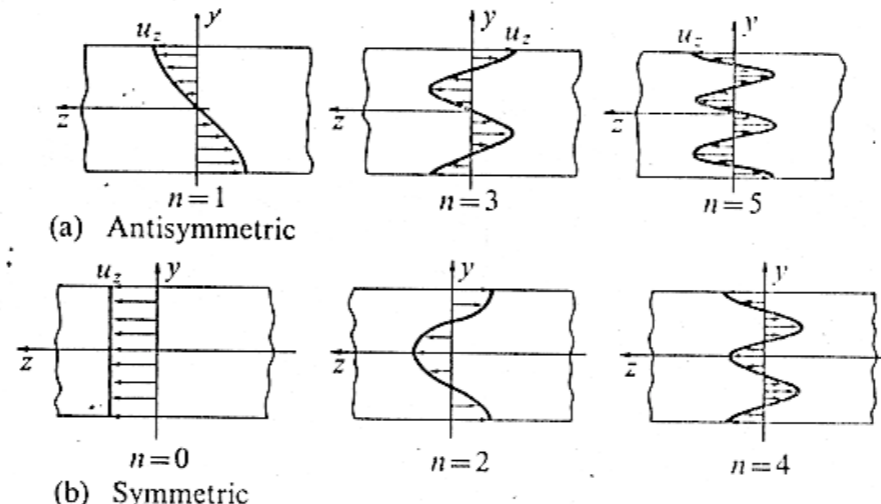
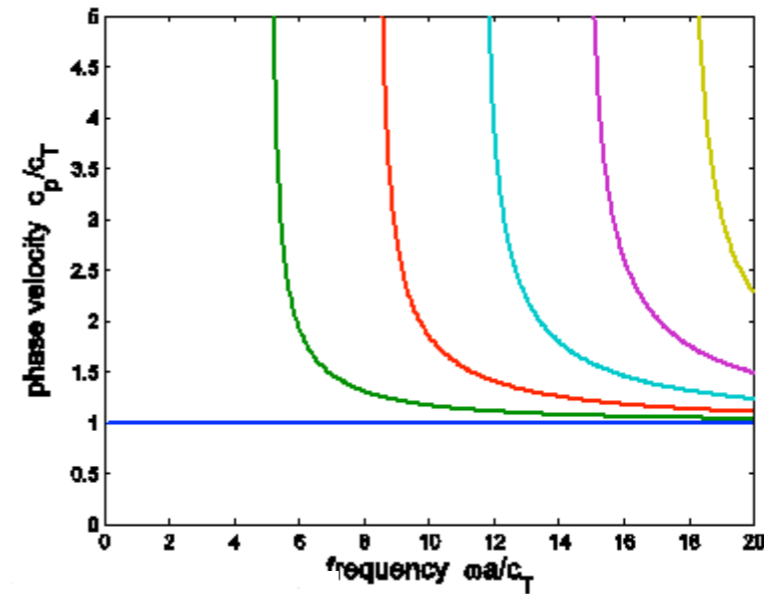
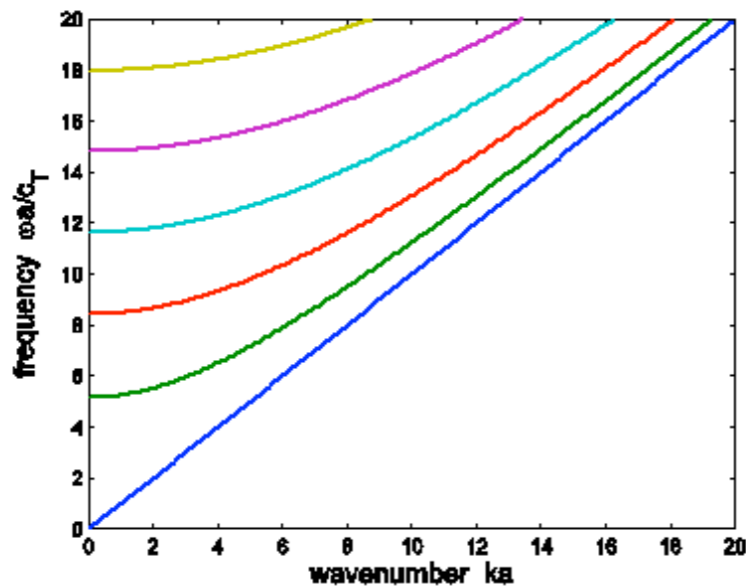
that leads to:  $k r_{\beta} 2h = n\pi$  with  $n=0,1,2,\dots$

NB: REMEMBER THE "STRING PROBLEM":  $kL=n\pi$

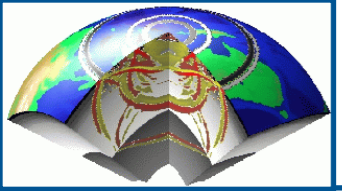


$$\omega^2 = k^2 \beta^2 + \left( \frac{n\pi\beta}{2h} \right)^2$$

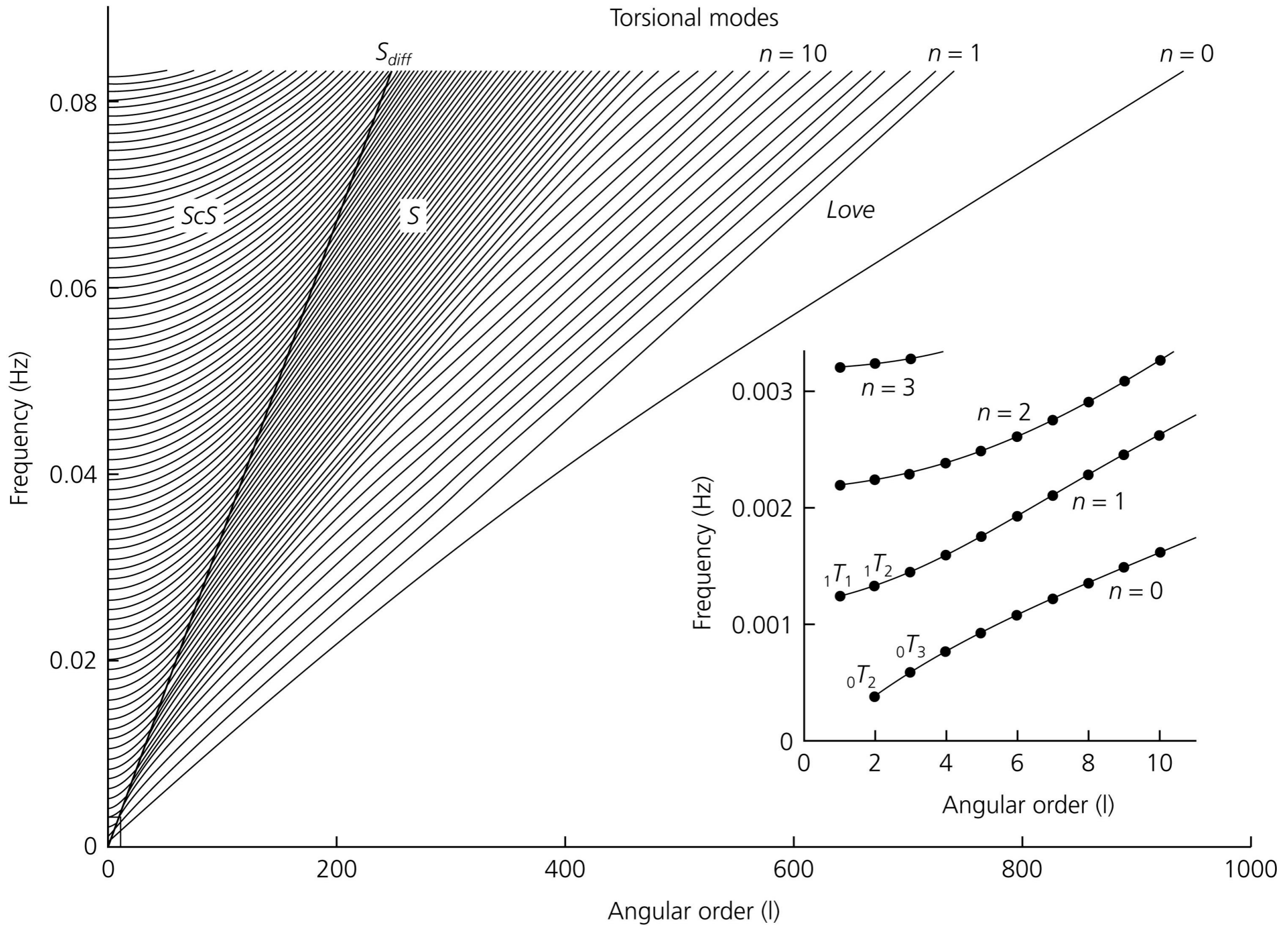
$$c = \frac{\beta}{\sqrt{1 - \left( \frac{n\pi\beta}{2h\omega} \right)^2}}$$





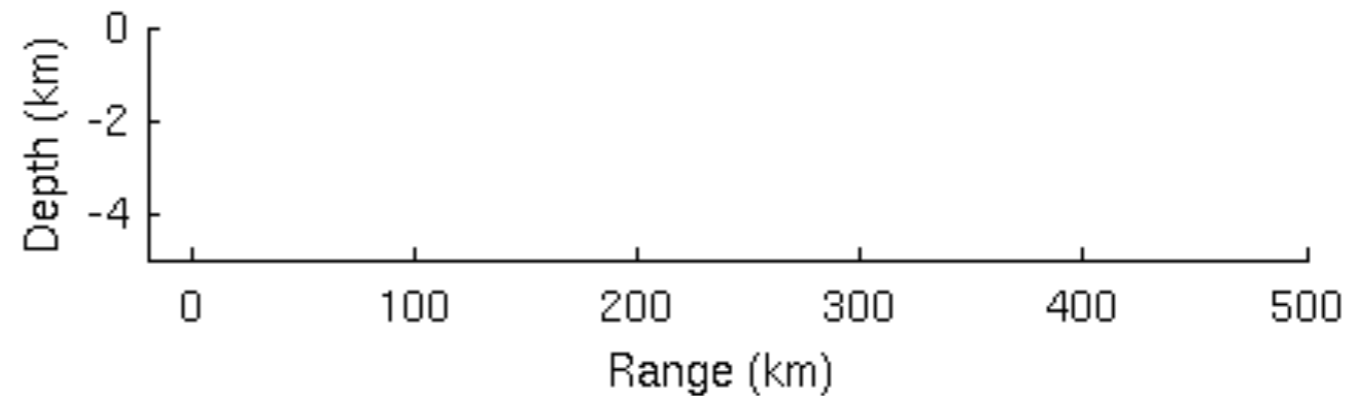
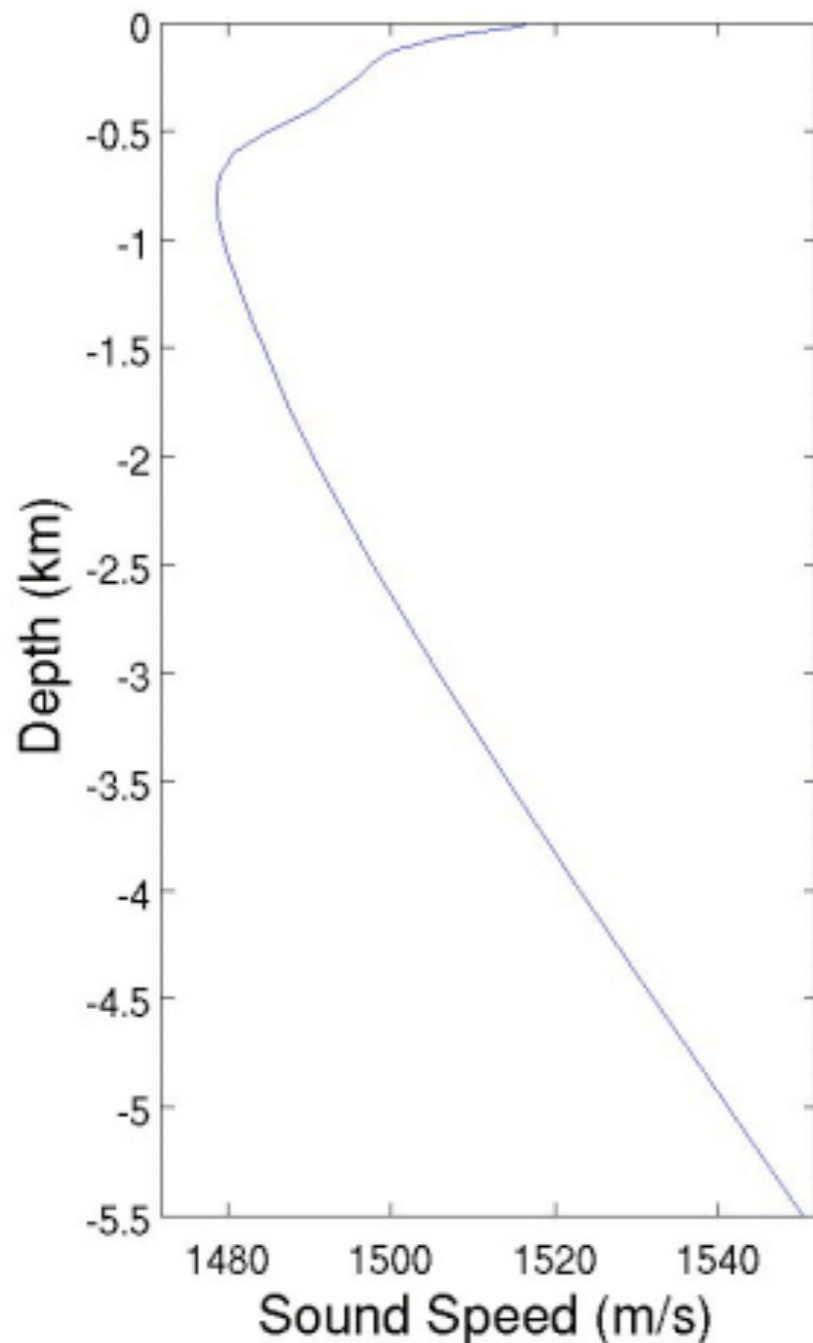


# Torsional modes dispersion



# Acoustic waveguides...

## SOFAR channel (Sound Fixing And Ranging channel)

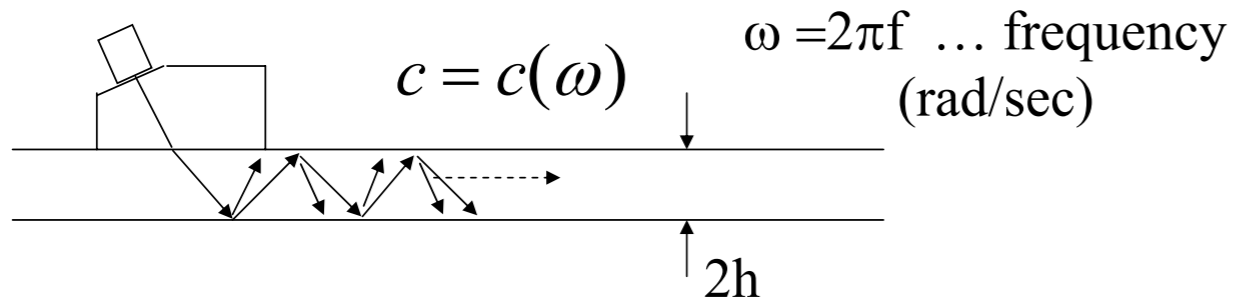


Sound speed as a function of depth at a position north of Hawaii in the [Pacific Ocean](#) derived from the 2005 [World Ocean Atlas](#). The SOFAR channel axis is at ca. 750-m depth.



# Waves in plates

In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.



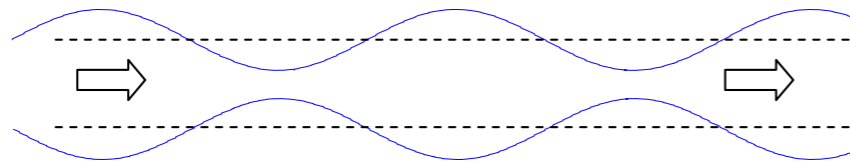
$$\phi = f(y) \exp[ik(x - ct)]$$

$$\psi = g(y) \exp[ik(x - ct)]$$

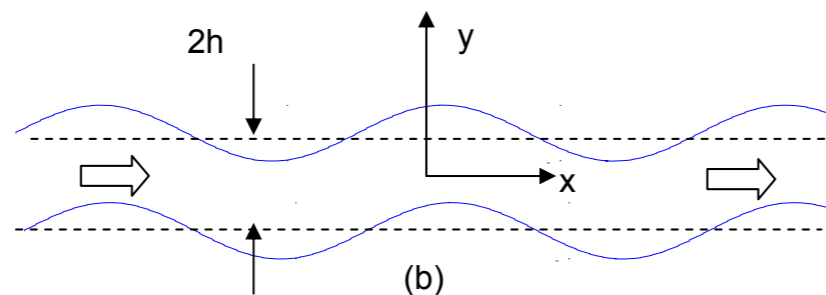
extensional waves

$$f = A \cosh(\alpha y)$$

$$g = B \sinh(\beta y)$$



(a)



(b)

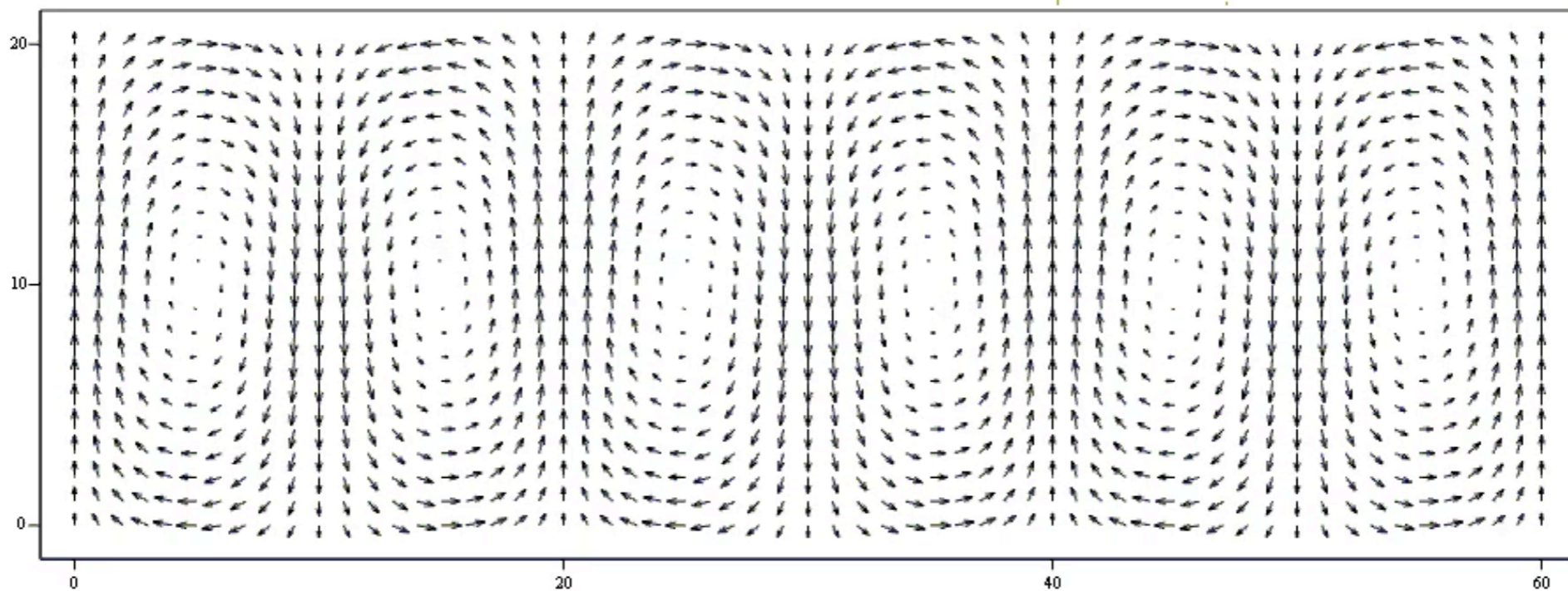
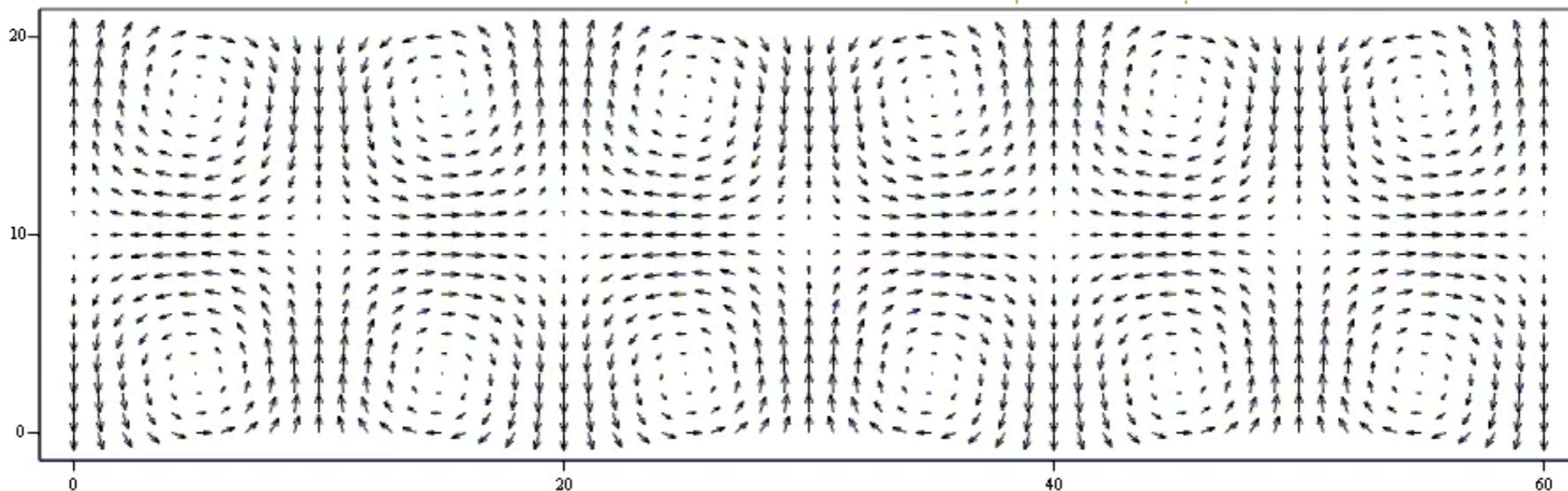
flexural waves

$$f = A' \sinh(\alpha y)$$

$$g = B' \cosh(\beta y)$$

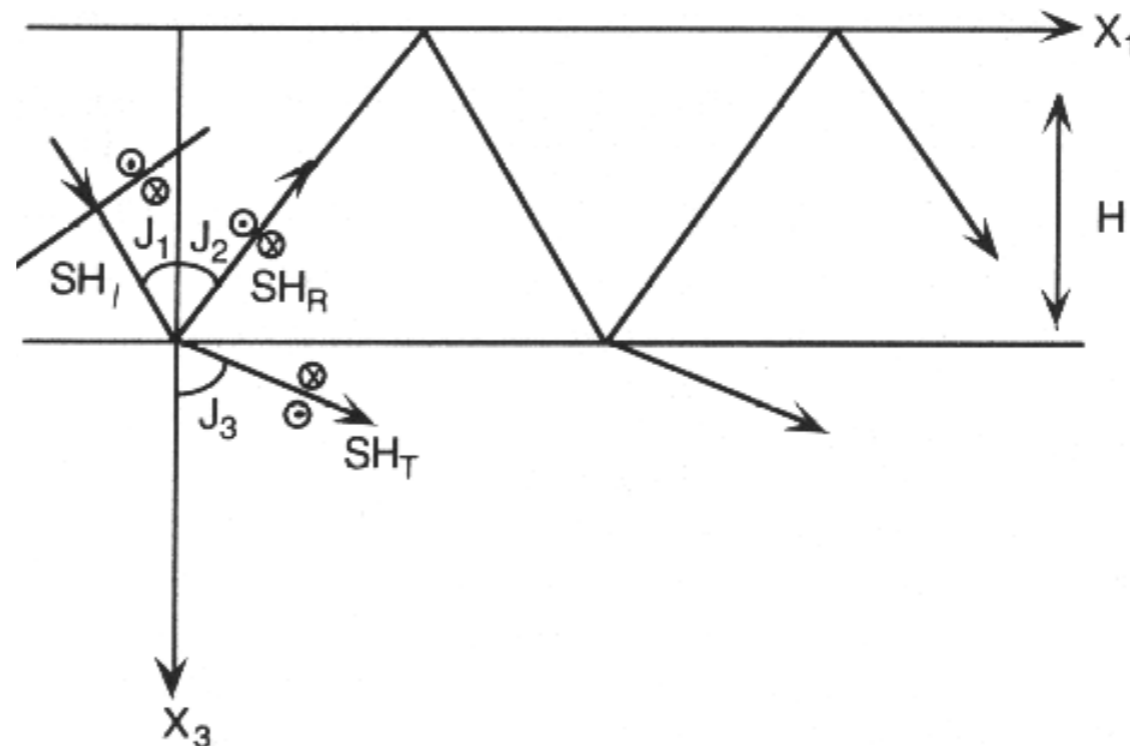
# Lamb waves

Lamb waves are waves of plane strain that occur in a free plate, and the traction force must vanish on the upper and lower surface of the plate. In a free plate, a line source along  $y$  axis and all wave vectors must lie in the  $x$ - $z$  plane. This requirement implies that response of the plate will be independent of the in-plane coordinate normal to the propagation direction.



# Love Waves: Geometry

In an elastic half-space no SH type surface waves exist. Why? Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer over a half space (Love, 1911) ?



## SOME PROBLEMS OF GEODYNAMICS

BEING AN ESSAY TO WHICH THE ADAMS PRIZE  
IN THE UNIVERSITY OF CAMBRIDGE  
WAS ADJUDGED IN 1911

BY

A. E. H. LOVE, M.A., D.Sc., F.R.S.

FORMERLY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE

HONORARY FELLOW OF QUEEN'S COLLEGE, OXFORD

SEDLERIAN PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF OXFORD

Cambridge:

at the University Press

1926

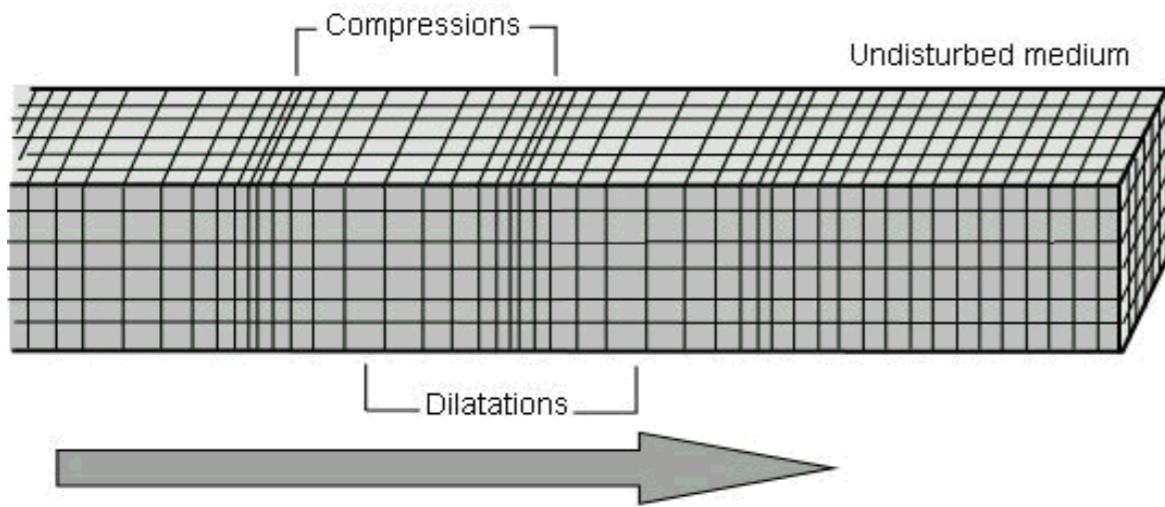
~

Repeated reflection in a layer over a half space.  
Interference between incident, reflected and transmitted SH waves.  
When the layer velocity is smaller than the halfspace velocity, then there is a critical angle beyond which SH reverberations will be totally trapped.

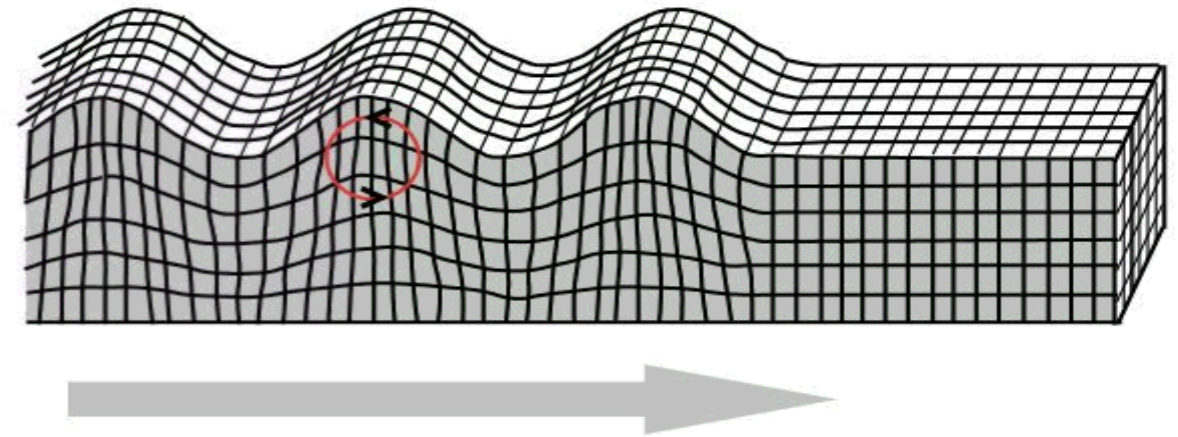


# Wavefields visualization

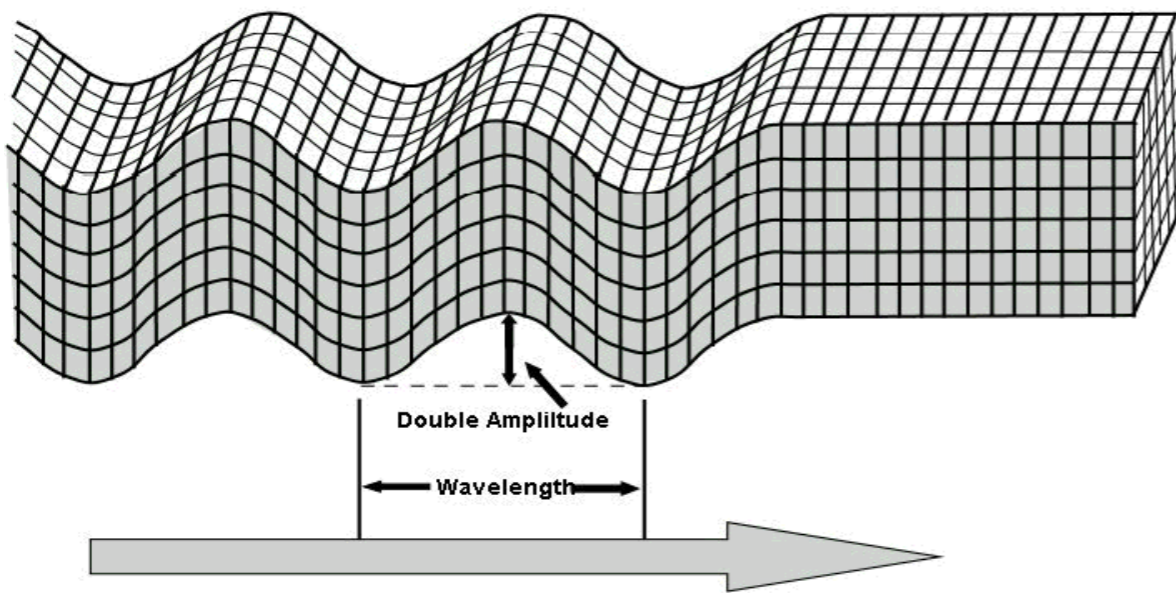
## P Wave



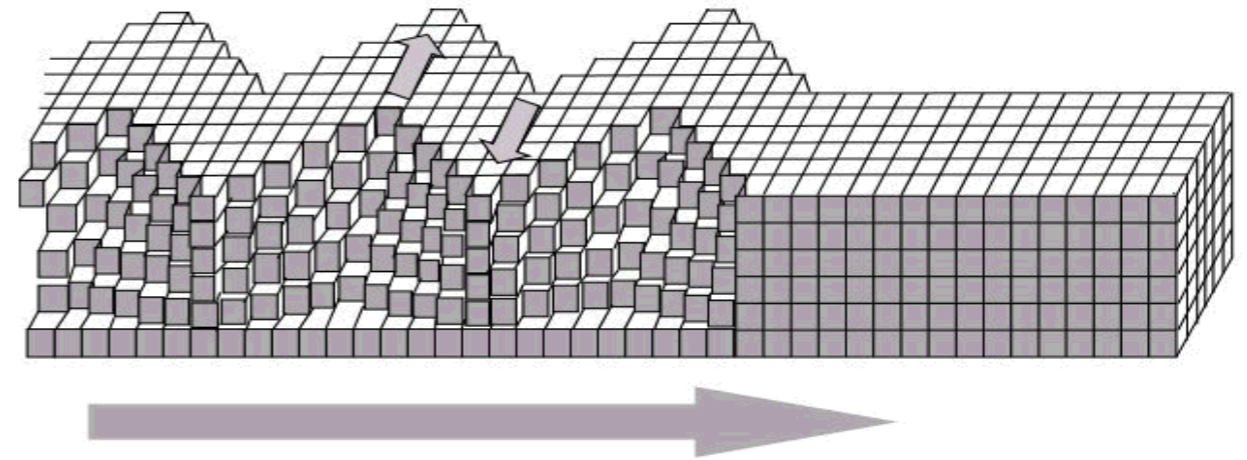
## Rayleigh Wave



## S Wave



## Love Wave



# Surface Wave Tomography

We describe a method to invert surface wave group or phase velocity measurements to estimate 2-D models of the distribution and strength of velocity variations.

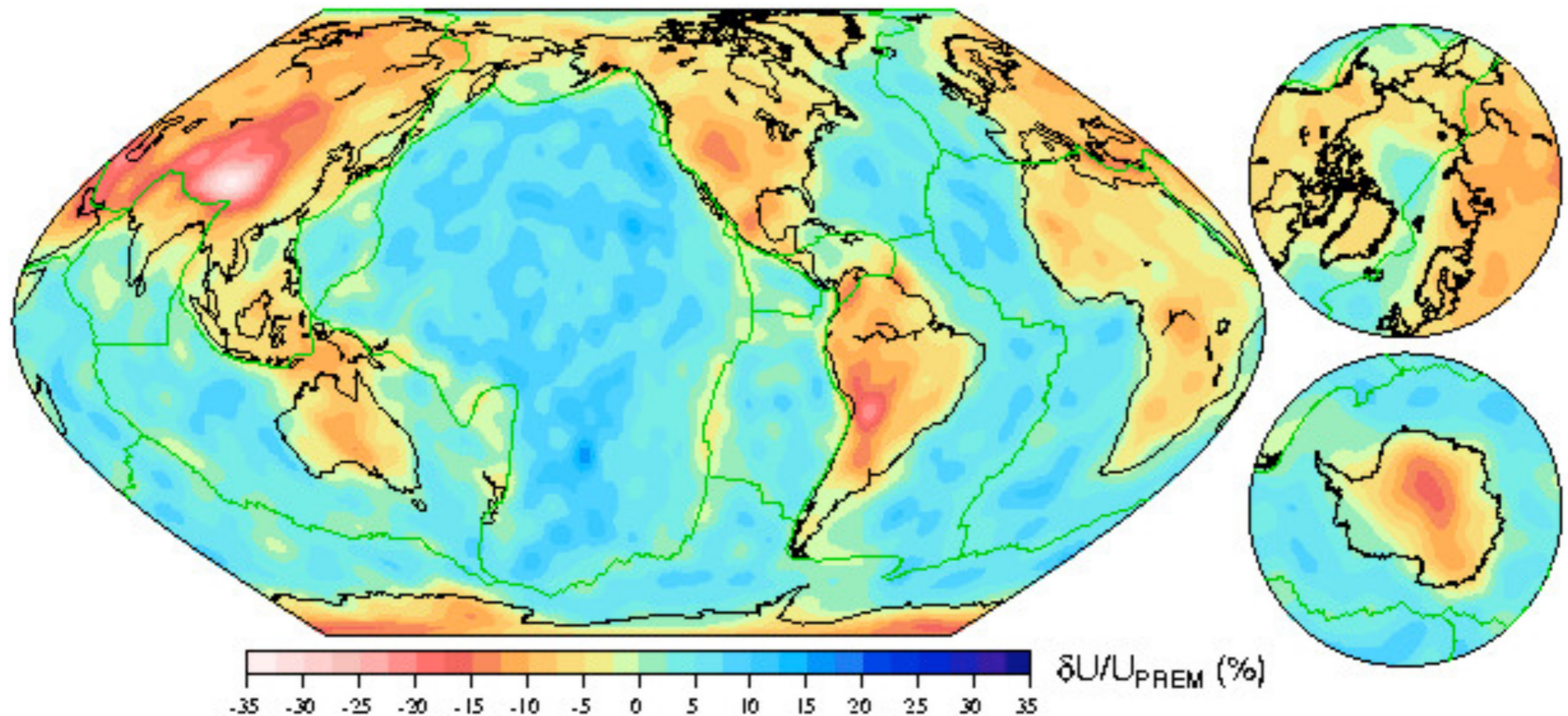
Using ray theory, the forward problem for surface wave tomography consists of predicting a frequency dependent travel time  $t_{R/L}(\omega)$ . For both Rayleigh (R) and Love (L) waves from a set of 2-D phase or group velocity maps,  $c(r, \omega)$ :

$$t_{R/L}(\omega) = \int_{\text{ray}} c_{R/L}^{-1}(r, \omega) ds$$

Where  $r=[\theta, \varphi]$  is the surface position vector,  $\theta$  and  $\varphi$  are colatitude and longitude, and ray specifies the path.



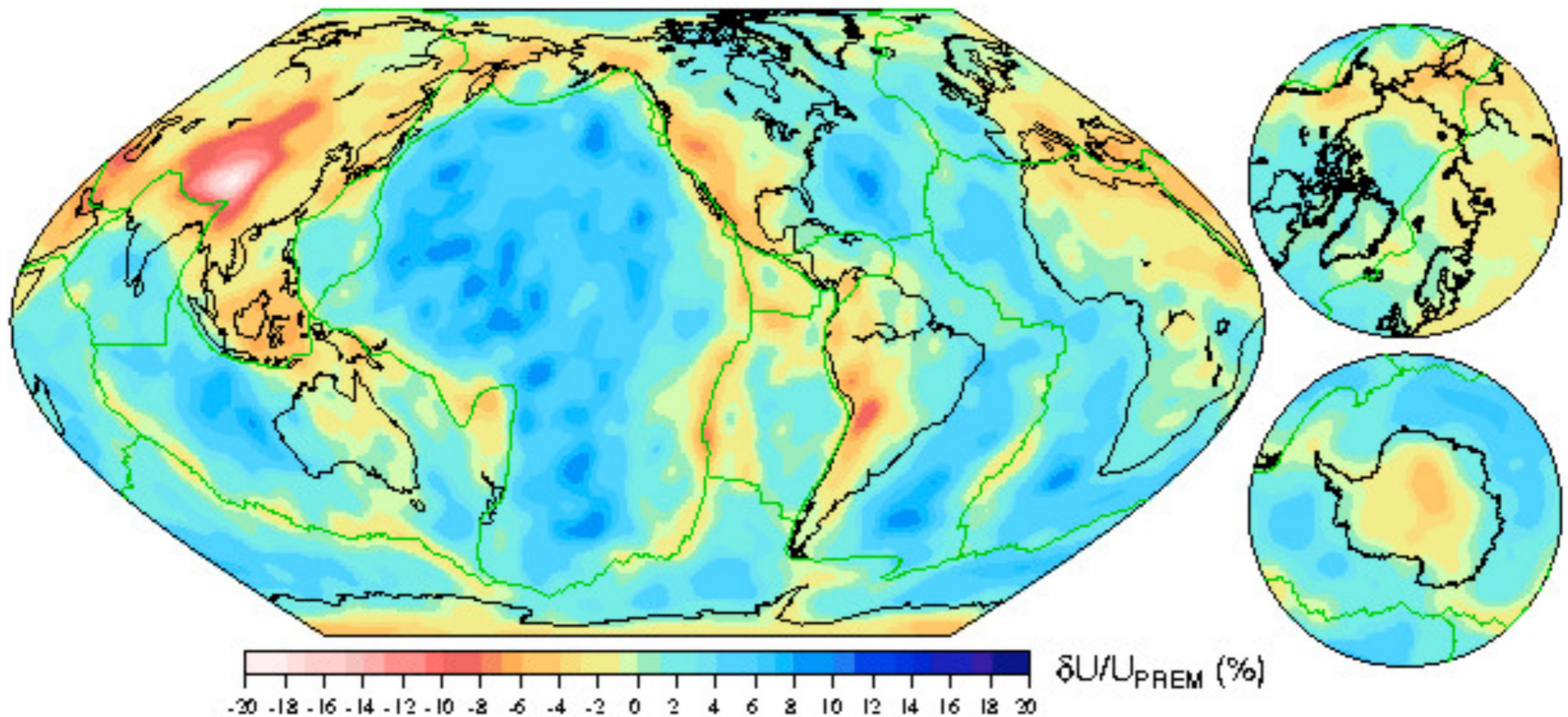
# Global scale



## RAYLEIGH WAVE 35s

Larson, E.W.F. and G. Ekström, *Global Models of Surface Wave Group Velocity*, *Pure Appl. Geophys.* **158** (8), 1377-1400, 2001.

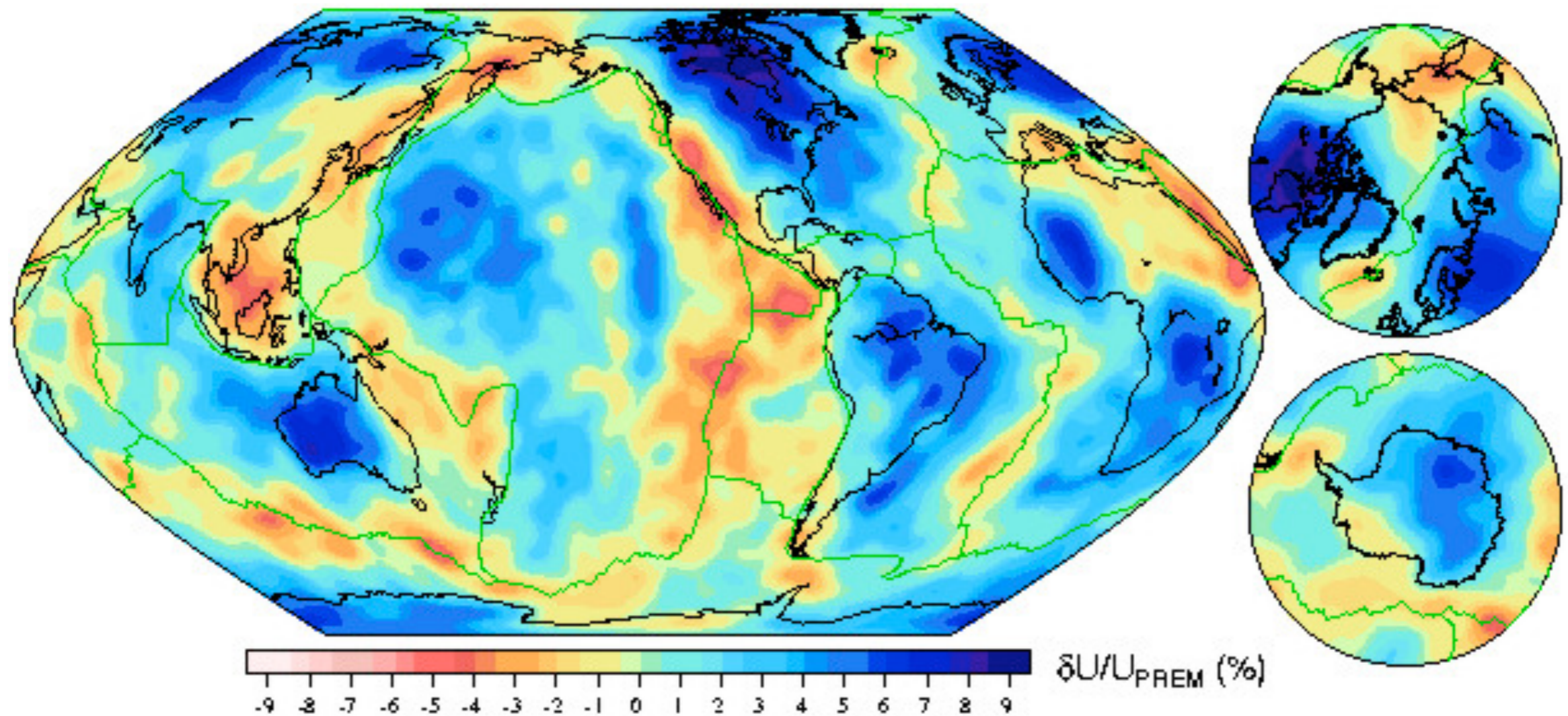




## RAYLEIGH WAVE 50s

Larson, E.W.F. and G. Ekström, *Global Models of Surface Wave Group Velocity*,  
*Pure Appl. Geophys.* **158** (8), 1377-1400, 2001.





## RAYLEIGH WAVE 100s

Larson, E.W.F. and G. Ekström, *Global Models of Surface Wave Group Velocity*,  
*Pure Appl. Geophys.* **158** (8), 1377-1400, 2001.





The reliability of the group velocity maps across large regions degrades sharply below 15 s and above 150-200 s for Rayleigh waves and 100-125 s for Love waves. Surface waves maps at and below 30 s period are particularly important because they provide significant constraints on crustal thickness by helping to resolve Moho depth from the average shear velocity of the crust. Although there have been numerous studies of surface wave dispersion that have produced measurements of group and/or phase velocities between 10 and 40 s period, these studies have typically been confined to areas of about 15° or less in lateral extent.

Phase and group velocity maps provide constraints on the shear velocity structure of the crust and uppermost mantle. Accurate high-resolution group velocity maps, in particular, are useful in monitoring clandestine nuclear tests.



Measurements of group velocities are much less sensitive to source effects than phase velocities because they derive from measurements of the wave packet envelopes rather than the constituent phases. This is particularly true at shorter periods and longer ranges. Group velocity sensitivity is compressed nearer to the surface than the related phase velocities, which should provide further help in resolving crustal from mantle structures.



# Surface waves

- Condition of existence:
  - Discontinuity (boundary waves, undispersed: Rayleigh, Stoneley)
  - Waveguide (interferential & dispersed: Love & Rayleigh)

T (s)	f (Hz)	$\lambda$ (km)	c (km/s)	d (km)	application
0.02-0.1	10-50	0.002-0.05	0.1-0.5	0.02	engineering, geophysics
0.2-1	1-5	0.15-1.50	0.1-1.5	0.2	upper sediments
5-10	0.1-0.2	7-30	2-3	5	sedimentary basins
10-35	0.03-0.1	30-100	3.0-3.5	40	crust
35-350	0.005-0.03	200-1000	4-5	300	upper mantle