# Physics 110A
# Helmholtz's theorem for vector functions

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This handout is a modification of Appendix B in Griffiths.

## I. THEOREM 1

Suppose we have a vector function of position $\mathbf{F}(\mathbf{r})$. Then we state without proof that, *quite generally*, $\mathbf{F}$ can be written as a gradient plus a curl, i.e.

$$\boxed{\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W}\,,} \tag{1}$$

where $U$ is a scalar function and $\mathbf{W}$ is a vector function (the minus sign in front of $\boldsymbol{\nabla} U$ is a convention).

Now this decomposition into a gradient and a curl is *not necessarily unique*. For example, suppose that $\mathbf{F}$ is a constant, e.g. $\mathbf{F} = k\,\hat{\mathbf{z}}$. Then Eq. (1) can be satisfied in many ways. Two simple choices are:

$$U = -z\,k, \qquad \mathbf{W} = 0, \tag{2}$$
$$U = 0, \qquad \mathbf{W} = x\,k\,\hat{\mathbf{y}}. \tag{3}$$

Note that, in this example, $\boldsymbol{\nabla} \cdot \mathbf{F}$ and $\boldsymbol{\nabla} \times \mathbf{F}$ are both zero. This vector function $\mathbf{F}$ is just a constant, but one can cook up less trivial examples of functions with zero divergence and curl, e.g. $\mathbf{F} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$, $\mathbf{F} = \sin x \cosh y\,\hat{\mathbf{x}} - \cos x \sinh y\,\hat{\mathbf{y}}$. Note, however, that all these functions *do not vanish at infinity*. A very important theorem, derived after Eq. (17) below, states that

the only vector function with zero divergence and curl which vanishes at infinity is zero everywhere.

In its general form, Eq. (1) will not be used in this course, and is given only for completeness. A much more useful, related, theorem for functions $\mathbf{F}$ which vanish at infinity (and which also satisfy some other conditions) will be given in the next section.

## II. THEOREM 2, HELMHOLTZ'S THEOREM

Suppose we don't know a vector function $\mathbf{F}(\mathbf{r})$, but we do know its divergence and curl, i.e.

$$\boldsymbol{\nabla} \cdot \mathbf{F} = D, \tag{4a}$$
$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{C}, \tag{4b}$$

where $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are specified scalar and vector functions. Since the divergence of a curl is always zero, $\mathbf{C}$ must be divergenceless,

$$\boldsymbol{\nabla} \cdot \mathbf{C} = 0\,. \tag{5}$$

We would like to know if Eqs. (4) provide enough information to determine $\mathbf{F}$ *uniquely*.

Helmholtz's theorem says the answer is "yes" provided $D(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ vanish fast enough as $r \to \infty$ (it turns out faster than $1/r^2$) and we interested in the solution for $\mathbf{F}$ which vanishes as $r \to \infty$. Helmholtz's theorem also tells us how to calculate $\mathbf{F}$ in terms of $D$ and $\mathbf{C}$. The result is that[1]

$$\boxed{\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W}\,,} \tag{6}$$

where

$$\boxed{U(\mathbf{r}) = \frac{1}{4\pi}\int \frac{D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau',} \tag{7}$$

and

$$\boxed{\mathbf{W}(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau',} \tag{8}$$

in which the integrals are over all space.

Let's now try to understand this theorem. Taking the divergence of Eq. (6), and noting that the divergence of a curl is always zero, we get

$$\boldsymbol{\nabla}\cdot\mathbf{F} = -\nabla^2 U\,. \tag{9}$$

Acting with $\nabla^2$ on Eq. (7) gives

$$\boldsymbol{\nabla}\cdot\mathbf{F} = -\nabla^2 U = -\frac{1}{4\pi}\int D(\mathbf{r}')\,\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\,d\tau' = \int D(\mathbf{r}')\,\delta^{(3)}(\mathbf{r}-\mathbf{r}')\,d\tau' = D(\mathbf{r})\,, \tag{10}$$

in agreement with Eq. (4a). We used that the derivative is respect to $\mathbf{r}$ and that

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}')\,. \tag{11}$$

We have seen that the divergence is right, so what about the curl? Taking the curl of Eq. (6), and noting that the curl of a divergence is always zero, we get

$$\boldsymbol{\nabla}\times\mathbf{F} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{W}) = -\nabla^2\mathbf{W} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{W})\,. \tag{12}$$

Now

$$-\nabla^2\mathbf{W} = -\frac{1}{4\pi}\int \mathbf{C}(\mathbf{r}')\,\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\,d\tau' = \int \mathbf{C}(\mathbf{r}')\,\delta^{(3)}(\mathbf{r}-\mathbf{r}')\,d\tau' = \mathbf{C}(\mathbf{r})\,, \tag{13}$$

which agrees with Eq. (4b). However, we have to show that the second term on the right hand side of Eq. (12) is zero. This is true because $\boldsymbol{\nabla}\cdot\mathbf{W} = 0$ since

$$\begin{aligned} 4\pi\nabla\cdot\mathbf{W} &= \int \mathbf{C}(\mathbf{r}')\cdot\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\,d\tau' = -\int \mathbf{C}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\,d\tau' \\ &= \int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\boldsymbol{\nabla}'\cdot\mathbf{C}(\mathbf{r}')\,d\tau' - \oint \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{C}(\mathbf{r}')\cdot d\mathbf{a}'\,, \end{aligned} \tag{14}$$

[1] Of course Eq. (6) is the same as Eq. (1). The difference here is that the decomposition into a gradient and curl is *unique* and explicit expressions for $U$ and $\mathbf{W}$ are given in Eqs. (7) and (8).

where we used that

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{15}$$

(the derivative on the LHS is with respect to the components of $\mathbf{r}$ and on the RHS to the components of $\mathbf{r}'$) and integrated by parts to get the last line. The first term on the right hand side of Eq. (14) vanishes because $\boldsymbol{\nabla}\cdot\mathbf{C} = 0$, see Eq. (5), and the second term, a surface integral, vanishes if $\mathbf{C}$ tends to zero fast enough at infinity. Hence

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{W} = 0,} \tag{16}$$

and so, from Eqs. (12) and (13) we recover Eq. (4b).

Hence the divergence and curl of $\mathbf{F}$ are correct, but is the solution in Eq. (6) *unique*? In general, the answer is *no* because we can always add any vector function whose curl and divergence are both zero. However, if we require that $\mathbf{F}$ vanishes at infinity (which is usually the situation with electric and magnetic fields[2]), then we show in the next paragraph that the *only* function with zero divergence and curl is the trivial one whose value is zero everywhere, and hence $\mathbf{F}$ is uniquely given by Eq. (6).

To prove that the only vector function, $\mathbf{K}(\mathbf{r})$ say, with zero divergence and curl which vanishes at infinity is $\mathbf{K}(\mathbf{r}) = 0$ everywhere, note first that a quantity with zero curl can be written as a gradient, i.e. $\mathbf{K} = \boldsymbol{\nabla}\psi$, and the constraint that its divergence is zero means that $\psi$ satisfies Laplace's equation,

$$\nabla^2\psi = 0. \tag{17}$$

However, there is a uniqueness theorem for Laplace's equation (Boas Ch. 6) which states that the solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined up to an additive constant if the normal derivative of $\psi$ is specified on the boundary. Here the boundary is at infinity (which we can represent by a sphere of radius $R$ for $R \to \infty$). We have specified that $\mathbf{K}$ vanishes on the boundary, so, in particular, its radial (i.e. normal) component $K_r \equiv \partial\psi/\partial r$ is zero there. By inspection, a solution of Eq. (17) inside a sphere of radius $R$ which has boundary condition $\partial\psi/\partial r = 0$ at $r = R$ is $\psi =$ constant. Because of the uniqueness theorem, this is the *unique* solution. It follows that $\mathbf{K} \equiv \boldsymbol{\nabla}\psi = 0$ *everywhere*. This proves, as also stated in Sec. I, that

| the only vector function with zero divergence and curl which vanishes at infinity is zero everywhere. |
|---|

Note, it is straightforward to show on dimensional grounds (Griffiths, Appendix B) that $D(\mathbf{r}')$ and $\mathbf{C}(\mathbf{r}')$ need to vanish faster than $1/r'^2$ as $r' \to \infty$ in order that the integrals in Eqs. (7) and (8) converge. This is then is more than sufficient for the surface integral in Eq. (14) to vanish.

We have shown that $\mathbf{F}$ is uniquely given by Eq. (6). We now ask whether the "potentials" $U$ and $\mathbf{W}$ themselves are *uniquely* given by Eqs (7) and (8) or whether we could *change* $U$ and $\mathbf{W}$ while keeping $\boldsymbol{\nabla}U$ and $\boldsymbol{\nabla}\times\mathbf{W}$ unchanged. Since me know that $\boldsymbol{\nabla}U$ is uniquely determined, the only change we can make to the "scalar potential" $U$ is to add a constant. If we also specify that $U$ vanishes at infinity then the constant is zero so $U$ *is uniquely determined* by Eq. (7).

Similarly $\boldsymbol{\nabla}\times\mathbf{W}$ is uniquely determined, but one can still add to $\mathbf{W}$ the gradient of *any* scalar function. This invariance of the "field" $\mathbf{F}$ while the "vector potential" $\mathbf{W}$ changes by the

[2] There are some artificial textbook problems in which the charge or current distribution tends to infinity, such as an infinite wire or an infinite sheet, for which this is not true. Other means must then be found to determine the solution.

gradient of a scalar function is called "gauge invariance". It is common to choose a gauge such that $\boldsymbol{\nabla} \cdot \mathbf{W} = 0$. As shown in Griffiths, Sec. 5.4.1, this can always be done. Indeed, our expression for $\mathbf{W}$ in Eq. (8) is in this gauge, as shown by Eq. (16). We therefore ask whether we can change $\mathbf{W}$ while maintaining its gauge (zero divergence) and the specified value of the curl. In other words, can we add a change to $\mathbf{W}$ which has zero divergence and zero curl? As discussed below Eq. (17), if we require that $\mathbf{W}$ vanishes at infinity, the only such function is zero everywhere, in which case $\mathbf{W}$ *is unique*.

Hence we can now state precisely Helmhotz's theorem:

> If the divergence $D(\mathbf{r})$ and curl $\mathbf{C}(\mathbf{r})$ of a vector function $\mathbf{F}(\mathbf{r})$ are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if $\mathbf{F}(\mathbf{r})$ itself tends to zero as $r \to \infty$, then $\mathbf{F}(\mathbf{r})$ is *uniquely* given by Eq. (6), in which the "scalar potential" $U(\mathbf{r})$ is given by Eq. (7) and the "vector potential" $\mathbf{W}(\mathbf{r})$ by Eq. (8).
> Furthermore, the potentials themselves are unique if they vanish as $r \to \infty$ and $\mathbf{W}$ is in a gauge where $\boldsymbol{\nabla} \cdot \mathbf{W} = 0$.

Now we give two important examples of Helmholtz's theorem, one in electrostatics and the other in magnetostatics.

### A. Example in electrostatics

We have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{18a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{18b}$$

where $\mathbf{E}$ is the electric field and $\rho$ the charge density. Hence $\boxed{\mathbf{E} = -\boldsymbol{\nabla} V,}$ where the scalar potential $V$ is given by the familiar expression

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau',} \tag{19}$$

as discussed in the course.

### B. Example in magnetistatics

We have

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{20a}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{20b}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ the current density. Hence $\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A},}$ where the vector potential $\mathbf{A}$ is given, in the gauge with $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, by

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d\tau',} \tag{21}$$

which will also be discussed in the course.