SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

LINEAR SYSTEMS

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Linear Systems

$$\begin{array}{l} a x_1(t) \longrightarrow \\ b x_2(t) \longrightarrow \end{array} \boxed{\text{Linear System}} \begin{array}{l} \longrightarrow a y_1(t) \\ \longrightarrow b y_2(t) \end{array} \Bigg\} = a y_1(t) + b y_2(t)$$

$$\delta(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow \boxed{h(t) \quad \text{Impulse Response}}$$

(remember GF definition)

Since any input x(t) can be written as:

$$x(t) = \int x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \int x(\tau)h(t-\tau)\,d\tau = x * h$$

$$e^{i\omega t} \longrightarrow \boxed{\text{Linear System}} \longrightarrow \boxed{H(\omega)e^{i\omega t} \quad \text{Transfer Function}}$$

$$\int e^{i\omega\tau}h(t-\tau)\,d\tau = \int e^{i\omega(t-\tau)}h(\tau)\,d\tau = e^{i\omega t}\int e^{i\omega\tau}h(\tau)\,d\tau$$

$$X(\omega) = \int x(\tau)e^{i\omega t}\,dt \longrightarrow X(\omega)\cdot H(\omega)$$

Poles and Zeros

$$x(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t)$$

Differential Equation:

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \ldots + b_1 \frac{dx}{dt} + b_0 y$$

$$e^{i\omega t} \longrightarrow H(\omega) e^{i\omega t}$$

$$H(\omega) = \frac{\sum_{j=0}^{m} b_j (i\omega)^j}{\sum_{k=0}^{n} a_k (i\omega)^k}$$

Zeroes

Poles

Transfer function of the seismometer:

$$H(\omega) = \frac{Y(\omega)}{U(\omega)} = \frac{-(i\omega)^2}{(i\omega)^2 + 2\gamma(i\omega) + (\omega_0)^2}$$

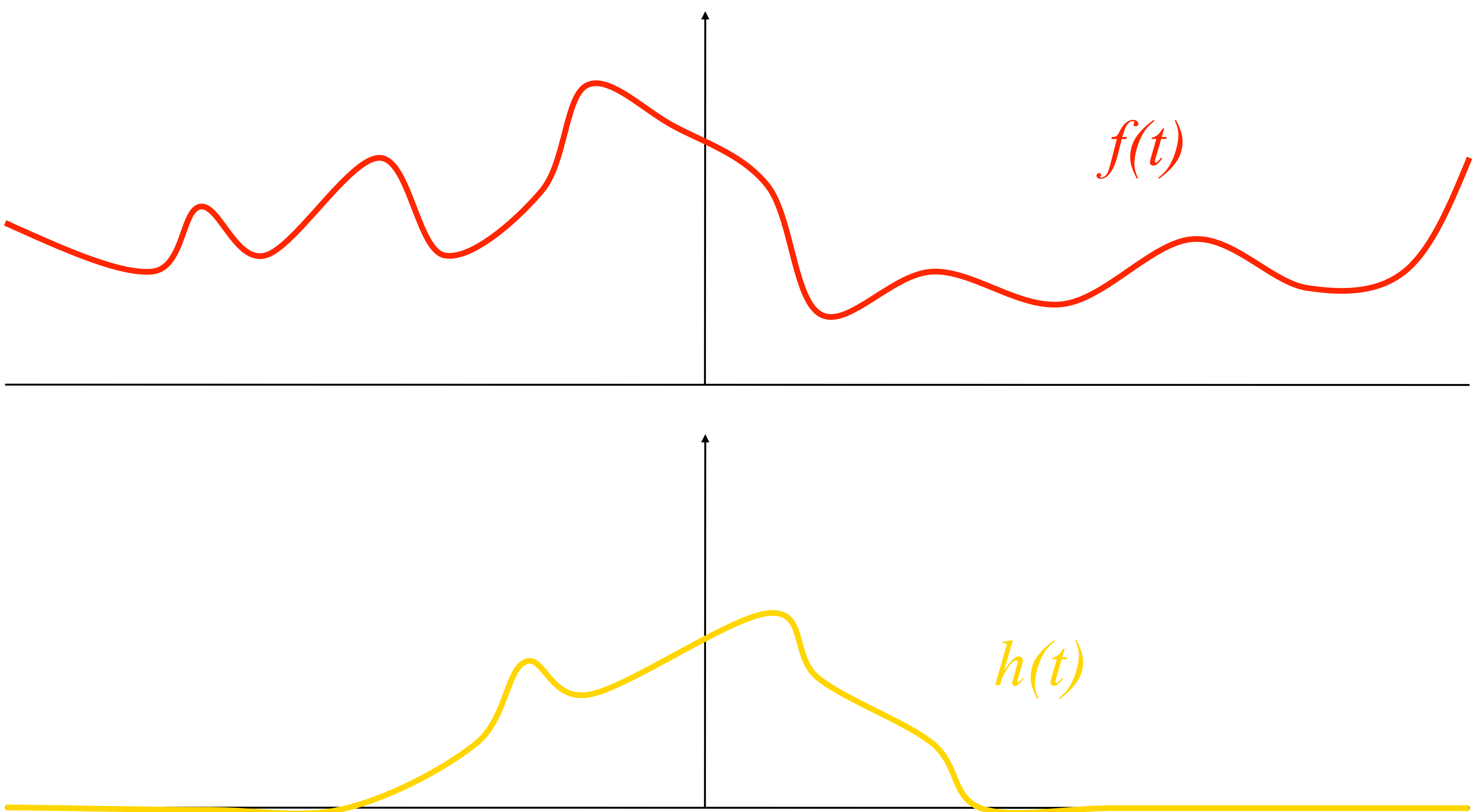
Convolution

- Definition:

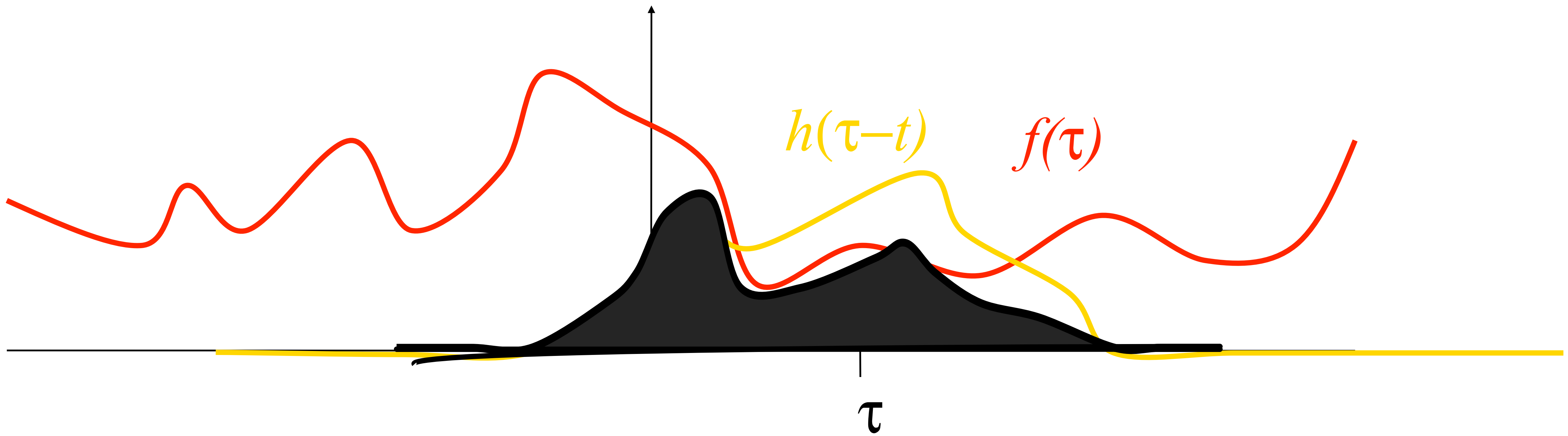
$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Convolution

- Definition: $$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$
- Pictorially



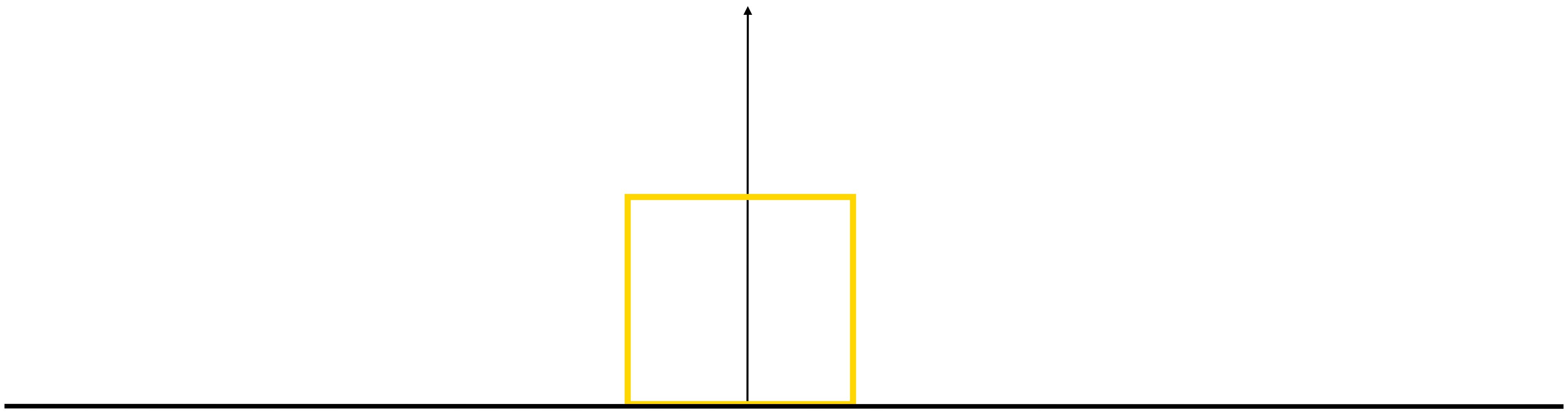
Convolution



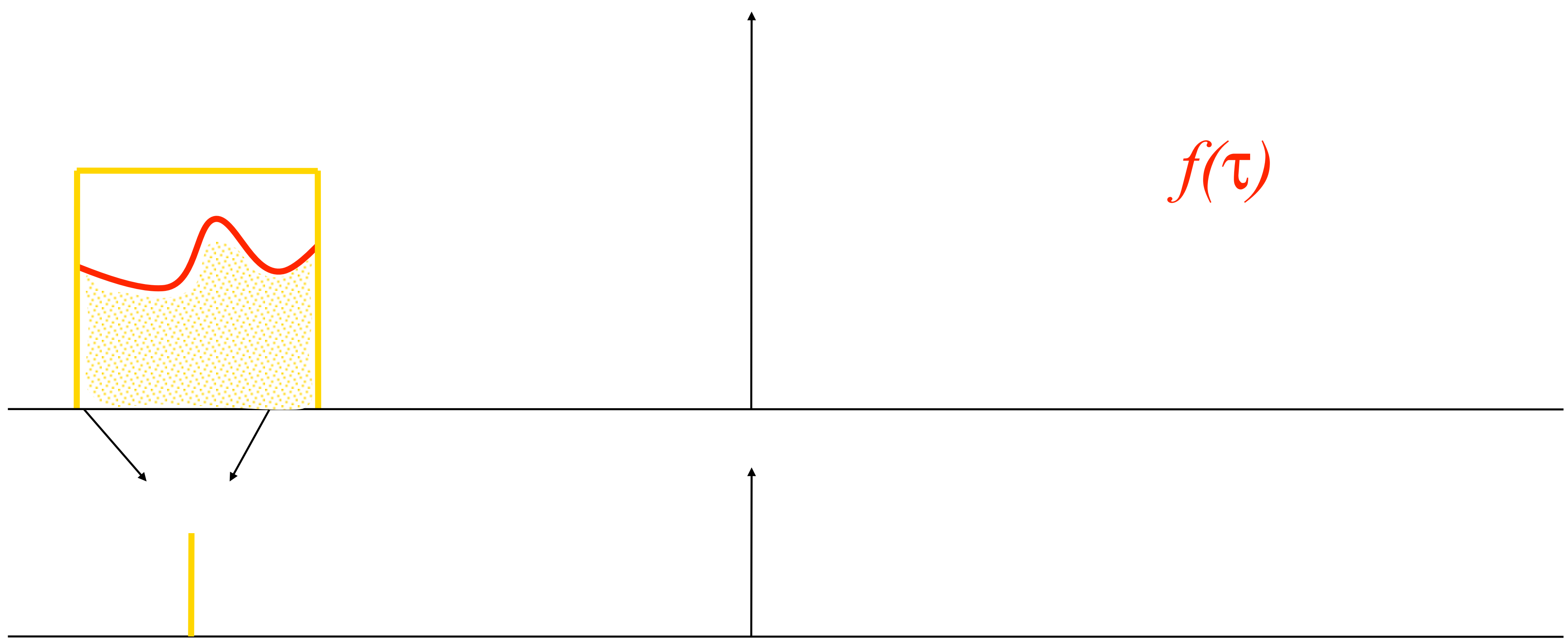
Convolution

- Consider the boxcar function (box filter):

$$h(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$



- This function windows our function f(t)



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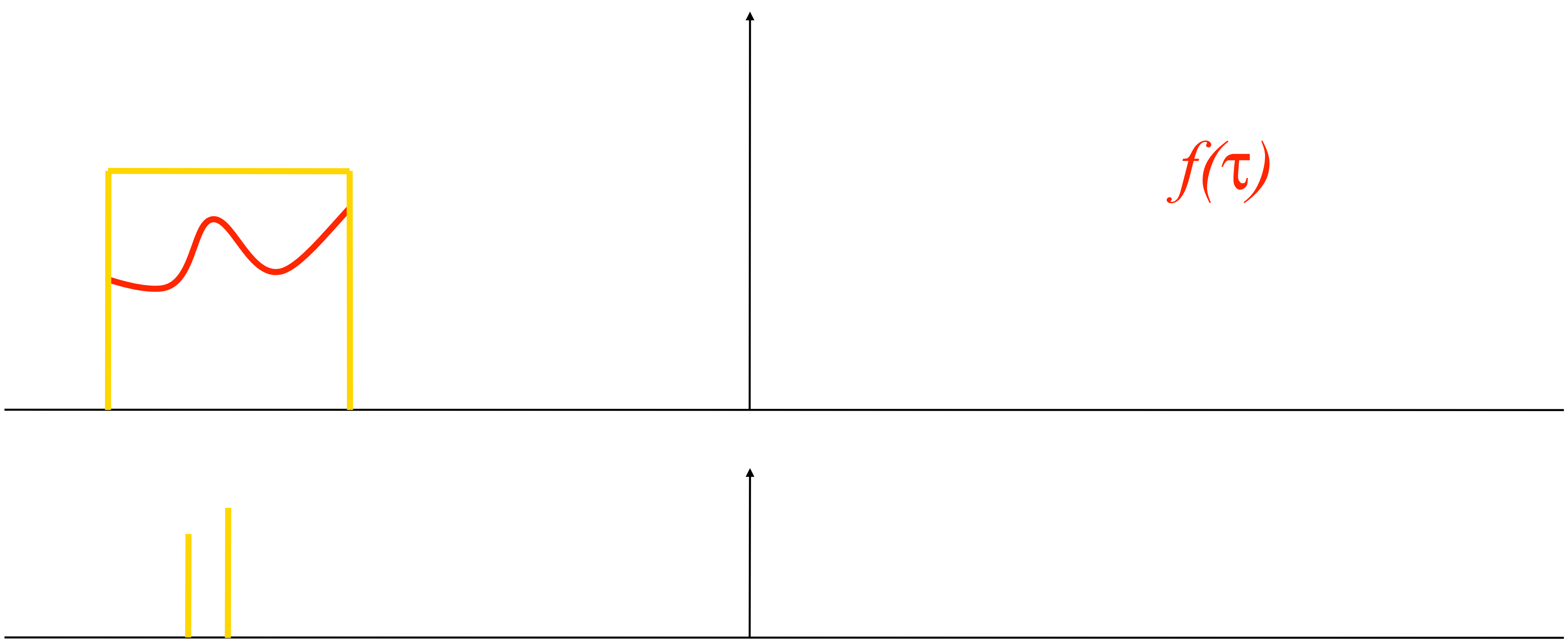

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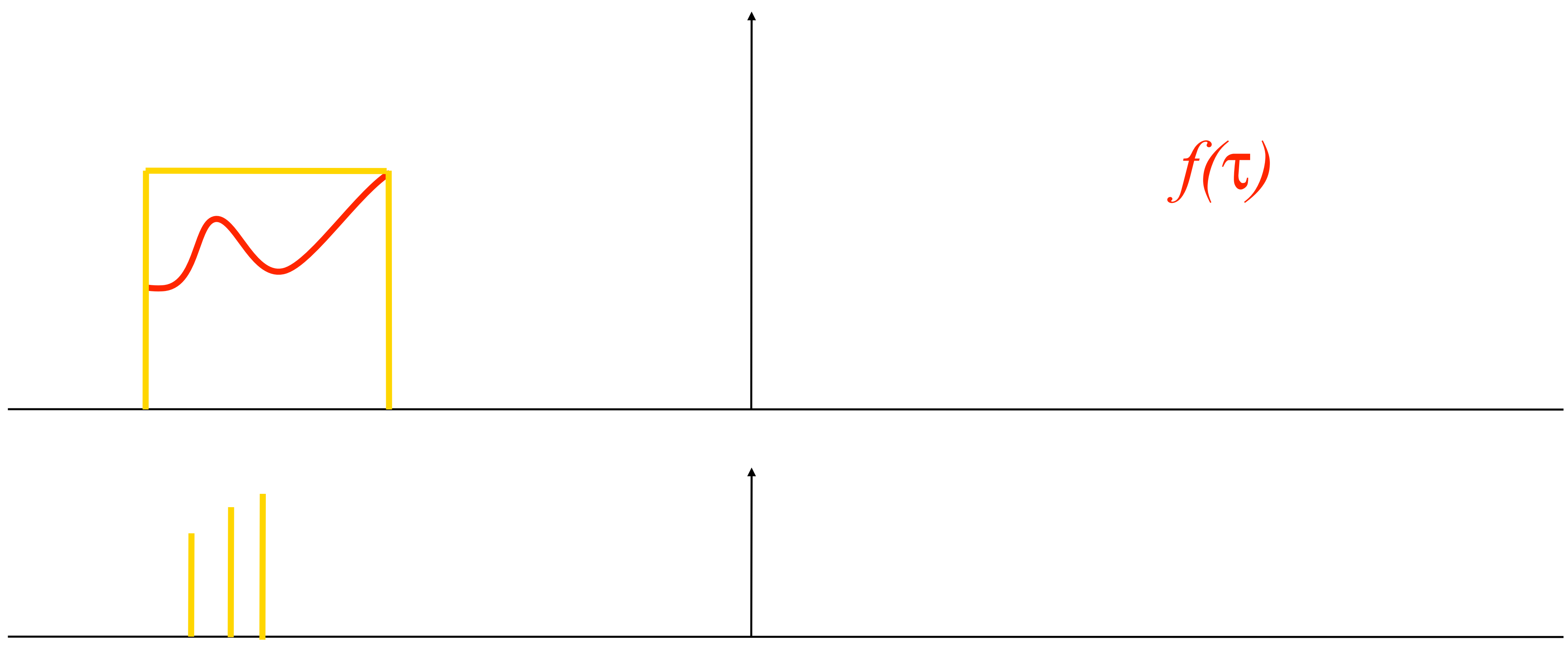

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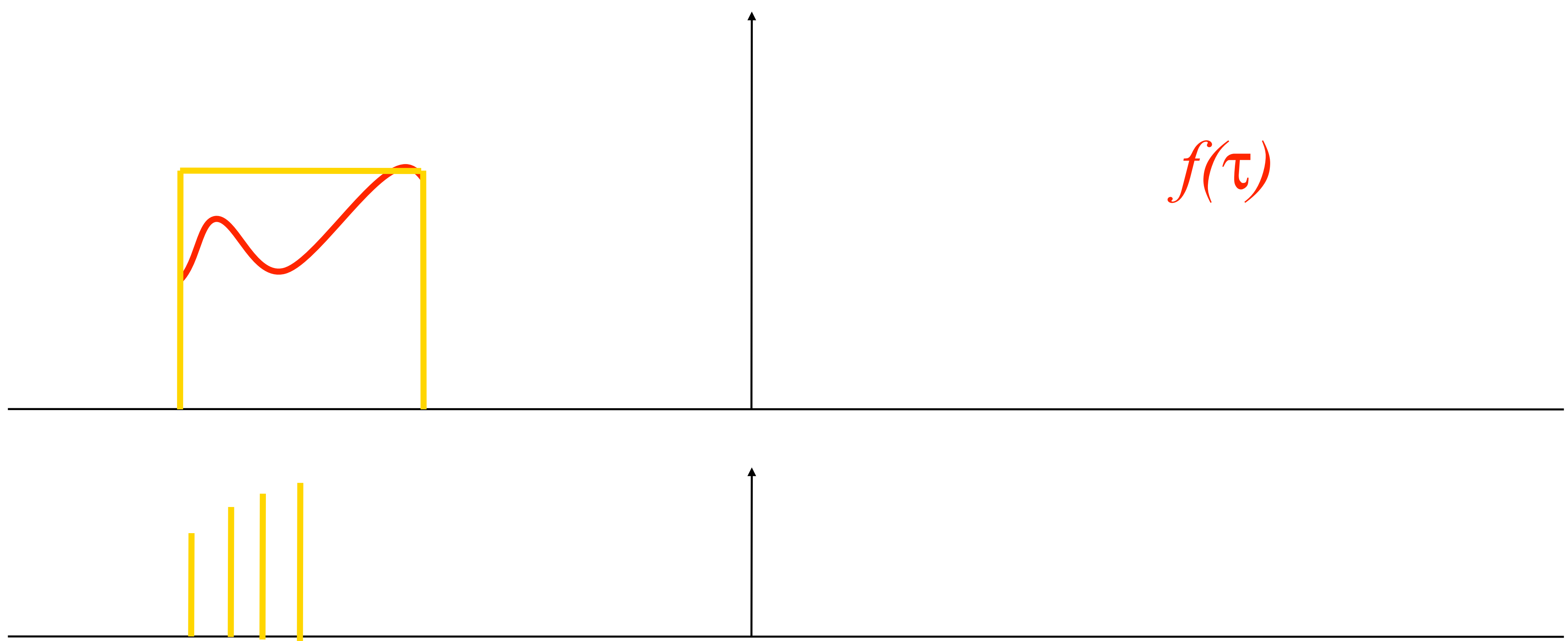

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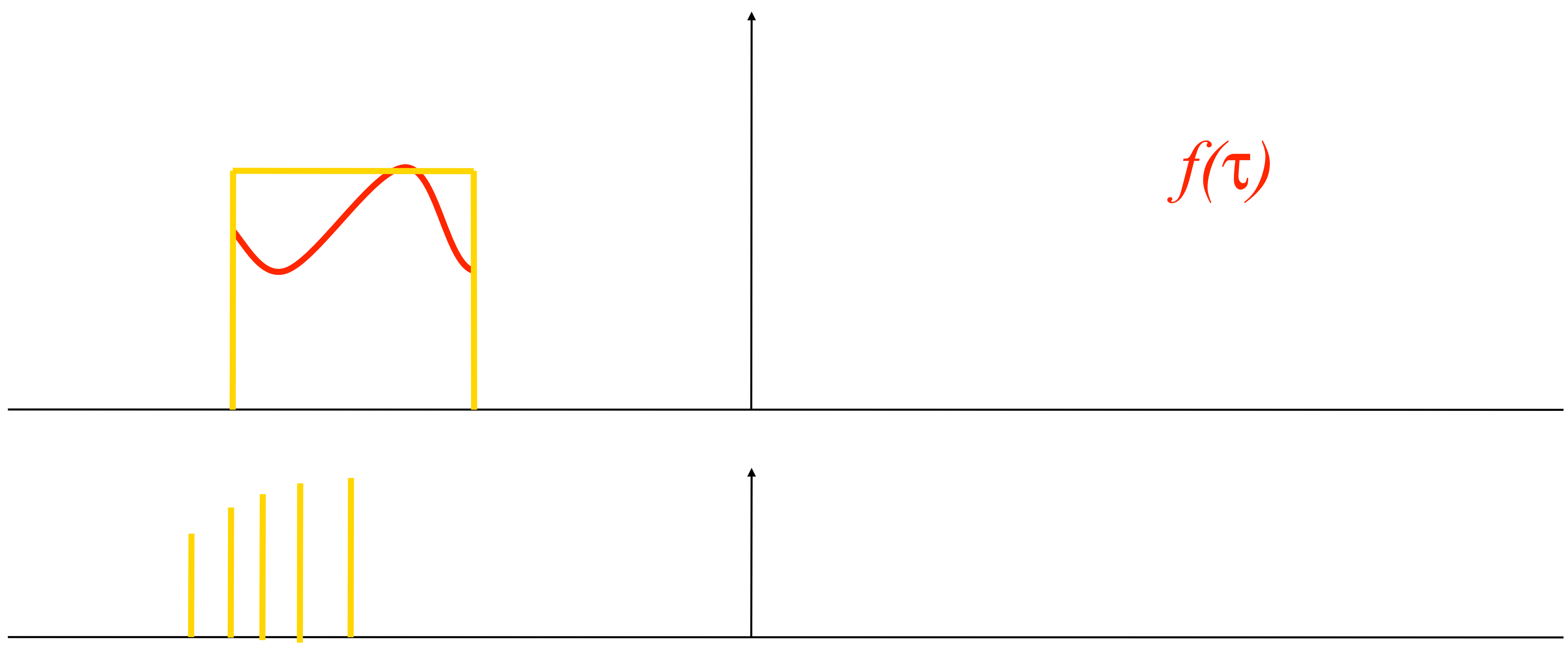

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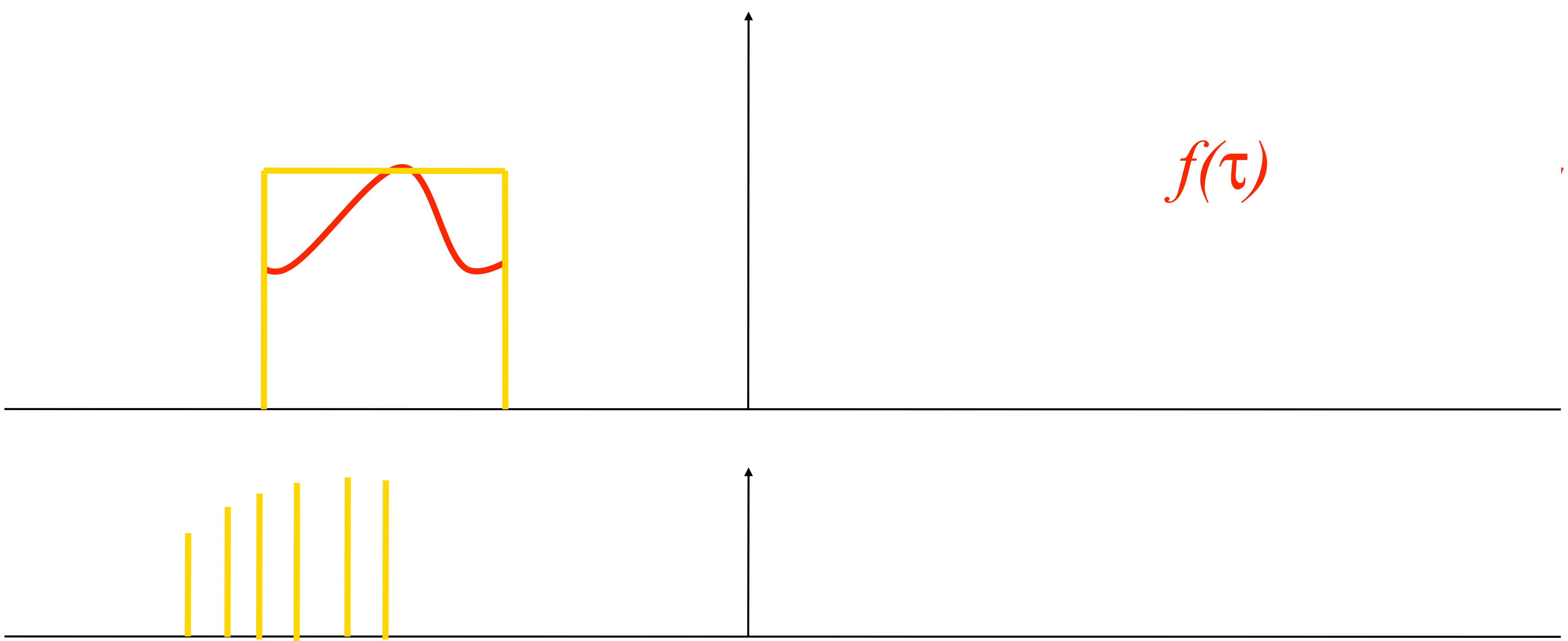

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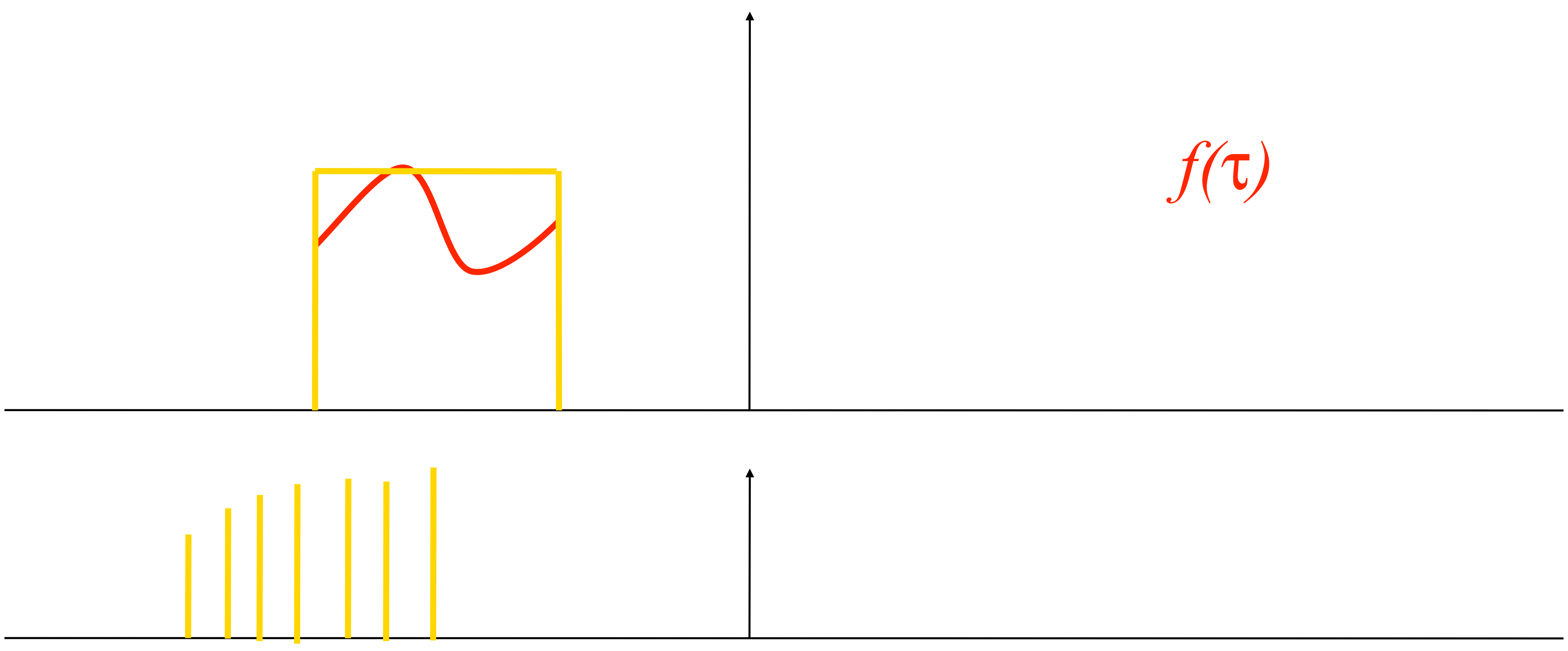

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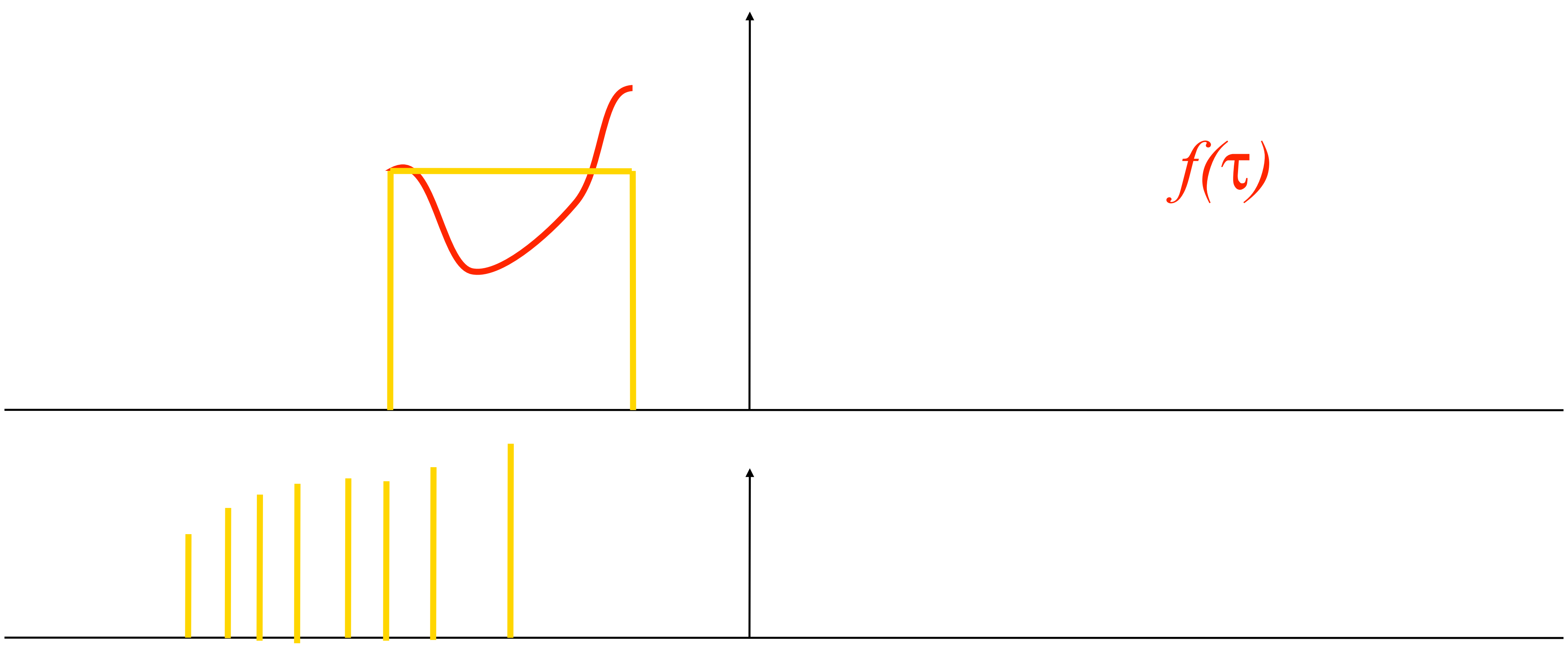

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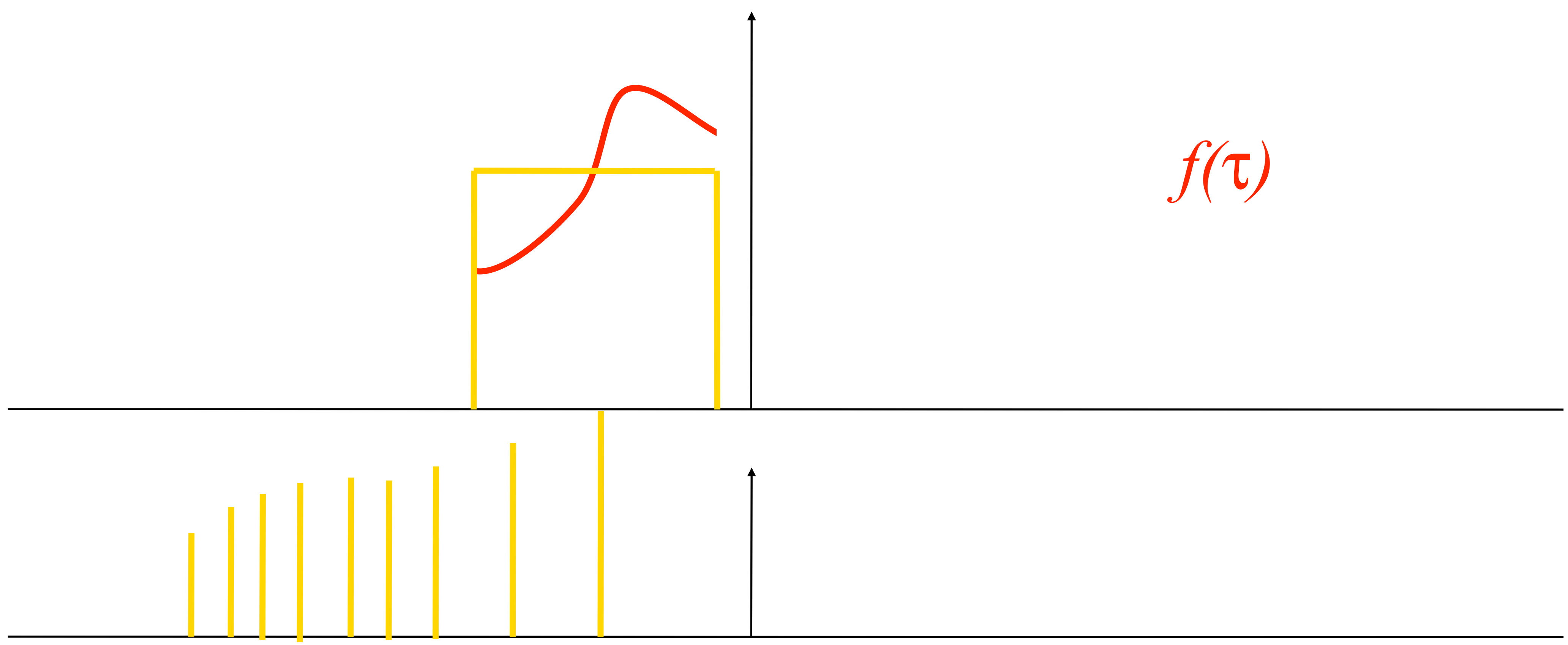

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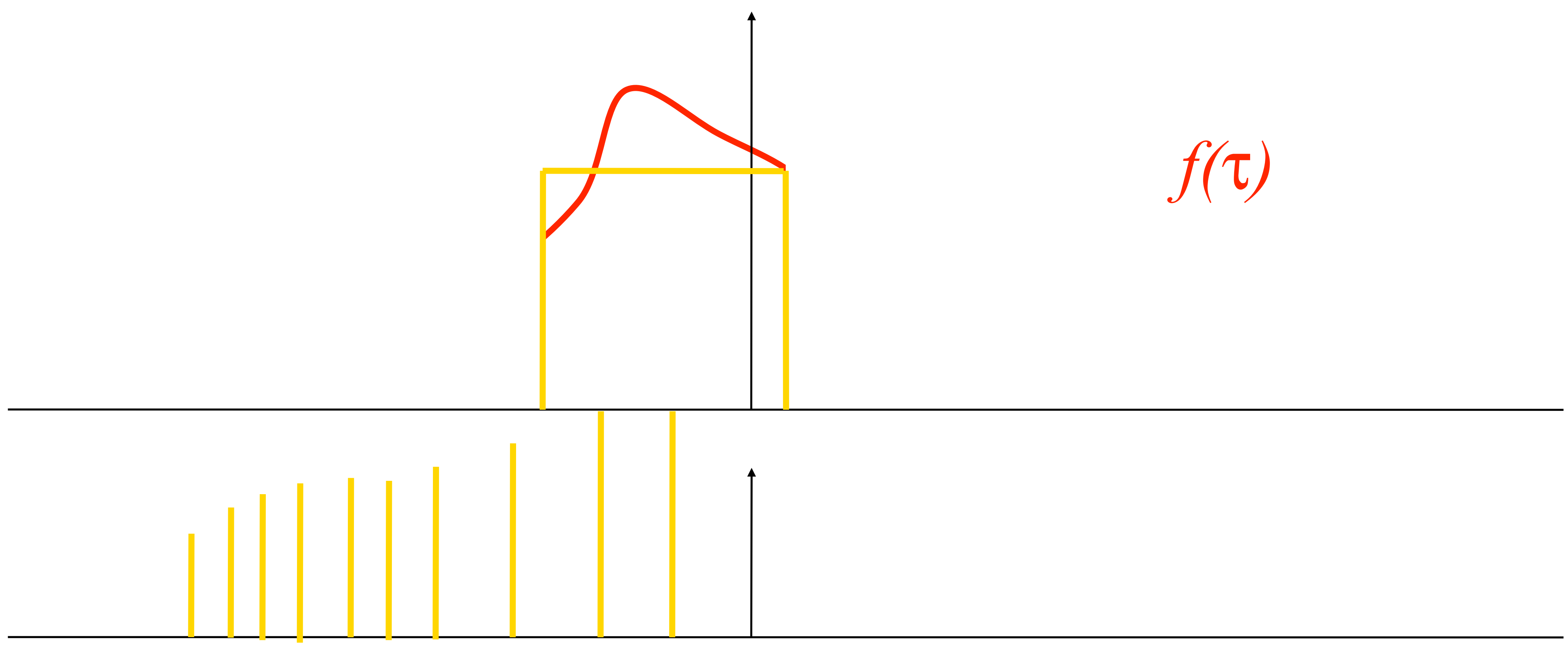

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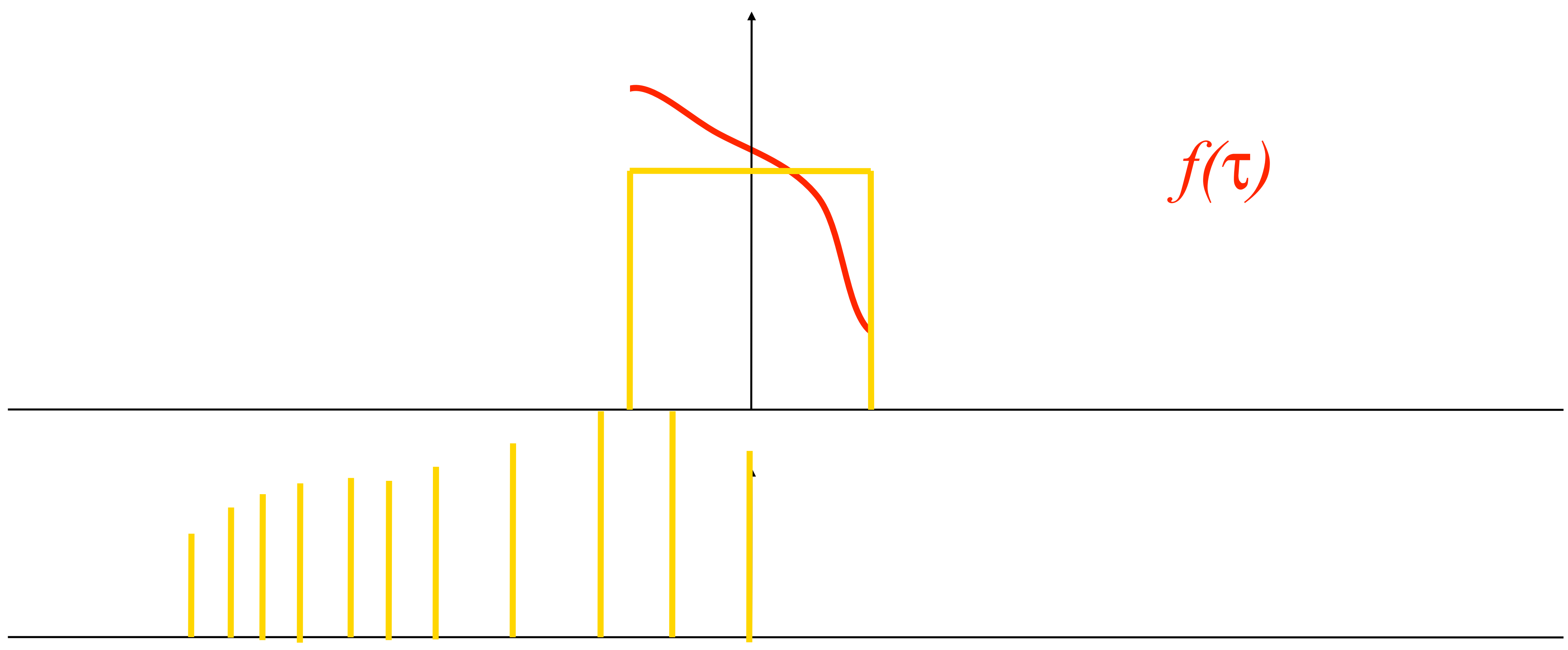

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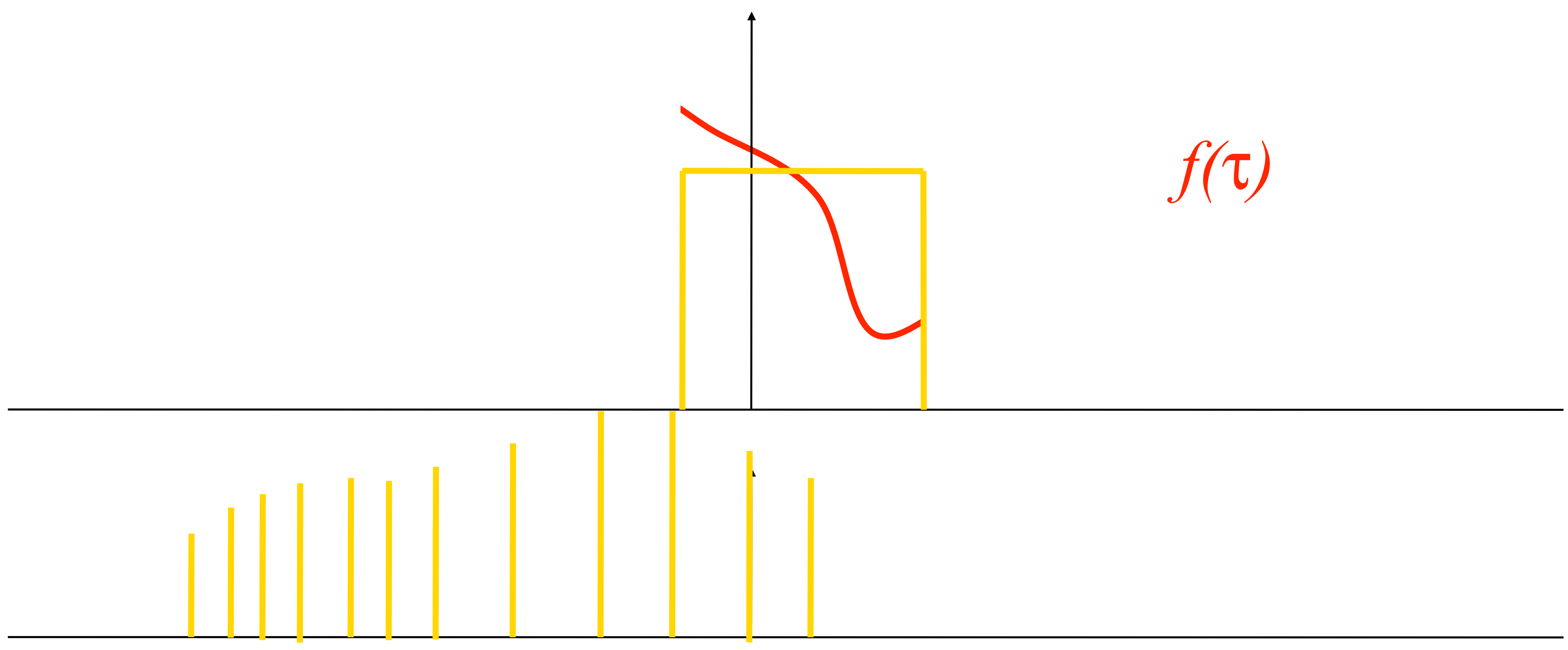

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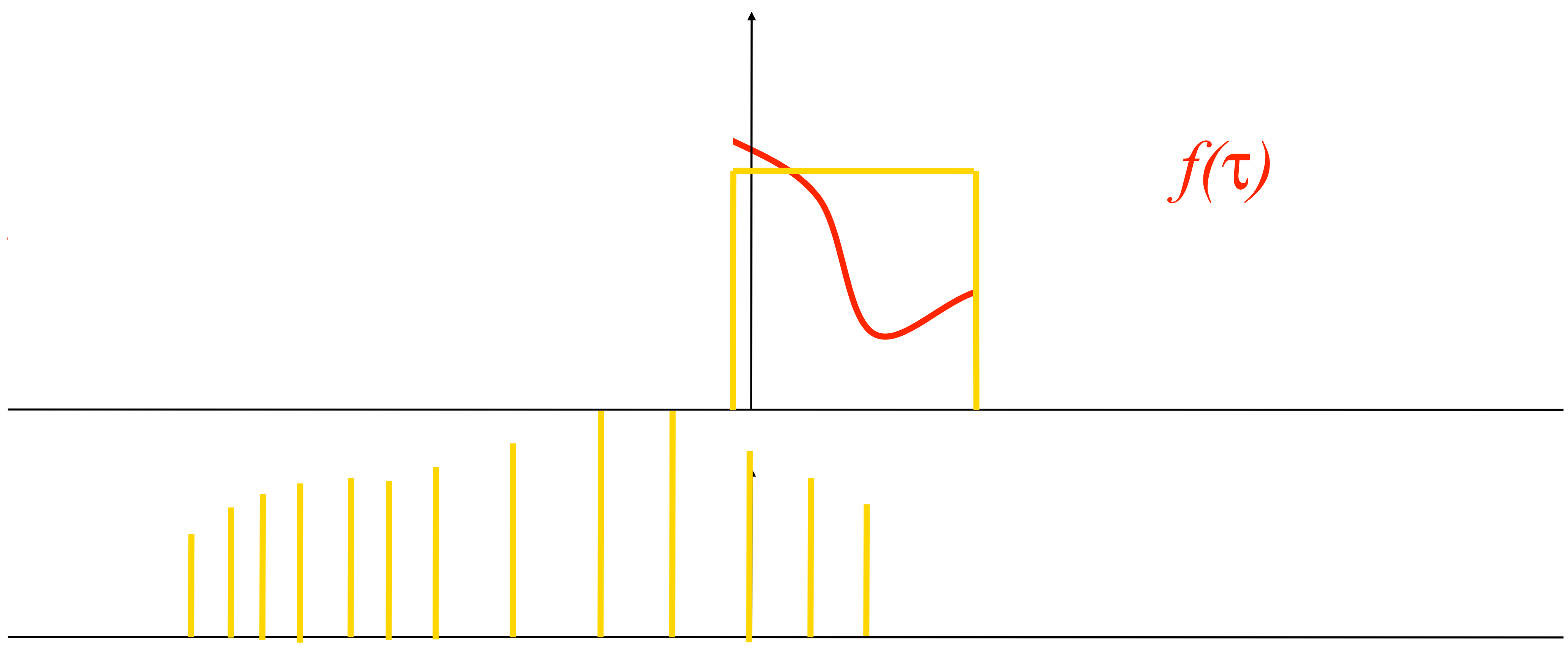

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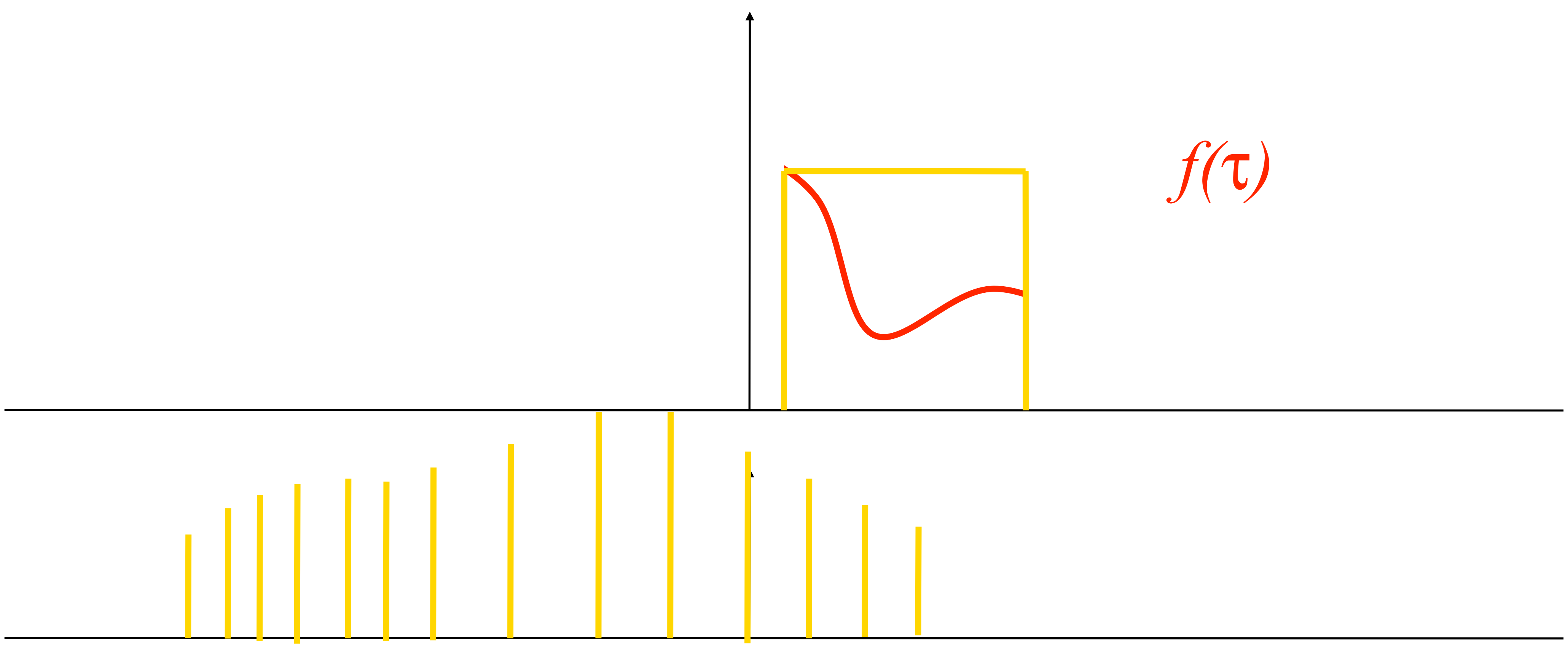

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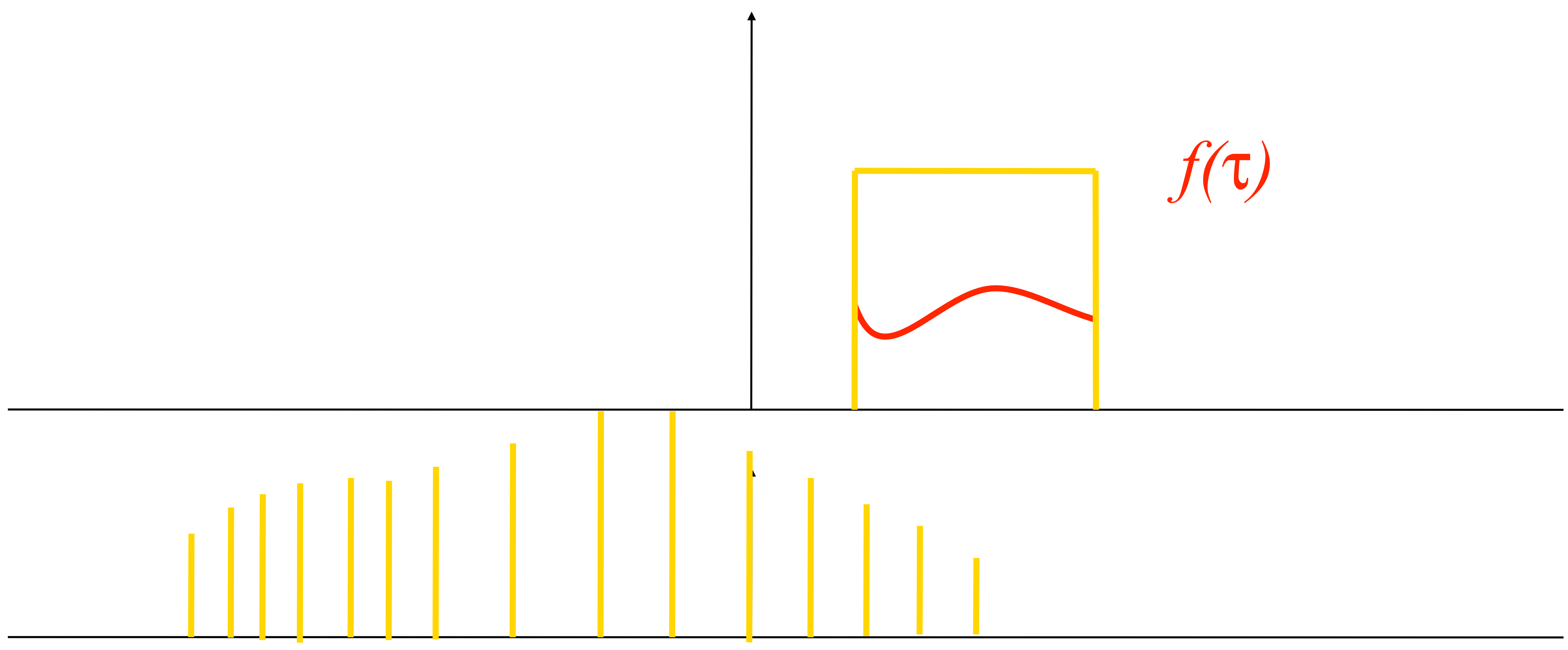

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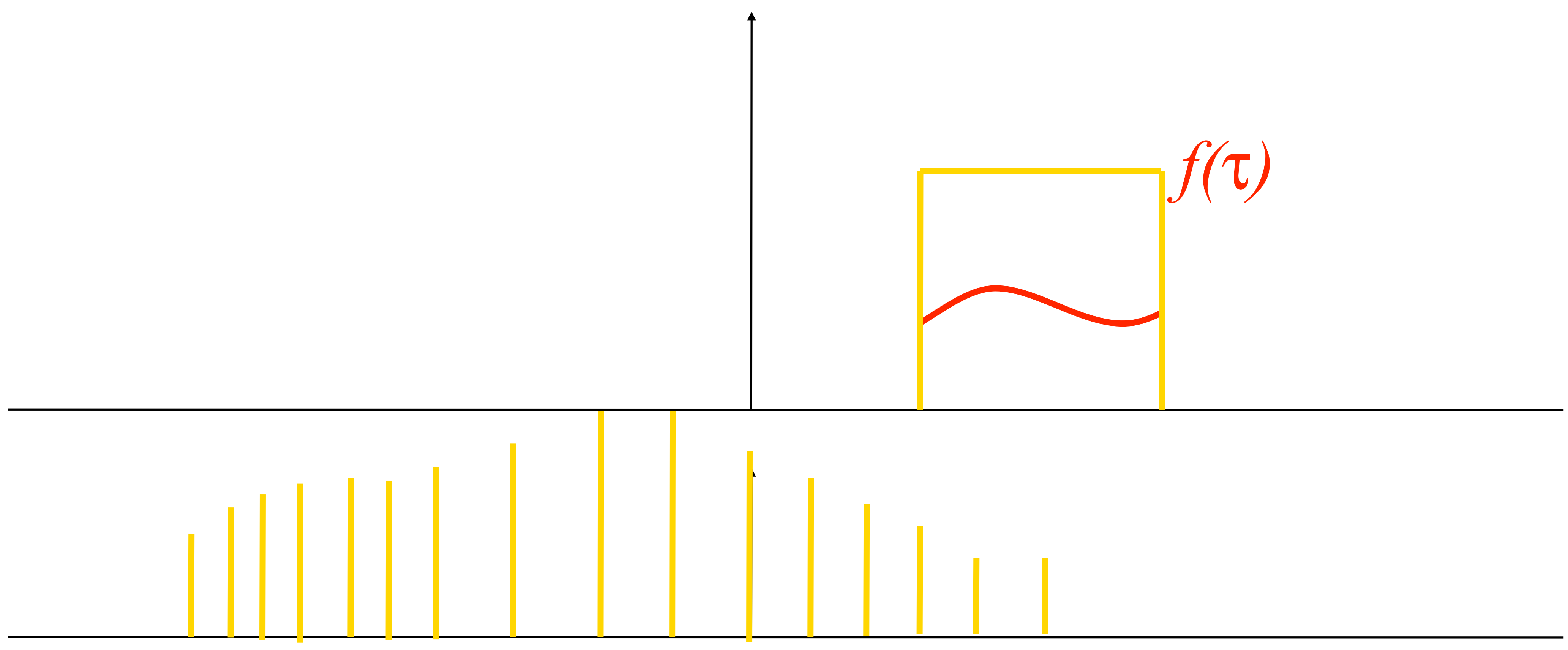

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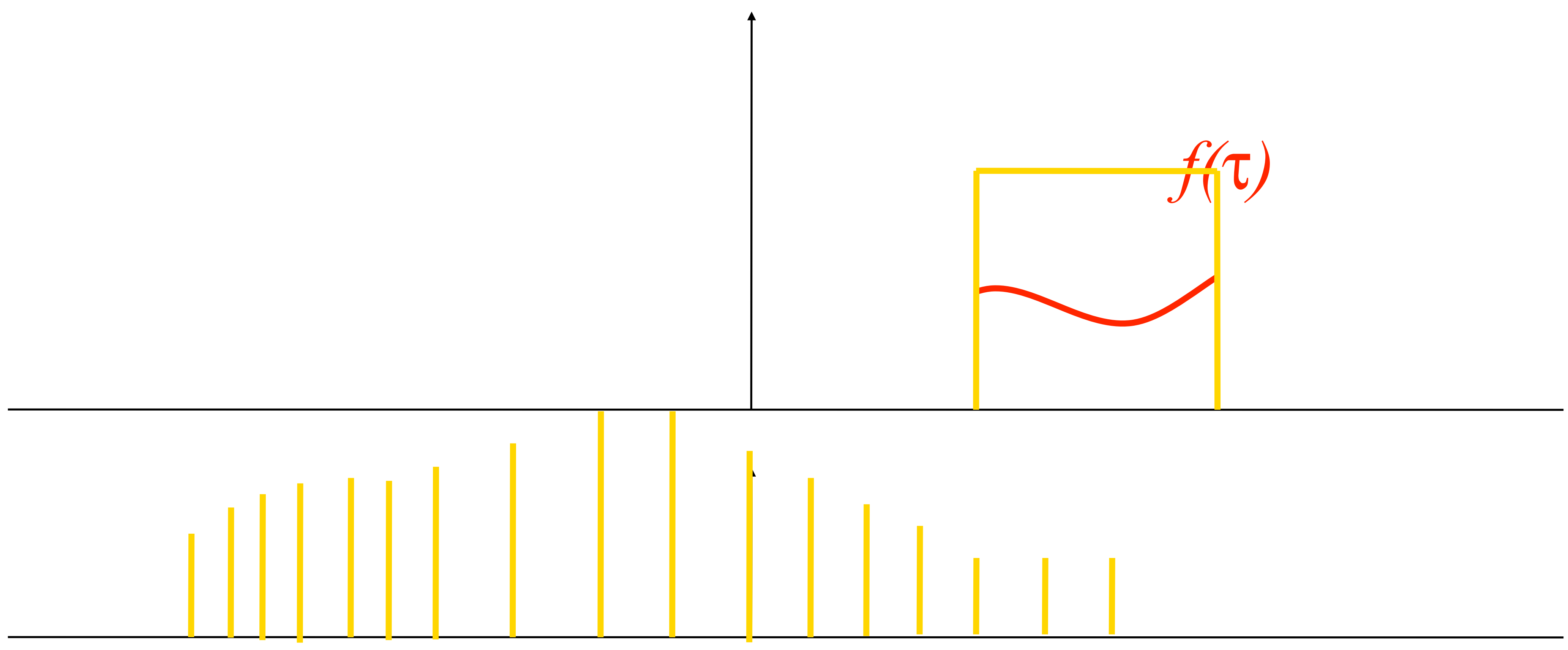

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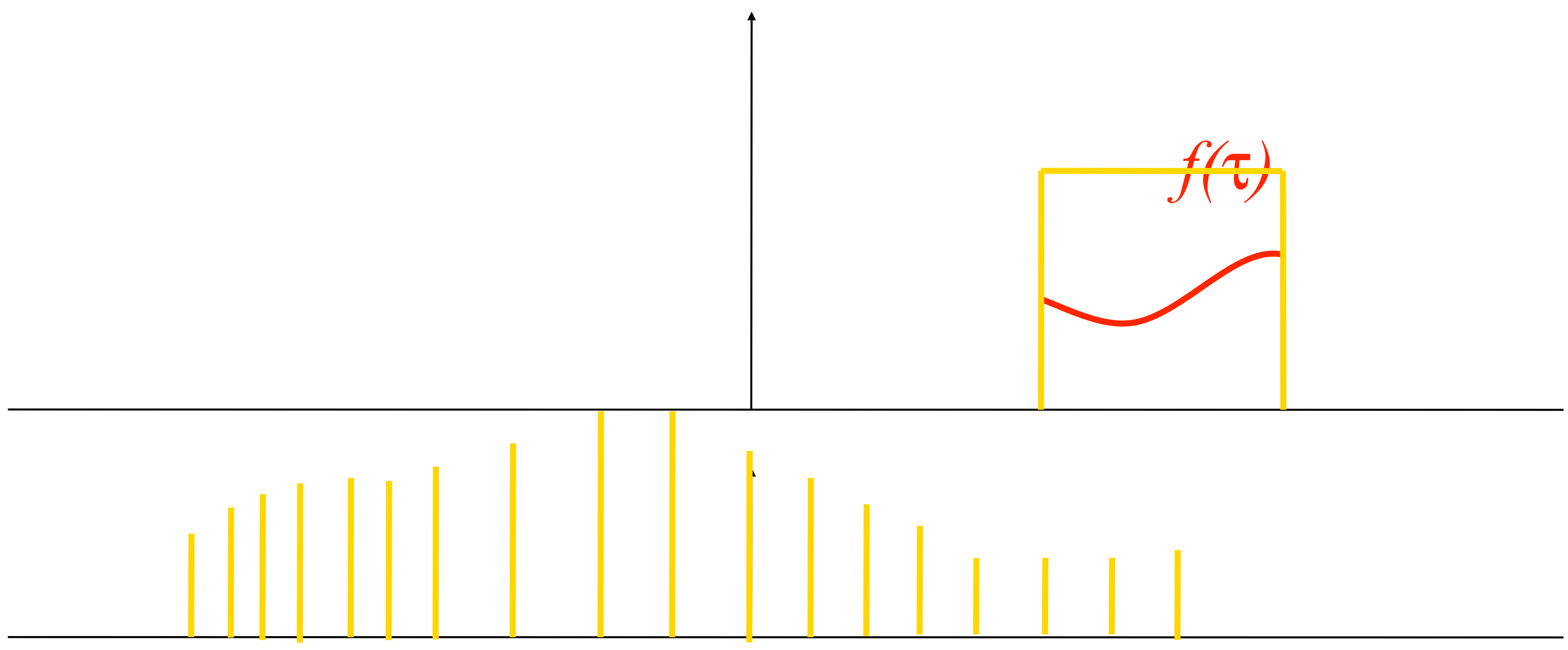

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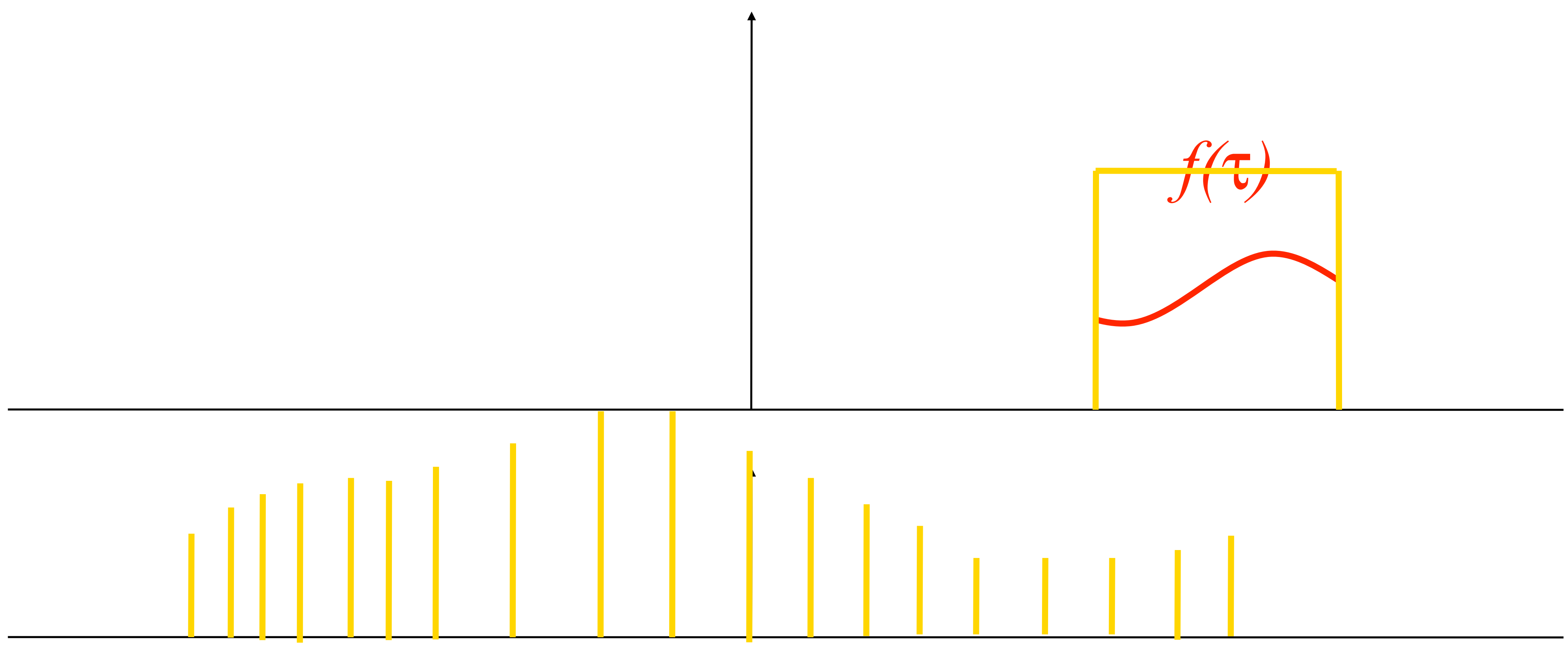

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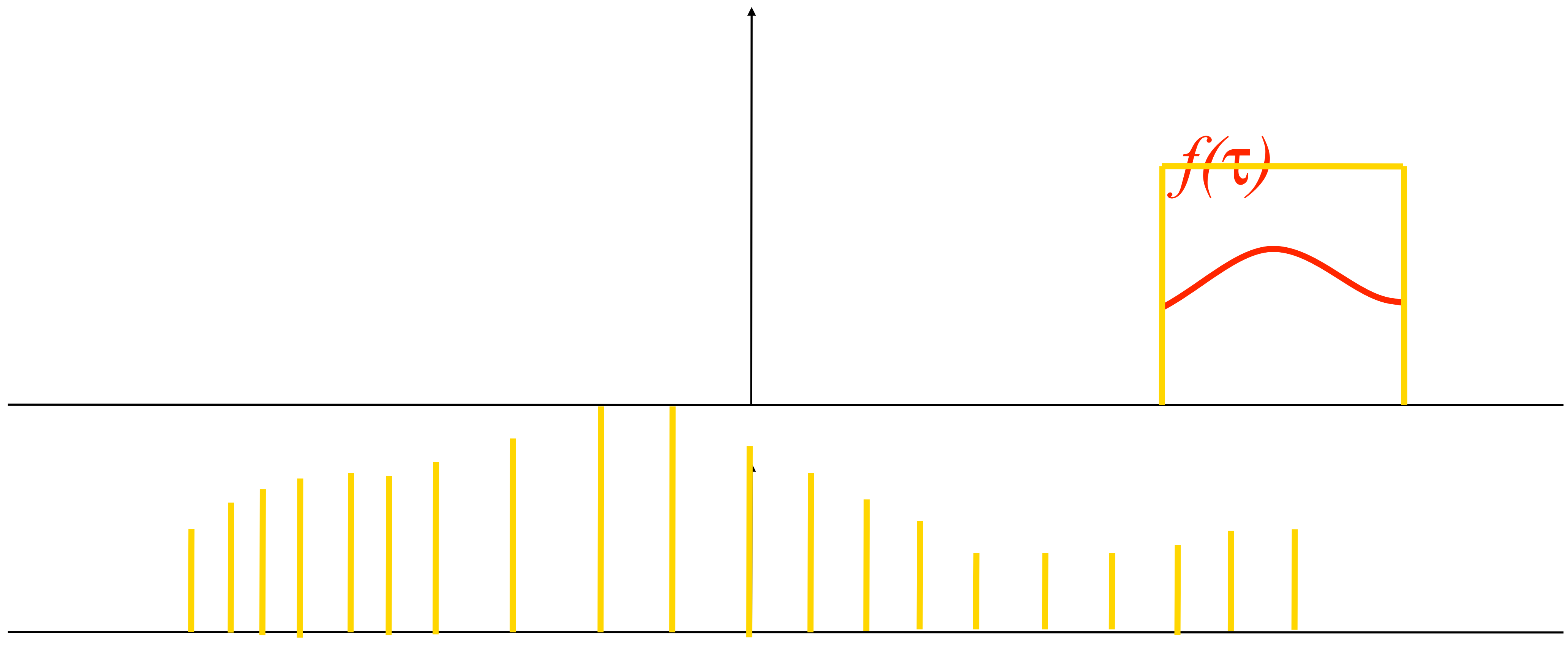

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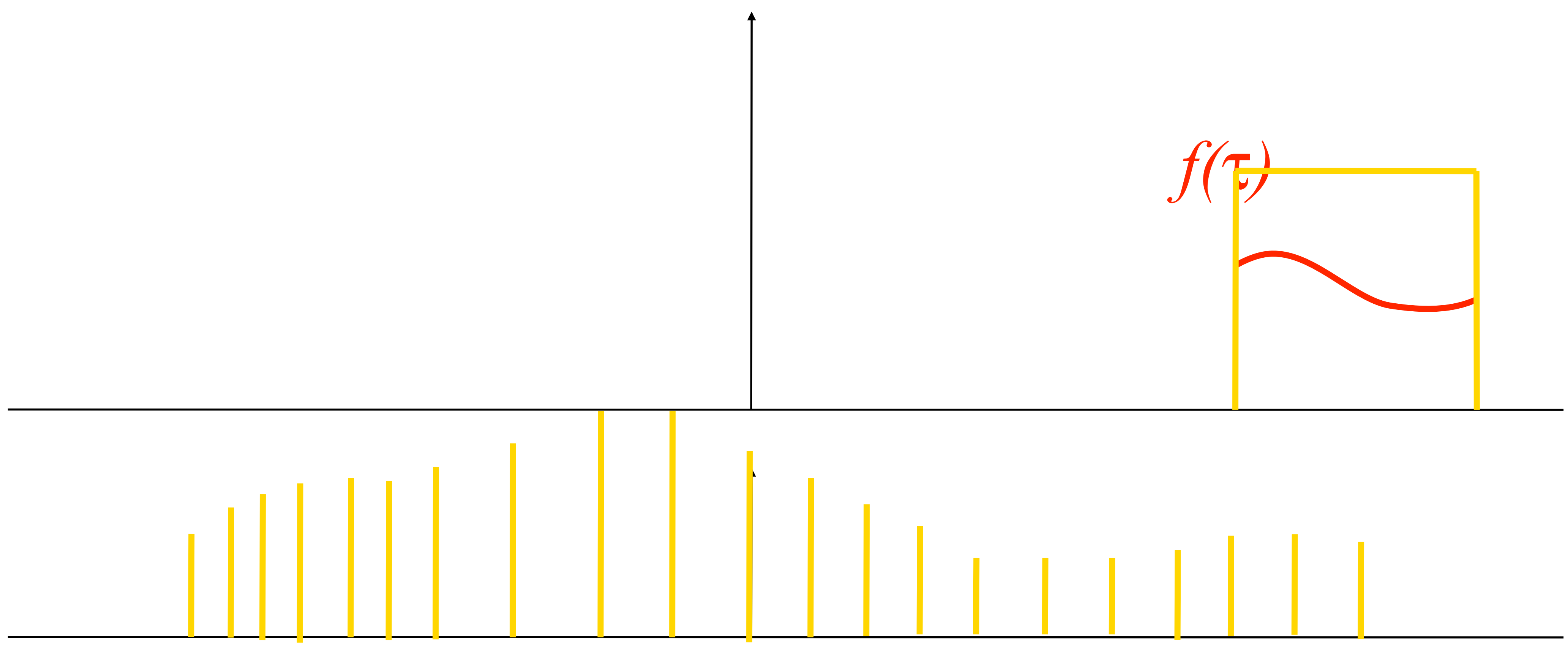

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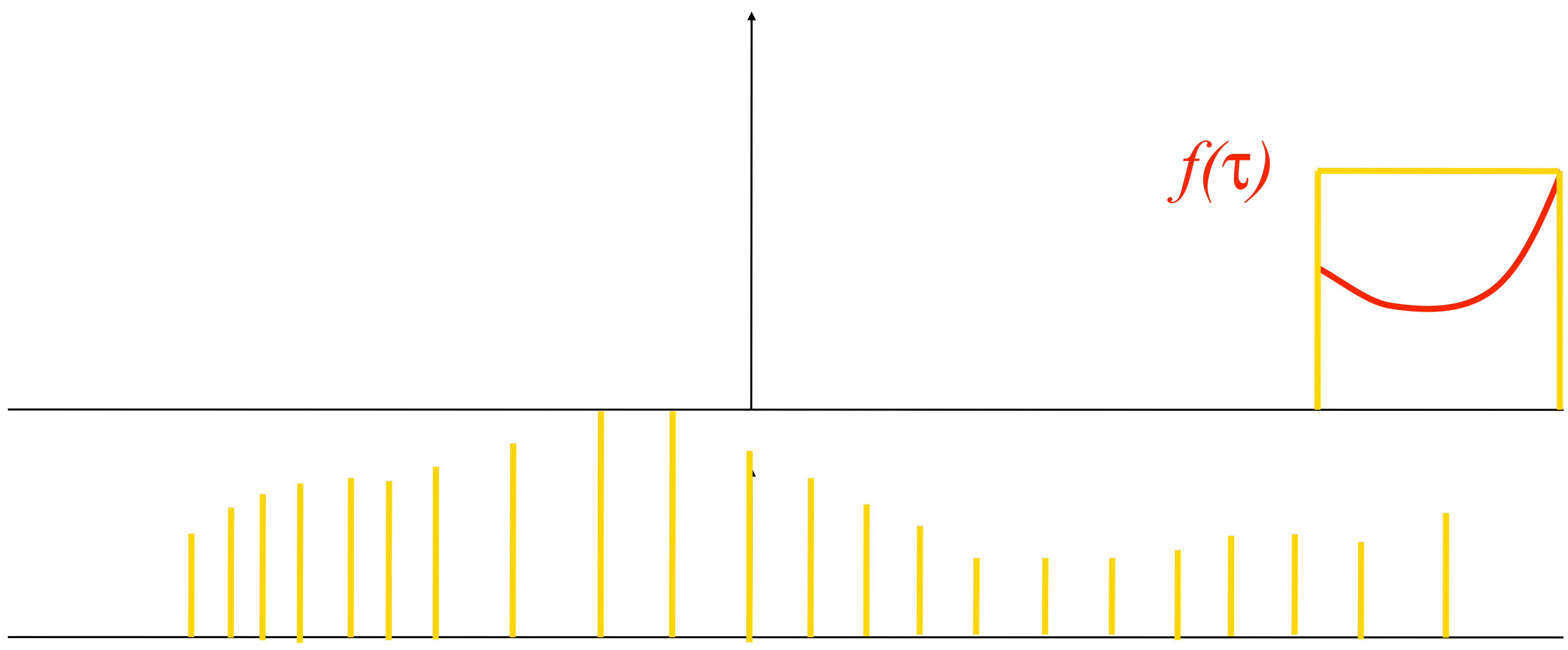
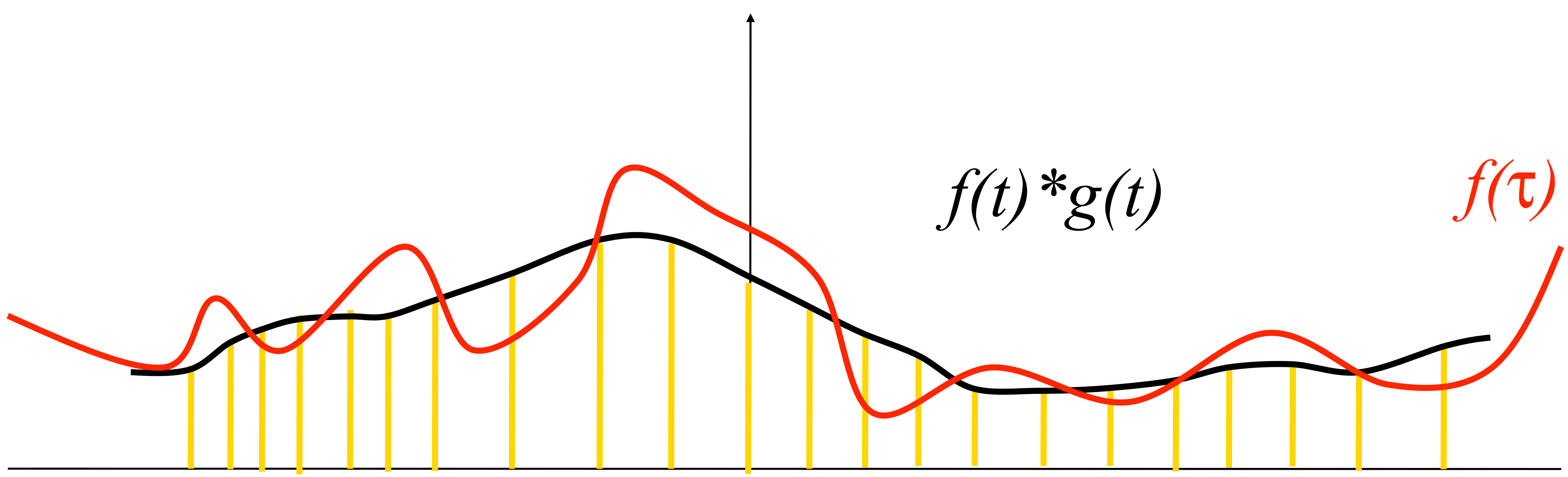


Convolution

- This particular convolution smooths out some of the high frequencies in f(t).

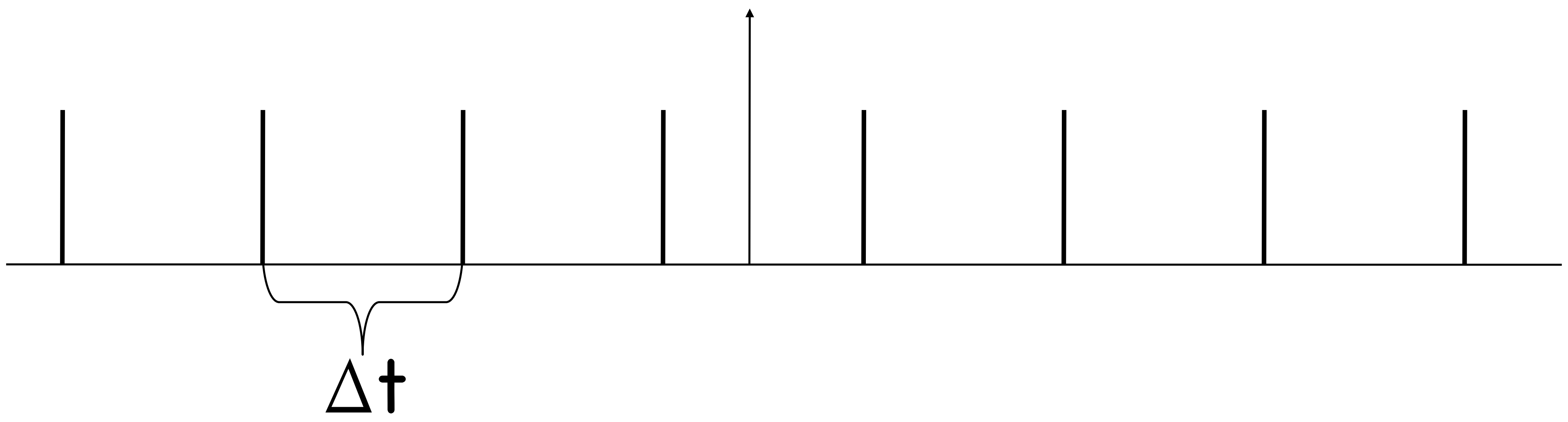


Sampling Function

- A Sampling Function or Impulse Train is defined by:

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t)$$

where Δt is the sample spacing.



Sampling Function

- The Fourier Transform of the Sampling Function is itself a sampling function.
- The sample spacing is the inverse.

$$S_{\Delta t}(t) \Leftrightarrow S_{\frac{1}{\Delta t}}(\omega)$$

Convolution Theorem

- The convolution theorem states that convolution in the spatial domain is equivalent to multiplication in the frequency domain, and viceversa.

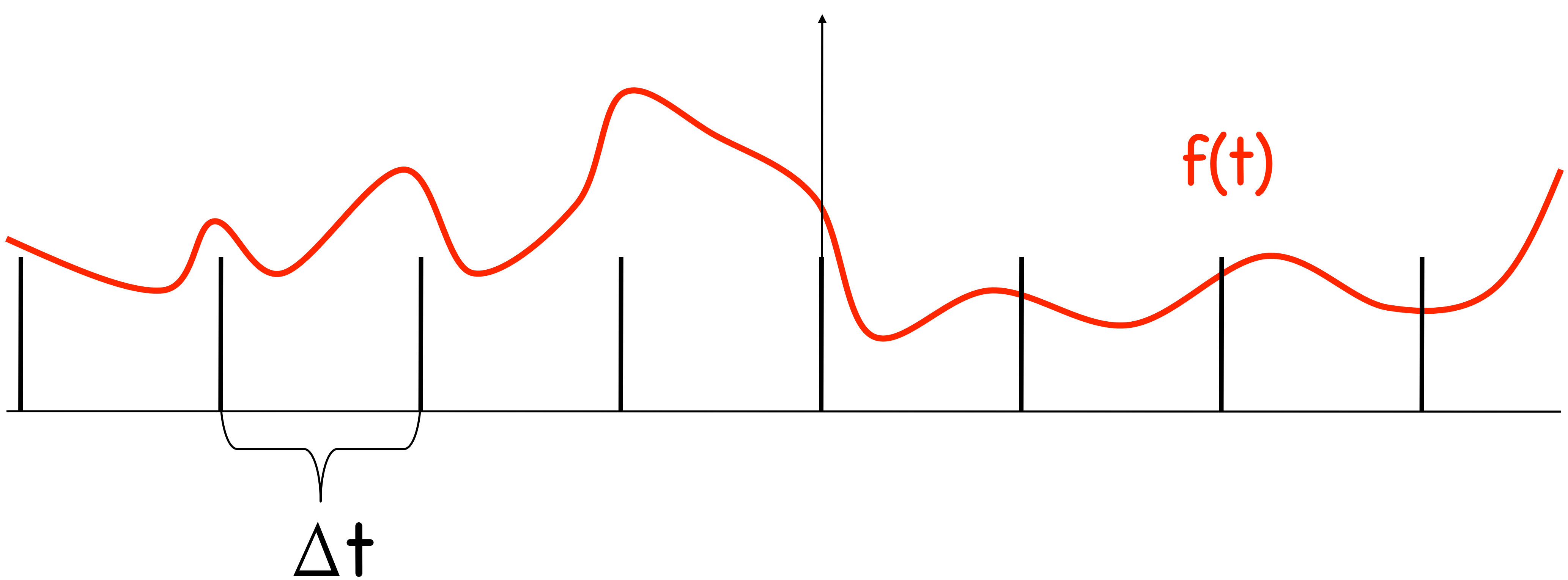
$$f(t) * g(t) \Leftrightarrow F(\omega) \cdot G(\omega)$$

$$f(t) \cdot g(t) \Leftrightarrow F(\omega) * G(\omega)$$

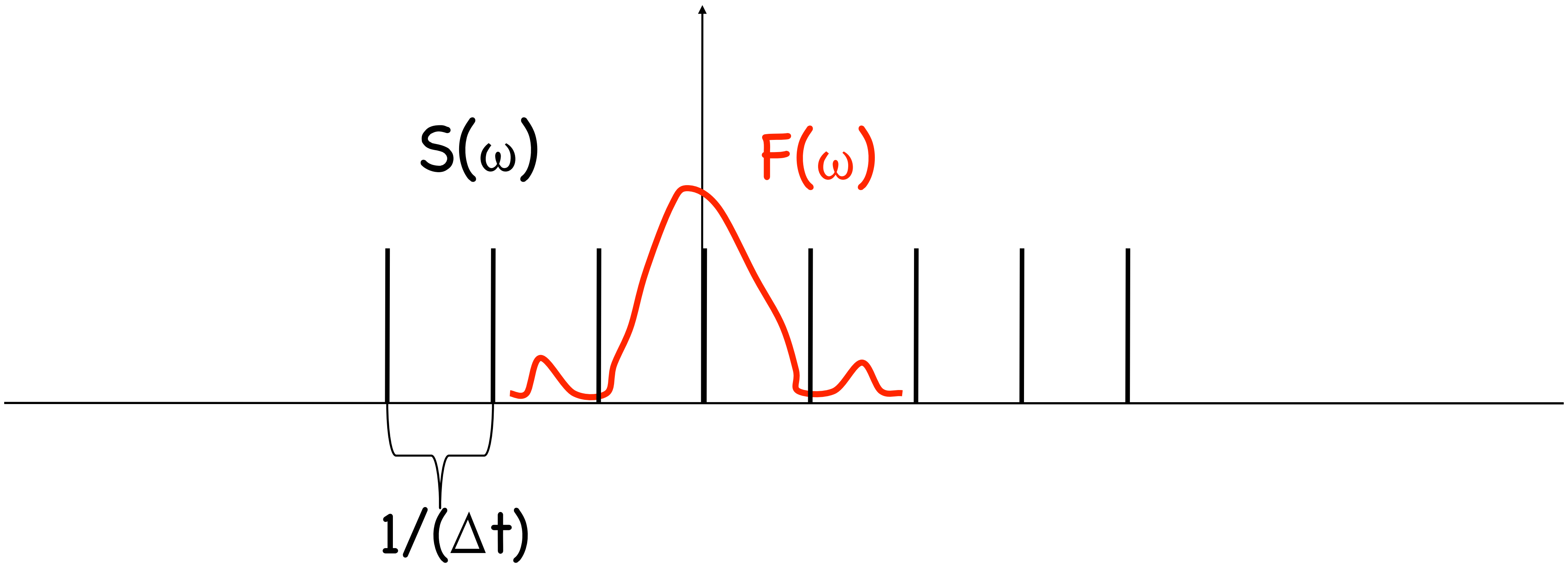
Convolution Theorem

- This powerful theorem can illustrate the problems with our point sampling and provide guidance on avoiding aliasing.

- Consider: $f(t) \cdot S_{\Delta t}(t)$

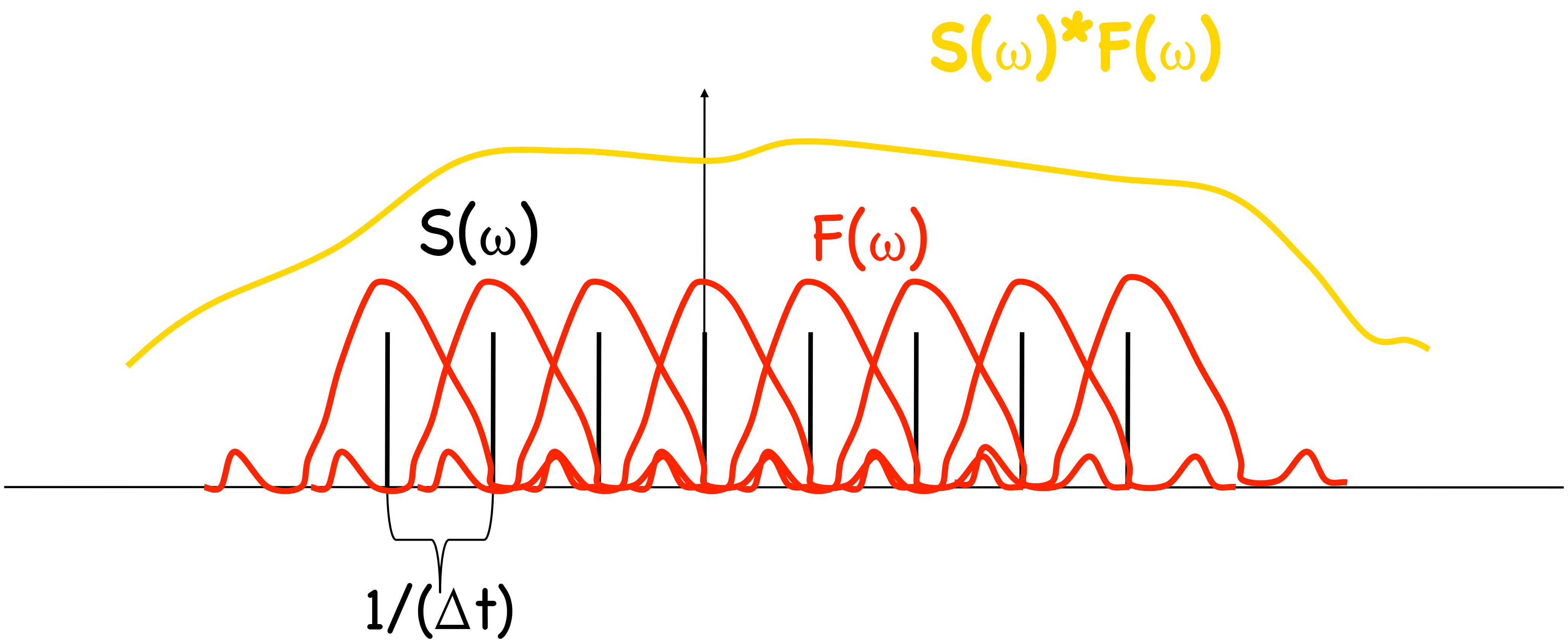


- What does this look like in the Fourier domain?



Convolution Theorem

- In Fourier domain we would convolve



Aliasing

- What this says, is that any frequencies greater than a certain amount will appear intermixed with other frequencies.
- In particular, the higher frequencies for the copy at $1/\Delta t$ intermix with the low frequencies centered at the origin.

Aliasing and Sampling

- Note, that the sampling process introduces frequencies out to infinity.
- We have also lost the function f(t), and now have only the discrete samples.
- This brings us to our next powerful theory.

Sampling Theorem

- The Shannon Sampling Theorem:

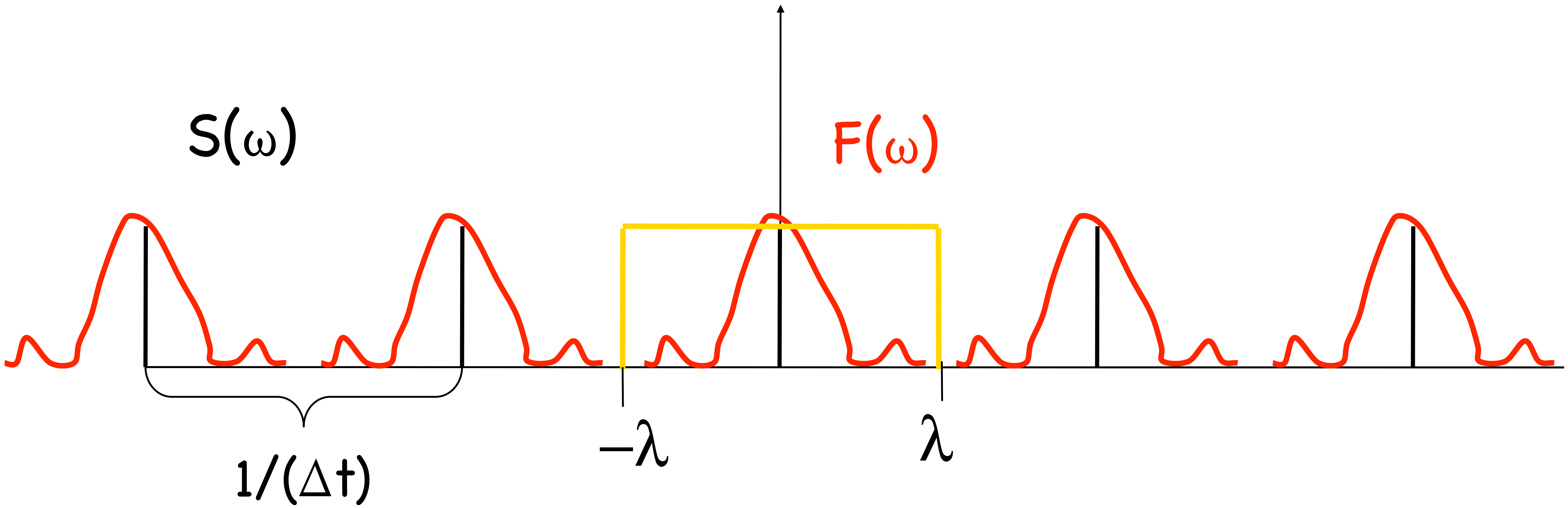
  A band-limited signal f(t), with a cutoff frequency of λ, that is sampled with a sampling spacing of Δt may be perfectly reconstructed from the discrete values $f[n\Delta t]$ by convolution with the sinc(t) function, provided the Nyquist limit: $\lambda<1/(2\Delta t)$

- Why is this?

  The Nyquist limit will ensure that the copies of $F(\omega)$ do not overlap in the frequency domain.

  We can completely reconstruct or determine f(t) from $F(\omega)$ using the Inverse Fourier Transform.

- In order to do this, we need to remove all of the shifted copies of $F(\omega)$ first.
- This is done by simply multiplying $F(\omega)$ by a box function of width 2λ.



Sampling Theory

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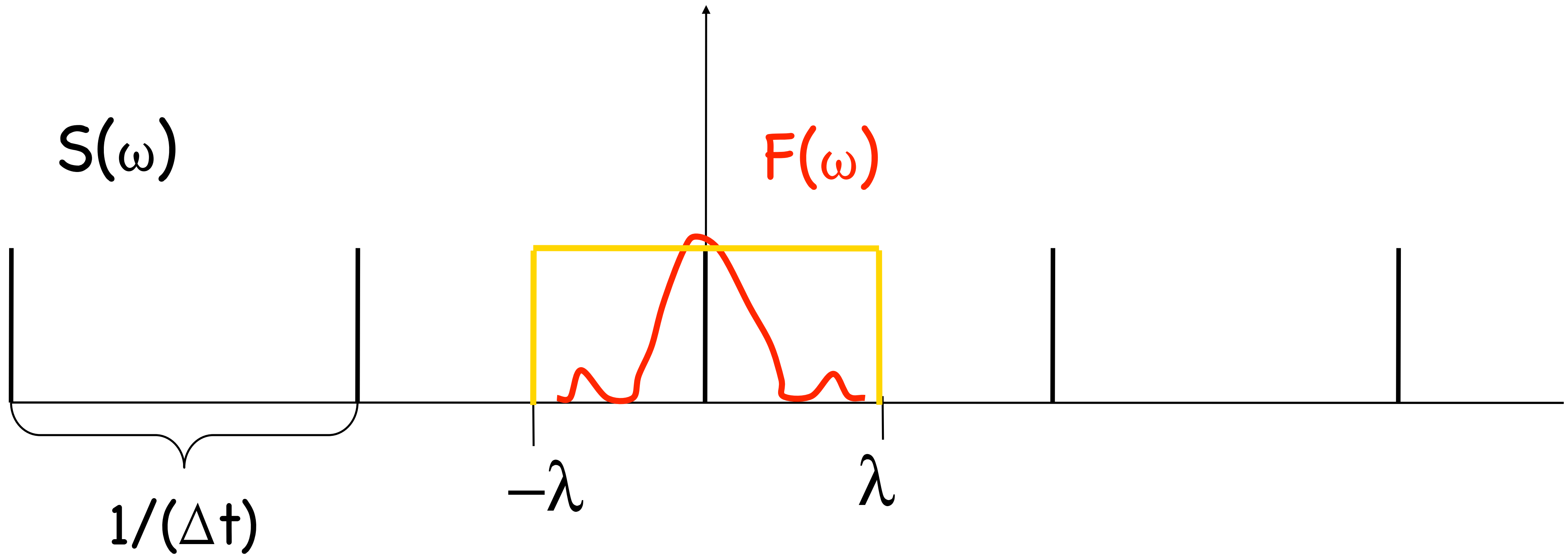


Sampling Theory

- So, given $f[n\Delta t]$ and an assumption that $f(t)$ does not have frequencies greater than $1/(2\Delta t)$, we can write the formula:

$$f[nT] = f(t) \cdot S_{\Delta t}(t) \Leftrightarrow F(\omega) * S_{\Delta t}(\omega)$$

$$F(\omega) = (F(\omega) * S_{\Delta t}(\omega)) \cdot Box_{1/(2\Delta t)}(\omega)$$

therefore,

$$f(t) = f[n\Delta t] * \mathrm{sinc}(t)$$

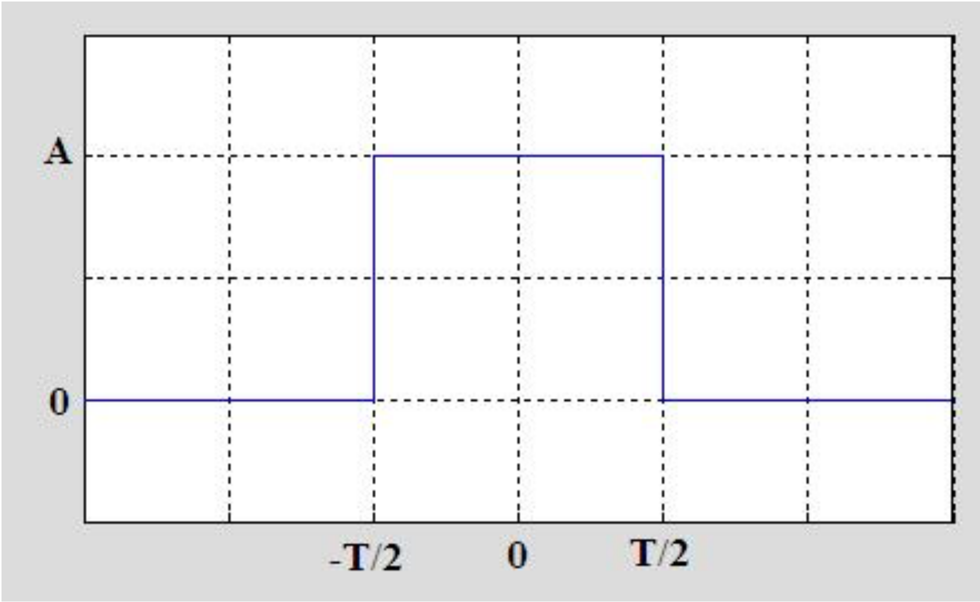
http://www.thefouriertransform.com/pairs/box.php

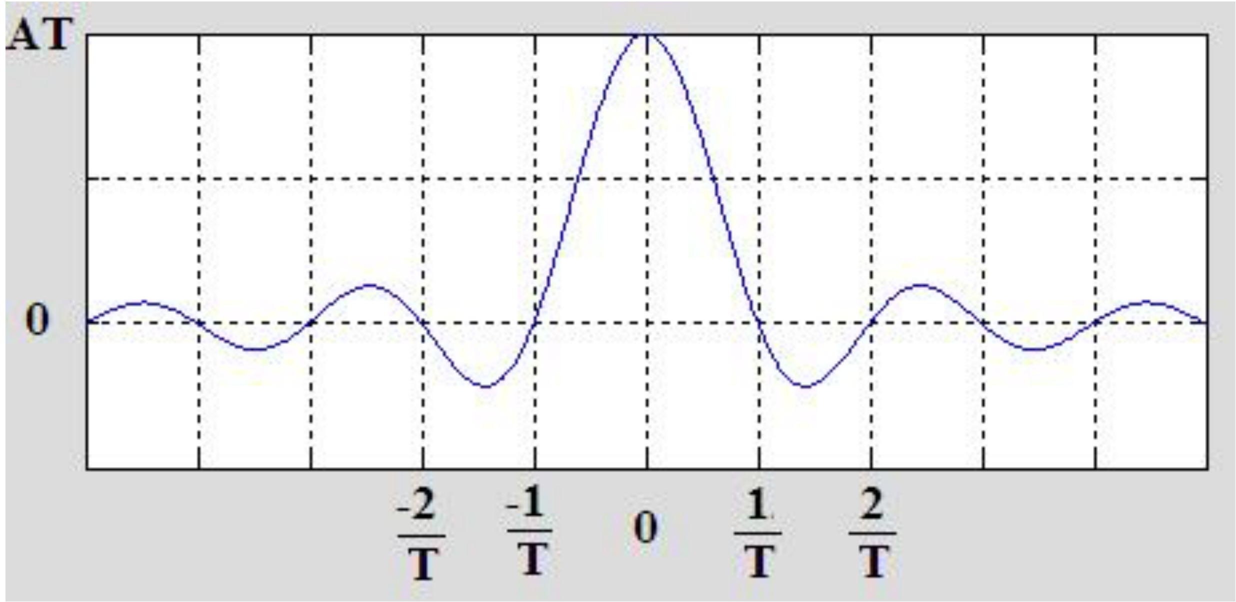
http://195.134.76.37/applets/AppletNyquist/Appl_Nyquist2.html

FT of Boxcar

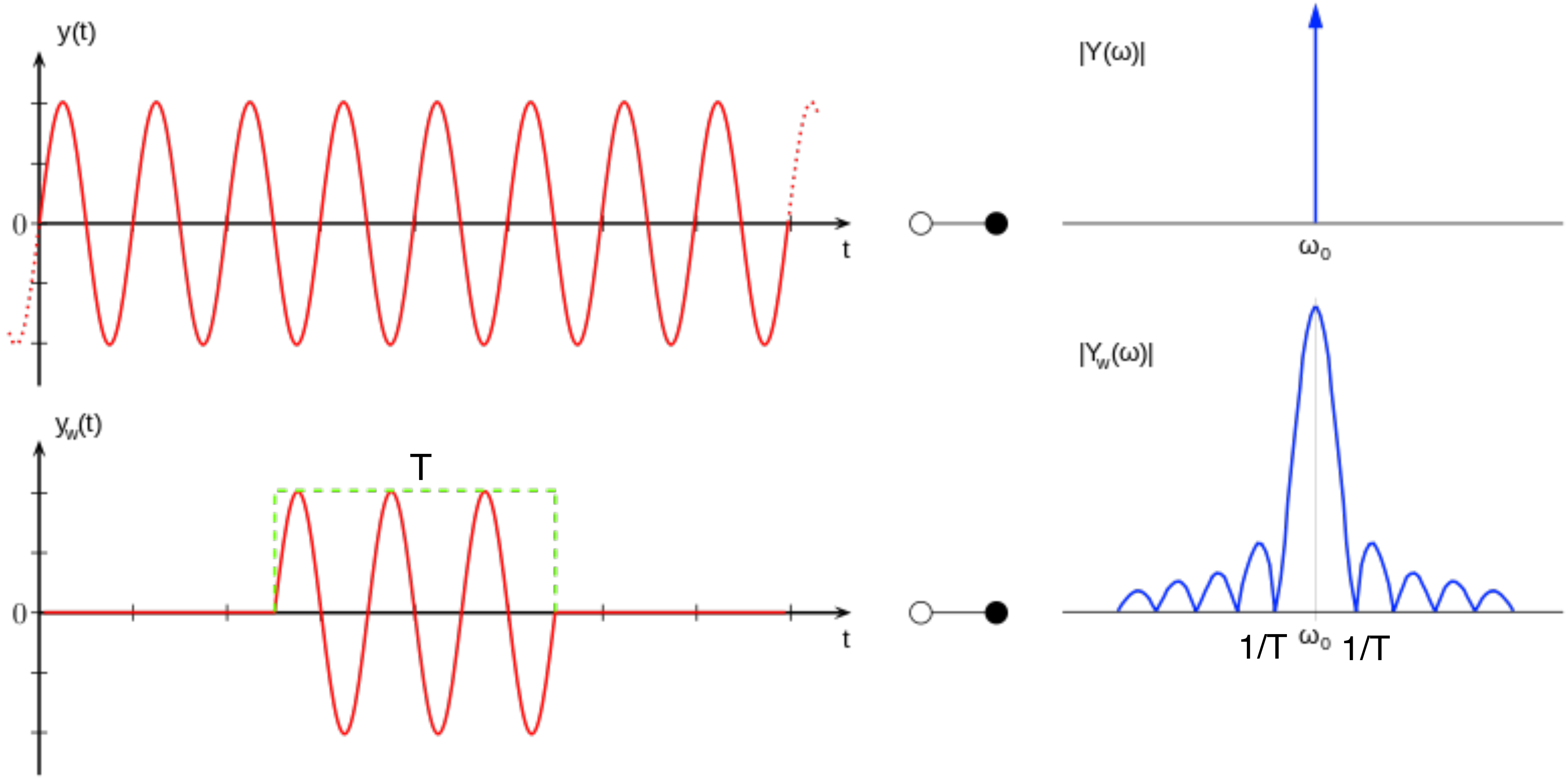
$$\int_{-\infty}^{+\infty} B_T(t)e^{-i\omega t}dt = \int_{-T/2}^{+T/2} e^{-i\omega t}\,dt = \frac{\sin(\pi fT)}{\pi fT}$$

$$\propto \mathrm{sinc}(\pi fT) = \mathrm{sinc}(\omega T / 2)$$





Spectral leakage



- Resolving power in frequency domain is related to maximum duration in time domain: $$\Delta f \geq \frac{1}{2\pi T}\left(= \frac{1}{2\pi N \Delta t}\right)$$

- Resolving power in time domain decides maximum resolvable frequency: $$\Delta f \geq \frac{1}{2\Delta t}$$

$$\Delta f \Delta t \geq \frac{1}{2\sqrt{\pi N}}$$

https://www.youtube.com/watch?v=MBnnXbOM5S4