Frequency Analysis

To use transfer functions, we must first decompose a signal into its component frequencies.

Basic idea: any signal can be written as the sum of sines and cosines of different frequencies.

The mathematical tool for doing this is the *Fourier Transform*.

General Idea of Transforms

Suppose that you have an orthonormal (orthogonal, unit length) basis set of vectors $\{\overline{e}_k\}$.

Any vector in the space spanned by this basis set can be represented as a weighted sum of those basis vectors:

$$\overline{v} = \sum_k a_k \, \overline{e}_k$$

To get the weights:

$$a_k = \overline{v} \cdot \overline{e}_k$$

In other words, the vector can be *transformed* into the weights a_i.

Likewise, the transformation can be *inverted* by turning the weights back into the vector.

Linear Algebra with Functions

The inner (dot) product of two vectors is the sum of the point-wise multiplication of each component:

$$\overline{u} \cdot \overline{v} = \sum_j \overline{u}[j] \, \overline{v}[j]$$

Can't we do the same thing with functions?

$$f \cdot g = \int_{-\infty}^{\infty} f(x) \, g(x) \, dx$$

Functions satisfy all of the linear algebraic requirements of vectors.

Transforms with Functions

Just as we transformed vectors, we can also transform functions:

	Vectors $\{\overline{e}_k\}$	Functions $\{e_k(t)\}$
Transform	$a_k = \overline{v} \cdot \overline{e}_k$ $\sum_j \overline{v}[j]\ e_k[j]$	$a_k = f \cdot e_k$ $= \int_{-\infty}^{\infty} f(t)\ e_k(t)\ dt$
Inverse	$\overline{v} = \sum_k a_k\ \overline{e}_k$	$f(t) = \sum_k a_k\ e_k(t)$

Basis Set: Generalized Harmonics

The set of generalized harmonics we discussed earlier form an orthonormal basis set for functions:

$$\{e^{i2\pi st}\}$$

where each harmonic has a different frequency s.

Remember:

$$e^{i2\pi st} = \cos(2\pi st) + i\sin(2\pi st)$$

The real part is a cosine of frequency s.

The imaginary part is a sine of frequency s.

The Fourier Series

	All Functions $\{e_k(t)\}$	Harmonics $\{e^{i2\pi st}\}$
Transform	$a_k = f \cdot e_k$ $= \int_{-\infty}^{\infty} f(t)\, e_k(t)\, dt$	$a_k = f \cdot e^{i2\pi s_k t}$ $= \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi s_k t}\, dt$
Inverse	$f(t) = \sum_k a_k\, e_k(t)$	$f(t) = \sum_k a_k\, e^{i2\pi s_k t}$

The Fourier Transform

Most tasks need an infinite number of basis functions (frequencies), each with their own weight $F(s)$:

	Fourier Series	Fourier Transform
Transform	$a_k = f \cdot e^{i2\pi s_k t}$ $= \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi s_k t}\, dt$	$F(s) = f \cdot e^{i2\pi st}$ $= \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi st}\, dt$
Inverse	$f(t) = \sum_k a_k\, e^{i2\pi s_k t}$	$f(t) = \int_{-\infty}^{\infty} F(s)\, e^{i2\pi st}\, ds$

The Fourier Transform

To get the weights (amount of each frequency):

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi st}\, dt$$

$F(s)$ **is the Fourier Transform of** $f(t)$**:** $\mathcal{F}(f(t)) = F(s)$

To turn the weights back into the signal (invert the transform):

$$f(t) = \int_{-\infty}^{\infty} F(s)\, e^{i2\pi st}\, ds$$

$f(t)$ **is the Inverse Fourier Transform of** $F(s)$**:** $\mathcal{F}^{-1}(F(s)) = f(t)$

What's All This Complex Arithmetic Mean?

Fourier Transform:

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi st}\, dt$$

Remember Euler's Formula (Notation):

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So,

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t)\left[\cos(-2\pi st) + i\sin(-2\pi st)\right]\, dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(-2\pi st)\, dt + i\int_{-\infty}^{\infty} f(t)\sin(-2\pi st)\, dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(2\pi st)\, dt - i\int_{-\infty}^{\infty} f(t)\sin(2\pi st)\, dt
\end{aligned}$$

Magnitude and Phase

Remember: complex numbers can be thought of as (real,imaginary) or (magnitude,phase).

$$\text{Magnitude:} \quad |F| = \left[\Re(F)^2 + \Im(F)^2\right]^{1/2}$$

$$\text{Phase:} \quad \phi(F) = \tan^{-1} \frac{\Im(F)}{\Re(F)}$$

Intuition:

Real part	How much of a cosine of that frequency you need
Imaginary part	How much of a sine of that frequency you need
Magnitude	Amplitude of combined cosine and sine
Phase	Relative proportions of sine and cosine

Odd and Even Functions

Even	Odd
$f(-t) = f(t)$	$f(-t) = -f(t)$
Symmetric	Anti-symmetric
Cosines	Sines
Transform is real*	Transform is imaginary*

* for real-valued signals

Sinusoids

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{1}{2}\left[\delta(s+\omega)+\delta(s-\omega)\right]$
$\sin(\omega t)$	$\frac{1}{2}i\left[\delta(s+\omega)-\delta(s-\omega)\right]$

Constant Functions

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
1	$\delta(s)$
a	$a\ \delta(s)$

Delta Functions

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\delta(t)$	1

Square Pulse

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\begin{cases} 1 & \text{if } -a/2 \leq t \leq a/2 \\ 0 & \text{otherwise} \end{cases}$	$\text{sinc}(a\pi s) = \frac{\sin(a\pi s)}{a\pi s}$

Triangle

Spatial Domain $f(t)$	Frequency Domain $F(s)$
$\begin{cases} 1 - \|t\| & \text{if } -a \leq t \leq a \\ 0 & \text{otherwise} \end{cases}$	$\text{sinc}^2(a\pi s)$

Comb

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\delta(t \bmod k)$	$\delta(s \bmod 1/k)$

Gaussian

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$e^{-\pi t^2}$	$e^{-\pi s^2}$

Differentiation

Spatial Domain	Frequency Domain
$f(t)$	$F(s)$
$\frac{d}{dt}$	$2\pi i s$

Some Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(s)$	
Cosine	$\cos(2\pi\omega t)$	Shifted Deltas	$\frac{1}{2}\left[\delta(s+\omega)+\delta(s-\omega)\right]$
Sine	$\sin(2\pi\omega t)$	Shifted Deltas	$\frac{1}{2}i\left[\delta(s+\omega)-\delta(s-\omega)\right]$
Unit Function	1	Delta Function	$\delta(s)$
Constant	a	Delta Function	$a\delta(s)$
Delta Function	$\delta(t)$	Unit Function	1
Comb	$\delta(t \bmod k)$	Comb	$\delta(s \bmod 1/k)$

More Common Fourier Transform Pairs

Spatial Domain $f(t)$		Frequency Domain $F(s)$	
Square Pulse	1 if $-a/2 \le t \le a/2$; 0 otherwise	Sinc Function	$\mathrm{sinc}(a\pi s)$
Triangle	$1-\lvert t\rvert$ if $-a \le t \le a$; 0 otherwise	Sinc Squared	$\mathrm{sinc}^2(a\pi s)$
Gaussian	$e^{-\pi t^2}$	Gaussian	$e^{-\pi s^2}$
Differentiation	$\frac{d}{dt}$	Ramp	$2\pi i s$

Properties: Notation

Let $\mathcal{F}$ denote the Fourier Transform:

$$F = \mathcal{F}(f)$$

Let $\mathcal{F}^{-1}$ denote the Inverse Fourier Transform:

$$f = \mathcal{F}^{-1}(F)$$

Properties: Linearity

Adding two functions together adds their Fourier Transforms together:

$$\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$$

Multiplying a function by a scalar constant multiplies its Fourier Transform by the same constant:

$$\mathcal{F}(af) = a \;\; \mathcal{F}(f)$$

Properties: Translation

Translating a function leaves the magnitude unchanged and adds a constant to the phase.

If

$$\begin{aligned} f_2 &= f_1(t-a) \\ F_1 &= \mathcal{F}(f_1) \\ F_2 &= \mathcal{F}(f_2) \end{aligned}$$

then

$$\begin{aligned} |F_2| &= |F_1| \\ \phi(F_2) &= \phi(F_1) - 2\pi s a \end{aligned}$$

Intuition: magnitude tells you ”how much”, phase tells you ”where”.

Change of Scale

Frequency and distance (period) are inversely proportional.

So, if

$$\begin{aligned} f_2 &= f(at) \\ F_1 &= \mathcal{F}(f_1) \\ F_2 &= \mathcal{F}(f_2) \end{aligned}$$

then

$$F_2(s) = F(s/a)$$

Rayleigh's Theorem

Total "energy" (sum of squares) is the same in either domain:

$$\int_{-\infty}^{\infty} |f(t)|^2 \; dt = \int_{-\infty}^{\infty} |F(s)|^2 \; ds$$

Linear Systems and Responses

	Time/Spatial	Frequency
Input	f	F
Output	g	G
Impulse Response	h	
Transfer Function		H
Relationship	g = f * h	G = F H

The Convolution Theorem

Let F, G, and H denote the Fourier Transforms of signals f, g, and h respectively.

$g = f * h$	$g = fh$
implies	implies
$G = FH$	$G = F * H$

Convolution in one domain is multiplication in the other and vice versa.

Convolution Theorem

Thus,

$$\mathcal{F}(f(t) * g(t)) = \mathcal{F}(f(t))\mathcal{F}(g(t))$$

Likewise,

$$\mathcal{F}(f(t)g(t)) = \mathcal{F}(f(t)) * \mathcal{F}(g(t))$$

System Characterization

We can measure the transfer function by comparing the frequencies of the input and output signals:

$$H = F/G$$

Transfer Functions

Expressing $H(s)$ in polar (magnitude-phase) form:

$$H(s) = A(s)e^{i\phi(s)}$$

Recall that the magnitudes multiply and the phases add:

$$H(s)e^{i2\pi st} = A(s)e^{i2\pi s(t+\phi(s)}$$

$A(s)$ is the **Modulation Transfer Function (MTF)**

$\phi(s)$ is called the **Phase Transfer Function (PTF)**

The MTF and PTF are simply the magnitude and phase of the transfer function.

Active vs. Passive Systems

Systems can also be categorized by whether they diminish or amplify components:

Passive systems do not use energy, hence they only diminish signals, not amplify them:

$$|H(s)| \leq 1$$

Active systems use energy and can amplify signals:

$$|H(s)| \geq 1$$

Types of Systems

Systems can be characterized by the shape of their MTF:

Low-pass	lets low frequencies through better than high ones
High-pass	lets high frequencies through better than low ones
Band-pass	lets a particular range of frequencies through better than others