Linearity (Revisited)

A function f is **linear** iff

$$f(ax + by) = af(x) + bf(y)$$

1. $f(ax) = af(x)$
2. $f(x + y) = f(x) + f(y)$

Multiplying an Input to a Linear Transformation

This also means that

$$x(t) \quad \longrightarrow \quad y(t)$$

$$a\ \ x(t) \quad \longrightarrow \quad a\ \ y(t)$$

Applying a linear operation to an signal multiplied by a constant is the same as applying the transformation and then multiplying by that constant.

Adding Inputs to a Linear Transformation

This means that

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) \\ x_2(t) &\rightarrow y_2(t) \end{aligned}$$

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

Applying a linear operation to the sum of two signals is the same applying it to each separately and adding the results.

Systems

Linearity and shift invariance are nice properties for a signal-processing operation to have.

- input devices
- output devices
- processing.

A transformation that is linear and shift invariant is called a **system**.

Impulses

One way of probing what a system does is to test it on a single input point (a single spike in the signal, a single point of light, etc.)

Mathematically, a perfect single-point input is written as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\ dt = 1$$

This is called the Dirac **delta function**.

More on Impulse Functions

Multiplying a delta function by a constant multiplies the integrated area:

$$\int_{-\infty}^{\infty} a \;\; \delta(t) \;\; dt = a$$

Impulse Responses

Because a system is shift-invariant, it responds the same everywhere:

$$\delta(t) \rightarrow h(t)$$

implies

$$\delta(t+T) \rightarrow h(t+T)$$

This response $h(t)$ is called the **impulse response**.

Impulse Responses

Because a system is linear, the response to a multiplied impulse is the same multiple of the response:

$$\delta(t) \rightarrow h(t)$$

implies

$$a\ \ \delta(t) \rightarrow a\ \ h(t)$$

Impulse Responses

Because a system is linear, the response to two impulses is the same as the sum of the two responses to them individually:

$$\begin{aligned} \delta(t) &\rightarrow h(t) \\ \delta(t+T) &\rightarrow h(t+T) \end{aligned}$$

implies

$$\delta(t) + \delta(t+T) \rightarrow h(t) + h(t+T)$$

Impulse Responses

Putting it all together:

$$\delta(t) \rightarrow h(t)$$

implies

$$a\ \ \delta(t) + b\ \ \delta(t+T) \rightarrow a\ \ h(t) + b\ \ h(t+T)$$

Implication: If you know the impulse response at any point, you know everything there is to know about the system!

Harmonics and Systems

$$x_1(t) = e^{i\omega t}$$
$$x_1(t) \rightarrow y_1(t)$$

$$y_1(t) = K(\omega, t)\ \ x_1(t)$$

(Just define $K(\omega, t) = \frac{y_1(t)}{x_1(t)}$)

Shifted Input Harmonic

$$
\begin{aligned}
x_2(t) &= e^{i\omega(t-T)} \\
x_2(t) &\rightarrow y_2(t)
\end{aligned}
$$

$$y_2(t) = K(\omega, t-T)\ x_2(t)$$

Shifted Input Harmonic

$$
\begin{aligned}
x_2(t) &= e^{i\omega(t-T)} \\
&= e^{i\omega t} e^{-i\omega T} \\
&= x_1(t) e^{-i\omega T}
\end{aligned}
$$

$$
x_2(t) \longrightarrow y_1(t) e^{-i\omega T}
$$

$$
\begin{aligned}
y_2(t) &= y_1(t) e^{-i\omega T} \\
&= K(\omega, t)\ \ x_1(t)\ \ e^{-i\omega T} \\
&= K(\omega, t)\ \ x_2(t)
\end{aligned}
$$

Shifted Input Harmonic

So, we have both

$$
\begin{aligned}
y_2(t) &= K(\omega, t)\ x_2(t) \\
y_2(t) &= K(\omega, t - T)\ x_2(t)
\end{aligned}
$$

Thus

$$K(\omega, t) = K(\omega, t - T)$$

So, K is just a constant function with respect to t:

$$K(\omega)$$

Harmonics and Systems

So, for any harmonic function

$$\begin{aligned} x(t) &= e^{i\omega t} \\ x(t) &\rightarrow y(t) \end{aligned}$$

$$y(t) = K(\omega)\ \ x(t)$$

When a system is applied to a harmonic signal, the result is the same harmonic signal multiplied by a constant that depends on the frequency.

Transfer Functions

We now have a second way to characterize systems: the function $K(\omega)$ that defines the degree to which harmonic inputs transfer to the output.

This is the **transfer function**.

Transfer Functions

Expressing $K(\omega)$ in polar (magnitude-phase) form:

$$K(\omega) = A(\omega)e^{i\phi(\omega))}$$

Recall that the magnitudes multiply and the phases add:

$$K(\omega)e^{i\omega t} = A(\omega)e^{i\omega(t+\phi(\omega))}$$

$A(\omega)$ is called the **Modulation Transfer Function (MTF)**

$\phi(\omega)$ is called the **Phase Transfer Function (PTF)**

The MTF and PTF are simply the magnitude and phase of the transfer function.

Impulse Response

Remember that we can entirely characterize a system by its *impulse response*:

$$\delta(t) \rightarrow h(x)$$

Problem: given an input signal $x(t)$, how do I determine the output $y(t)$?

Linearity and Shift Invariance

Because a system is linear:

$$a\ \ \delta(t) \rightarrow a\ \ h(x)$$

Because a system is shift invariant:

$$\delta(t-k) \rightarrow h(t-k)$$

Response to an Entire Signal

A signal $x(t)$ can be thought of as the sum of a whole lot of weighted shifted impulse functions:

$$x(t) = \int_{-\infty}^{\infty} a(\tau)\ \ \delta(t-\tau)\ \ d\tau$$

where

$\delta(t-\tau)$ is a delta function at τ

$a(\tau)$ is the weight of that delta function

(Read the integral simply as “summation”.)

Response to an Entire Signal

Because of linearity, each impulse goes through the system separately:

$$a(\tau)\ \ \delta(t-\tau) \rightarrow a(\tau)\ \ h(t-\tau)$$

This means

$$\int_{-\infty}^{\infty} a(\tau)\ \ \delta(t-\tau)\ \ d\tau \quad \rightarrow \quad \int_{-\infty}^{\infty} a(\tau)\ \ h(t-\tau)\ \ d\tau$$

Response to an Entire Signal

But isn't the weight $a(\tau)$ of the delta function at τ just $x(\tau)$?

So,

$$x(t) \;\; \rightarrow \;\; \int_{-\infty}^{\infty} x(\tau) \;\; h(t-\tau) \;\; d\tau$$

This operation is called the **convolution** of x and h.

Convolution

Convolution of input $x(t)$ by the impulse response $h(t)$ is written as

$$x(t) * h(t)$$

where

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \; h(t - \tau) \; d\tau$$

Response to an Entire Signal

So, the response of a system with impulse response $h(t)$ to input $x(t)$ is simply the convolution of $x(t)$ and $h(t)$:

$$x(t) \rightarrow x(t) * h(t)$$

Convolution of Discrete Functions

For a discrete function $x[n]$ and impulse response $h[n]$:

$$x[n] * h[n] = \sum_k x[k] \;\; h[n-k]$$

One Way to Think of Convolution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\ \ h(t-\tau)\ \ d\tau$$

$$x[n] * h[n] = \sum_{k} x[k]\ \ h[n-k]$$

Think of it this way:

- shift a copy of h to each position τ (or discrete position k)
- multiply by the value at that position $x(\tau)$ (or discrete sample $x[k]$)
- add shifted, multiplied copies for all τ (or discrete k)

Convolution - Another Way To Look At It

$$x[n] * h[n] = \sum_k \; f[k] \; h[n-k]$$

Think of it this way:

- flip the function h around zero
- shift a copy to output position n
- point-wise multiply for each position k the value of the function f and the *shifted inverted* copy of h
- add for all k and write that value at position n

Why Flip the Impulse Function?

An input at t produces a response at $t + \tau$ of $h(\tau)$.

Suppose I want to determine the output at t.
What effect does the input at $t + \tau$ have on t?

$$h(-\tau)$$

Convolution in Two Dimensions

In one dimension:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\ \ h(t-\tau)\ \ d\tau$$

In two dimensions:

$$I(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(\tau_x,\tau_y)\ \ h(x-\tau_x, y-\tau_y)\ \ d\tau_x\ \ d\tau_y$$

Or in discrete form:

$$I[x,y] * h[x,y] = \sum_j \sum_k I(j,k)\ \ h(x-j, y-k)$$

Properties of Convolution

- Commutative
- Associative
- Distributes over addition

In simple terms, convolution has the same mathematical properties as multiplication.

(This is no coincidence.)

Useful Functions

- Square
- Triangle
- Gaussian
- Step
- Impulse/Delta
- Comb (Shah Function)

Each has their two-dimensional equivalent.

Square

$$\Pi_a(t) = \begin{cases} 1 & \text{if } -a \leq t \leq a \\ 0 & \text{otherwise} \end{cases}$$

Triangle

$$\Lambda_a(t) = \begin{cases} 1 - |t/a| & \text{if } -a \leq t \leq a \\ 0 & \text{otherwise} \end{cases}$$

Gaussian

Gaussian (maximum value = 1)

$$G(t,\sigma) = e^{-t^2/2\sigma^2}$$

Normalized Gaussian (area = 1)

$$G(t,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

Convolving a Gaussian with another Gaussian:

$$G(t,\sigma_1) * G(t,\sigma_2) = G(t, \sqrt{\sigma_1^2 + \sigma_2^2})$$

Step Function

$$\text{Step}(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the derivative of a step function?

Impulse Function / Delta Function

We've seen the delta function before:

$$\delta(t) \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) \; dt = 1$$

Shifted delta function (impulse at $t = k$):

$$\delta(t - k)$$

What is a function $f(t)$ convolved with $\delta(t)$?

What is a function $f(t)$ convolved with $\delta(t - k)$?

Comb (Shah Function)

A set of equally-spaced impulses (sometimes called an *impulse train*)

$$\text{comb}_h(t) = \sum_k \delta(t - hk)$$

h is the spacing.

What is $f(t) * \text{comb}_h(t)$?