SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

FREE MODES OF THE EARTH

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Surface waves in an elastic half spaces: Rayleigh waves

- Potentials
- Free surface boundary conditions
- Solutions propagating along the surface, decaying with depth
- Lamb's problem

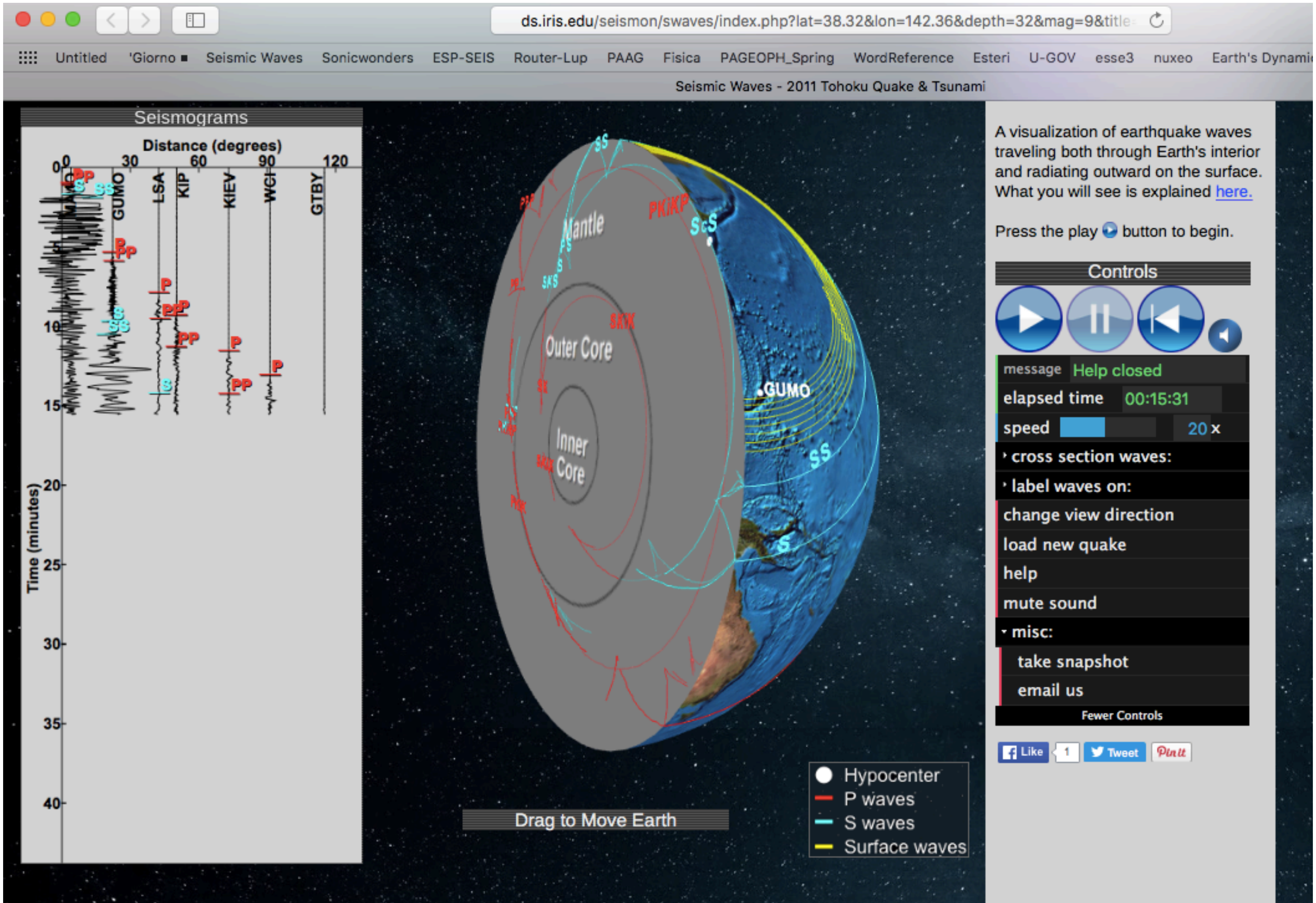
Surface waves in media with depth-dependent properties: Love waves

- Constructive interference in a low-velocity layer
- Dispersion curves
- Phase and Group velocity

Free Oscillations

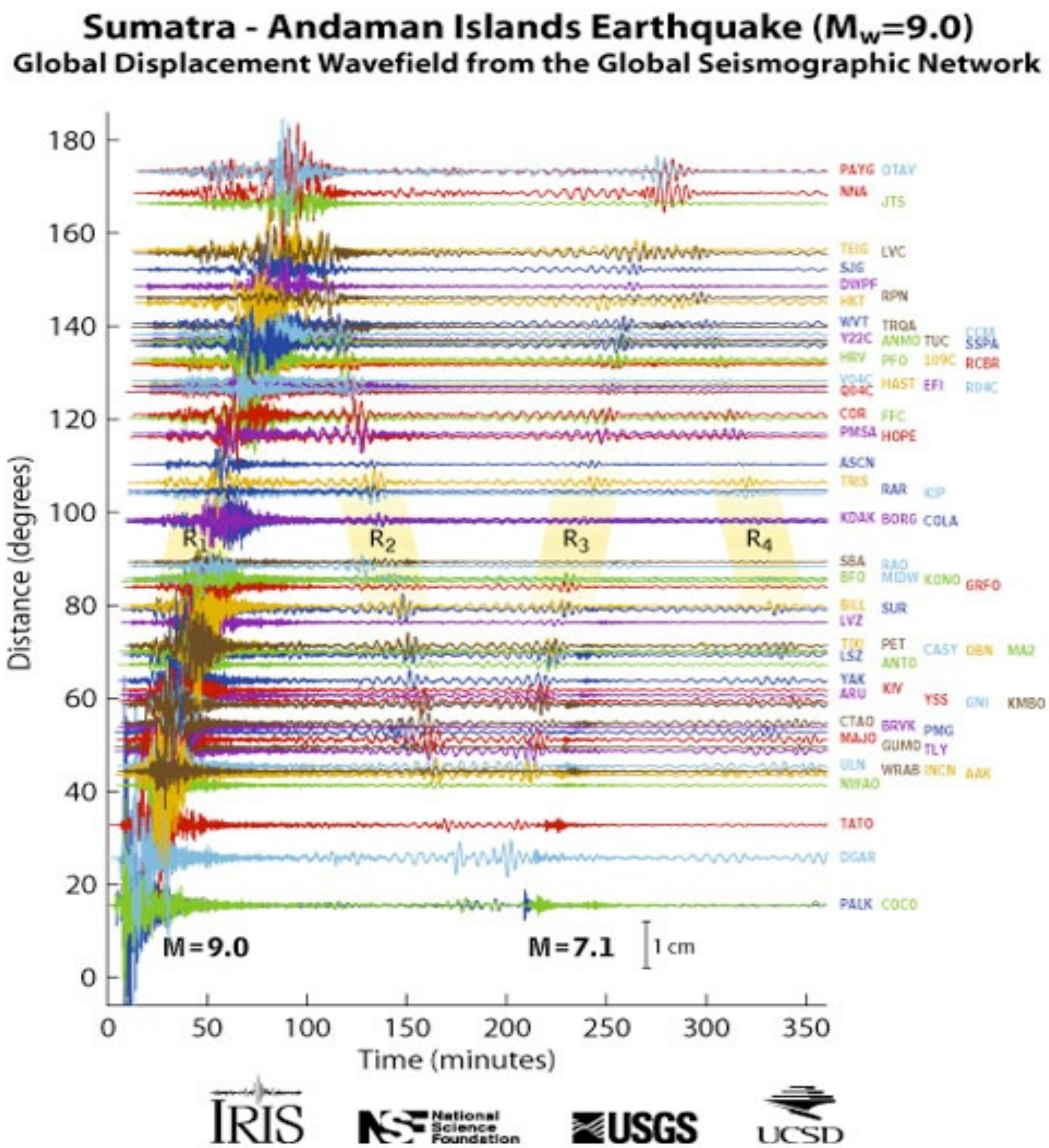
- Spherical Harmonics
- Modes of the Earth
- Rotational Splitting

Traveling surface waves



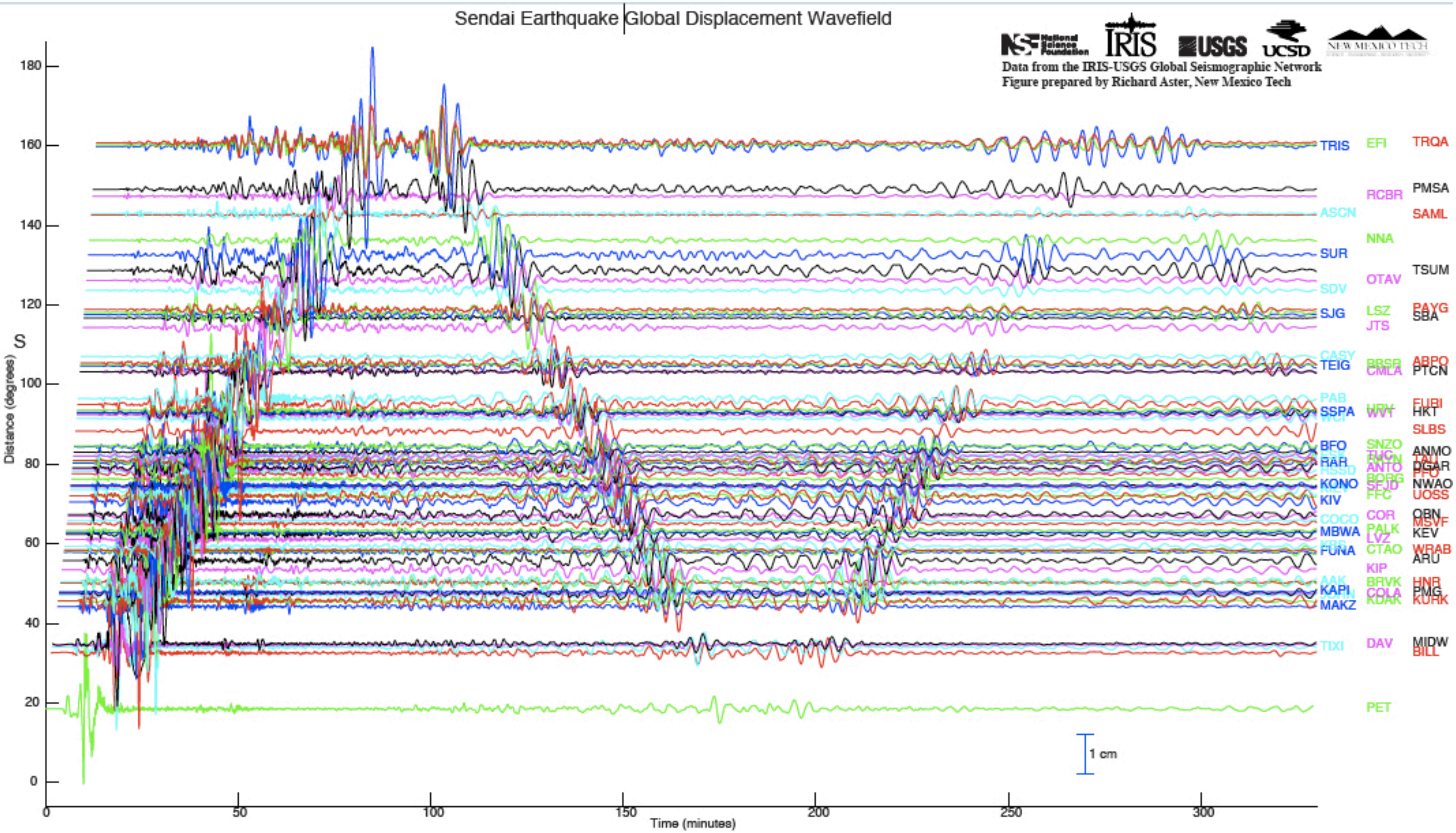
http://ds.iris.edu/seismon/swaves/index.php

Traveling surface waves



Vertical displacements of the Earth's surface recorded by seismometers.
The traces are arranged by distance from the epicenter in degrees. The earliest, lower amplitude, signal is that of the compressional (P) wave, which takes about 22 minutes to reach the other side of the planet (the antipode).
The largest amplitude signals are seismic surface waves that reach the antipode after about 100 minutes. The surface waves can be clearly seen to reinforce near the antipode (with the closest seismic stations in Ecuador), and to subsequently circle the planet to return to the epicentral region after about 200 minutes.
A major aftershock (magnitude 7.1) can be seen at the closest stations starting just after the 200 minute mark (note the relative size of this aftershock, which would be considered a major earthquake under ordinary circumstances, compared to the mainshock).

Traveling surface waves



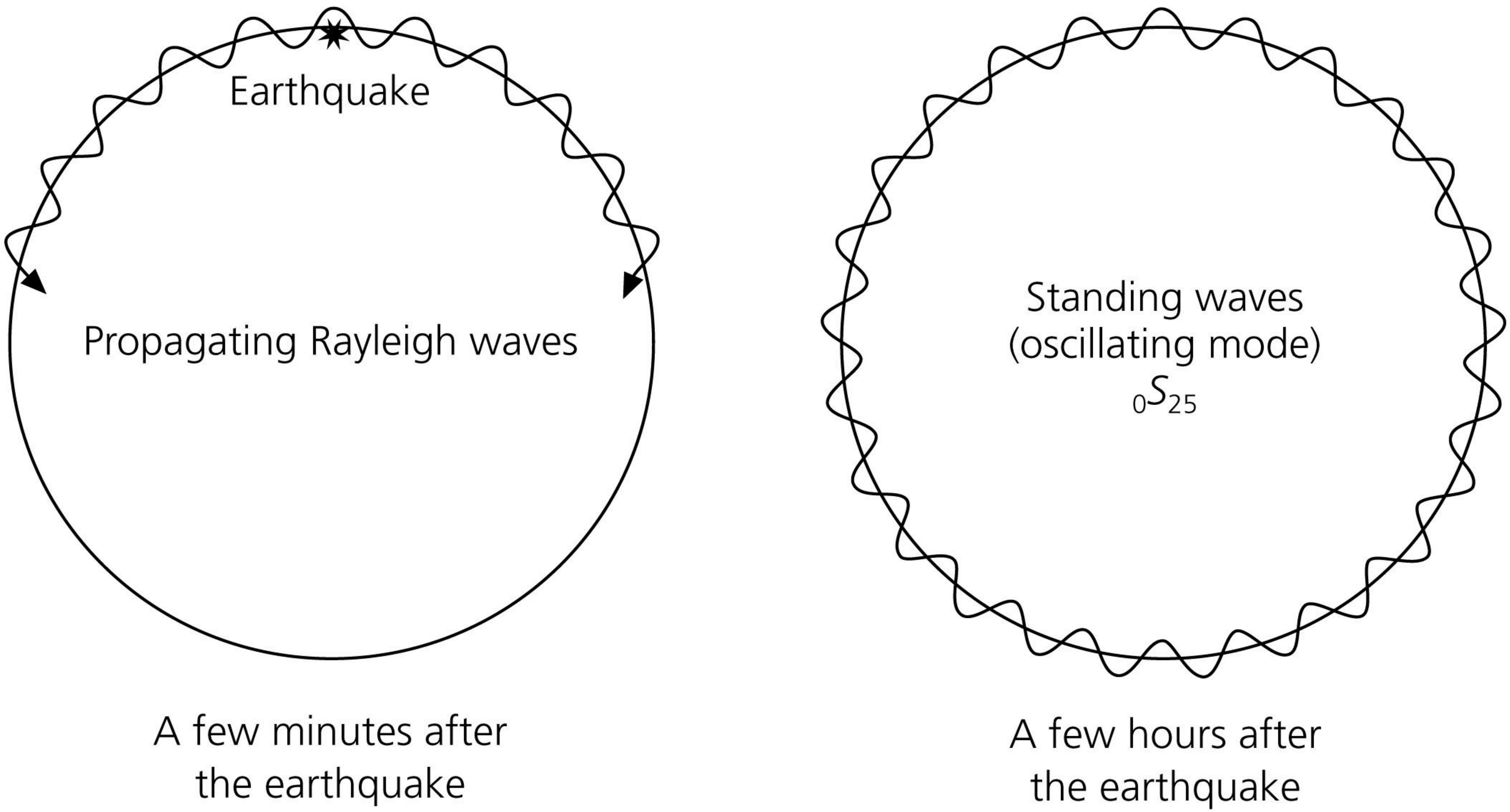
Traveling surface waves

http://www.princeton.edu/geosciences/tromp/index.xml

http://global.shakemovie.princeton.edu

Surface waves and free modes

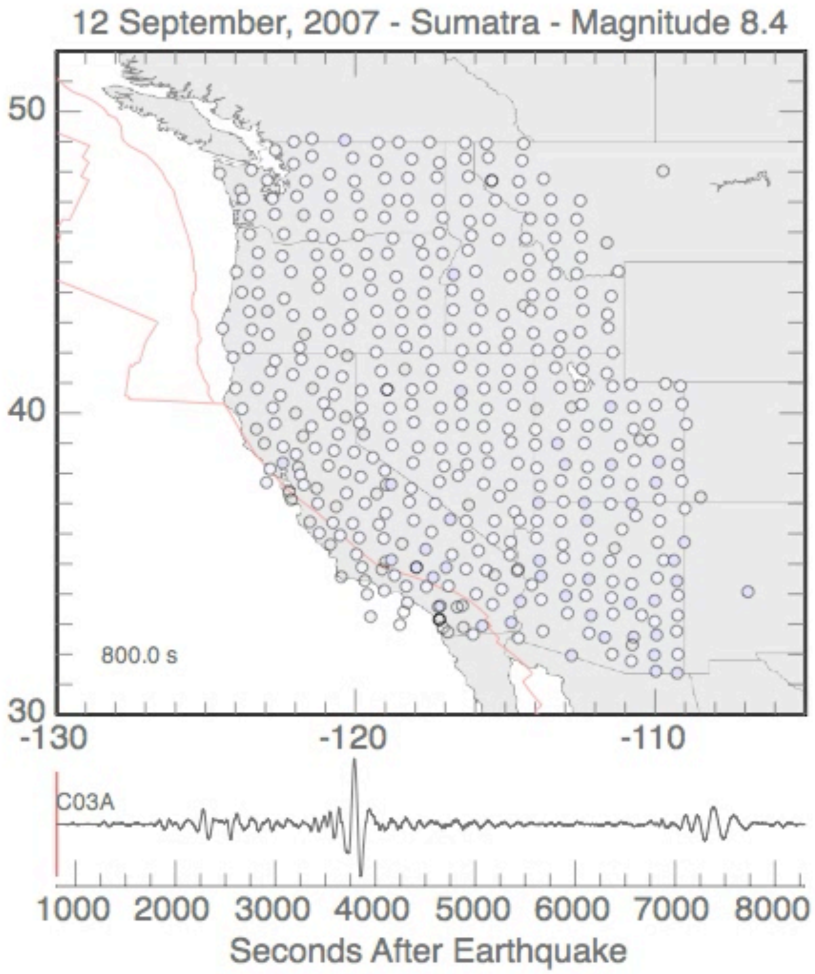
Figure 2.9-8: Cartoon of the equivalence of surface waves and normal modes.



https://jbrussell.github.io/other/normal_modes/

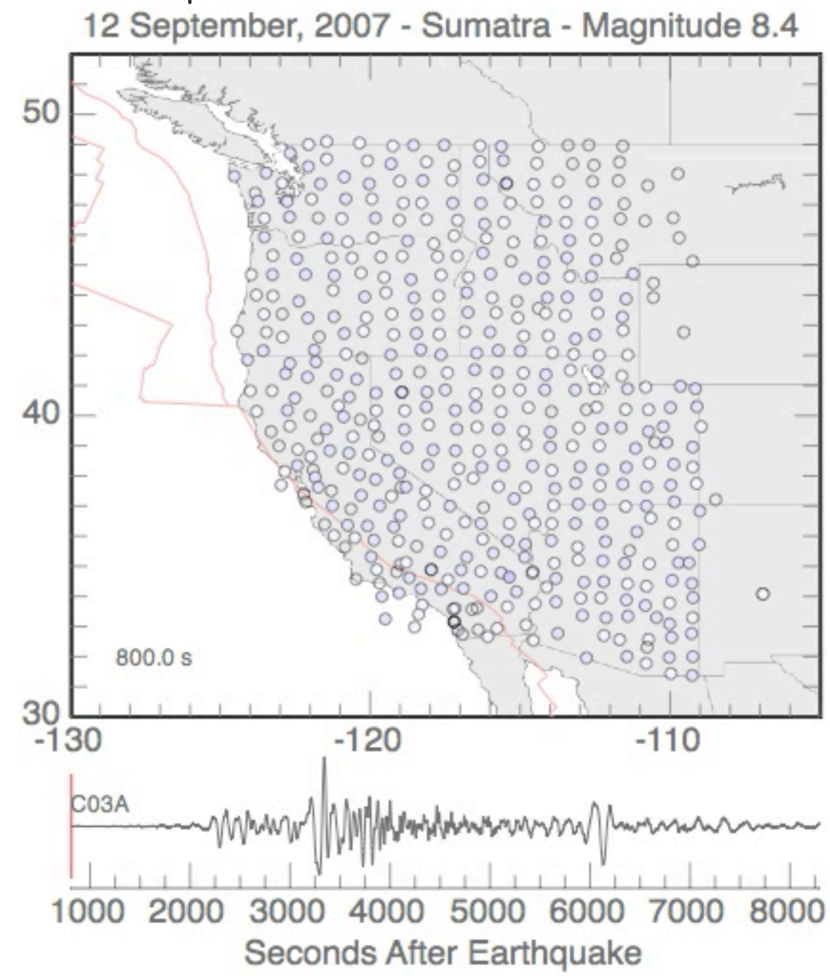
Traveling and standing waves

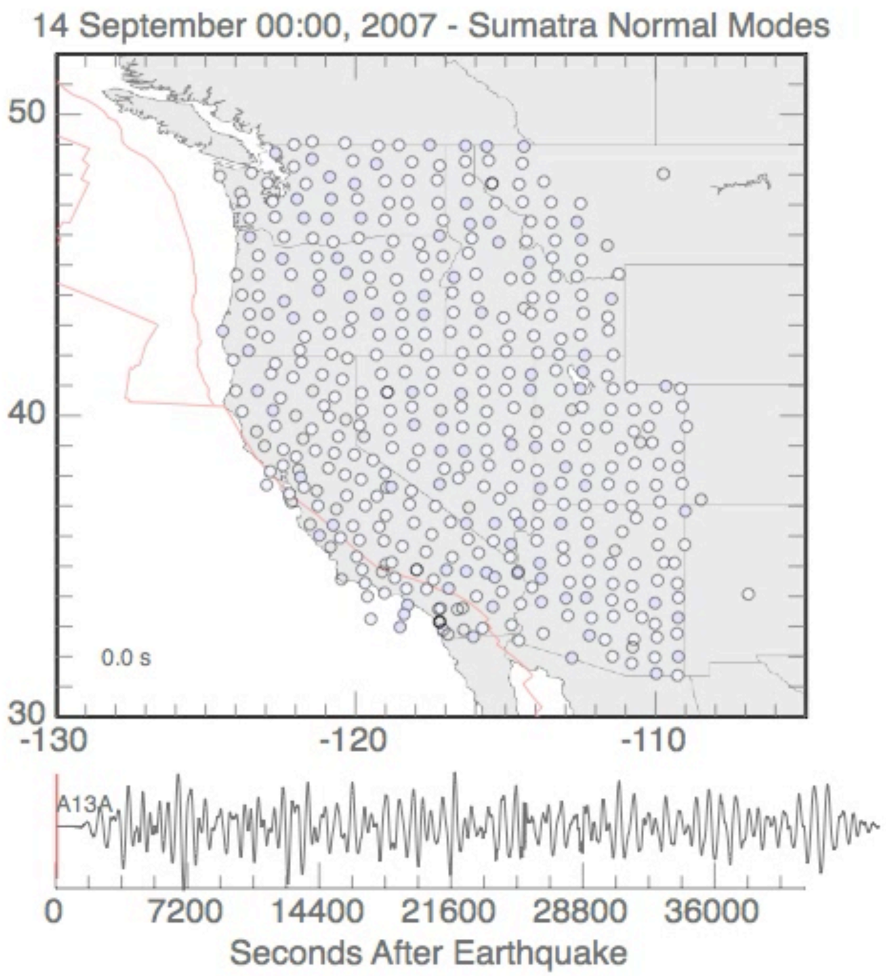
Periods from 200s-50s-R1-to-R2



Vertical ground velocity in the 750-50s period range observed across the western United States by the EarthScope transportable array and generated by the great 12 September, 2007 earthquake offshore southern Sumatra. Each circle represents a seismometer and the colors change to reflect variations in the signal amplitude crossing the array. Near the end of this animation you can see the waves that traveled the long way around Earth to reach the western United States (they propagate from NW to SE). Station 319A is located at the Douglas, AZ.

Tcomp: Periods from 200s-50s-G1-to-G2





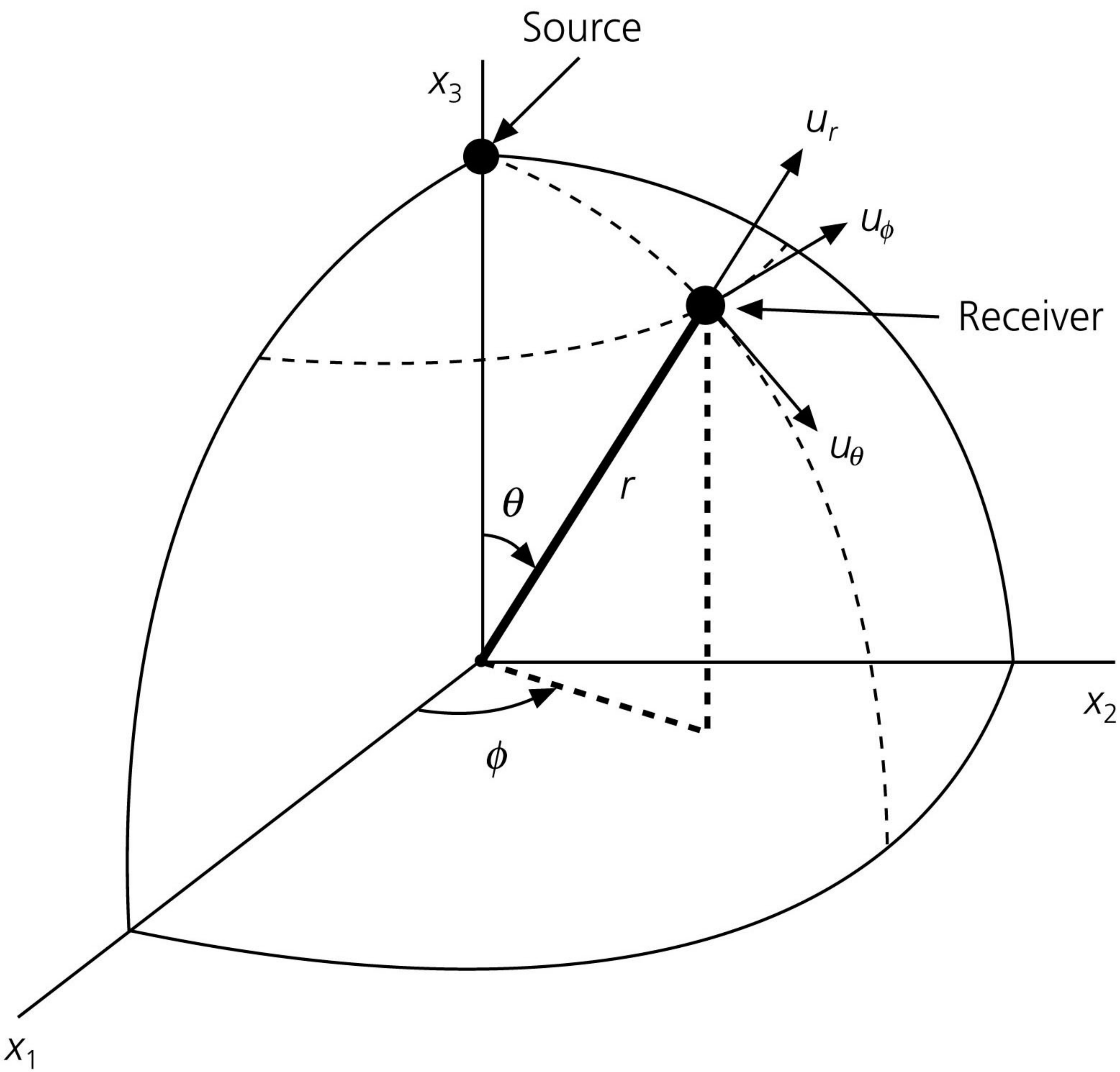
If you watch closely, you'll see the waves from the 06:01:34 Magnitude 6.4 aftershock sweep through. This movie starts several just over one day after the Mw 8.4 earthquake (the horizontal axis label is incorrect). The time step for this longer animation is 20 seconds per frame. Each second of this animation represents almost 7 minutes. I didn't screen the data so some seismic stations with glitches more or less have large amplitudes throughout the animation. The amplitude scale for this animation is about 1000 times smaller than the main-shock animations.

http://eqseis.geosc.psu.edu/~cammon/QA/sumatra12Sep2007.html

Seismic waves

Seismic domain	Wavefield type		Data	Application	Boundary conditions
Body waves	P-SV	SH	Travel times Waveforms (>1 Hz)	Local, regional tomography	Unbounded Free surface Interfaces
Surface waves	Rayleigh	Love	Dispersion Waveforms (0.05-1 Hz)	Local, regional tomography Crustal, lithospheric	Free surface Interfaces **Flat** geometry
Normal modes	Spheroidal	Torsional	Power spectra (mHz)	Global seismology	Free surface Interfaces **Spherical** geometry

Figure 2.9-1: Spherical coordinate geometry for normal modes.



Wave equation & Laplacian

- **Wave equation**

$$v^2\nabla^2 \mathbf{u} = v^2\Delta \mathbf{u} = \mathbf{u}_{tt}$$

- **Laplacian in Spherical system**

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Separation of variables

$$u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$$

$$\Phi''(\phi) + m^2\Phi(\phi) = 0$$

$$\Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$$

m is a positive integer

$$T''(t) + c^2k^2T(t) = 0$$

$$T(t) = A\cos(\omega t) + B\sin(\omega t)$$

$\omega = ck$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{1}{r^2}\frac{\partial}{dr}\left(r^2\frac{dR}{dr}\right) + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = 0$$

Legendre polynomials

Spherical harmonics: defined by an orthogonal set of functions called Legendre Polynomials

$\theta =$ angular distance from the pole (colatitude)
$\phi =$ azimuth around the pole (longitude)

Legendre polynomials: $$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

$l =$ *degree*, or *angular order*

The first several polynomials are

$P_0(x) = 1$

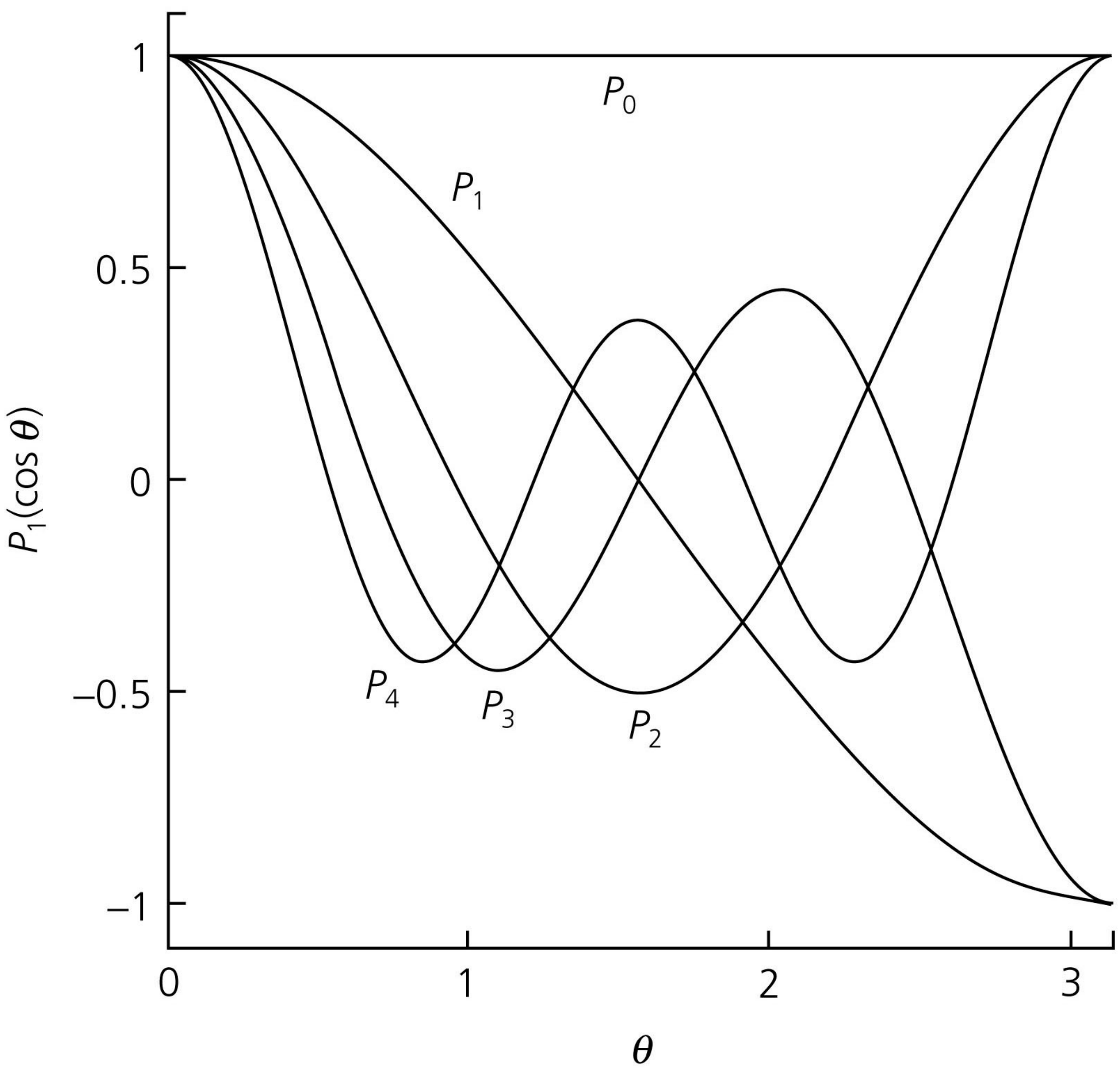
$P_1(x) = x$

$P_2(x) = (1/2)(3x^2 - 1)$

$P_3(x) = (1/2)(5x^3 - 3x)$

On a sphere, $x = \cos\theta$ so x ranges from $-1 \le x \le 1$

Figure 2.9-3: Examples of Legendre polynomials.



Spherical harmonics

The azimuthal variations are included by forming the *Associated Legendre functions,*

$$P_l^m(x) = \left[\frac{(1 - x^2)^{m/2}}{2^l l!} \right] \left[\frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l \right]$$

(the *azimuthal order*, m, varies over $-l \le m \le l$)

Fully normalized spherical harmonics:

$$Y_l^m(\theta, \phi) = (-1)^m \left[\left(\frac{2l + 1}{4\pi} \right) \frac{(l - m)!}{(l + m)!} \right]^{1/2} P_l^m(cos\theta) e^{im\phi}$$

Spherical harmonics

In mathematics, the **spherical harmonics** are the angular portion of an orthogonal set of solutions to Laplace's equation represented in a system of spherical coordinates.

Spherical harmonics are orthogonal:

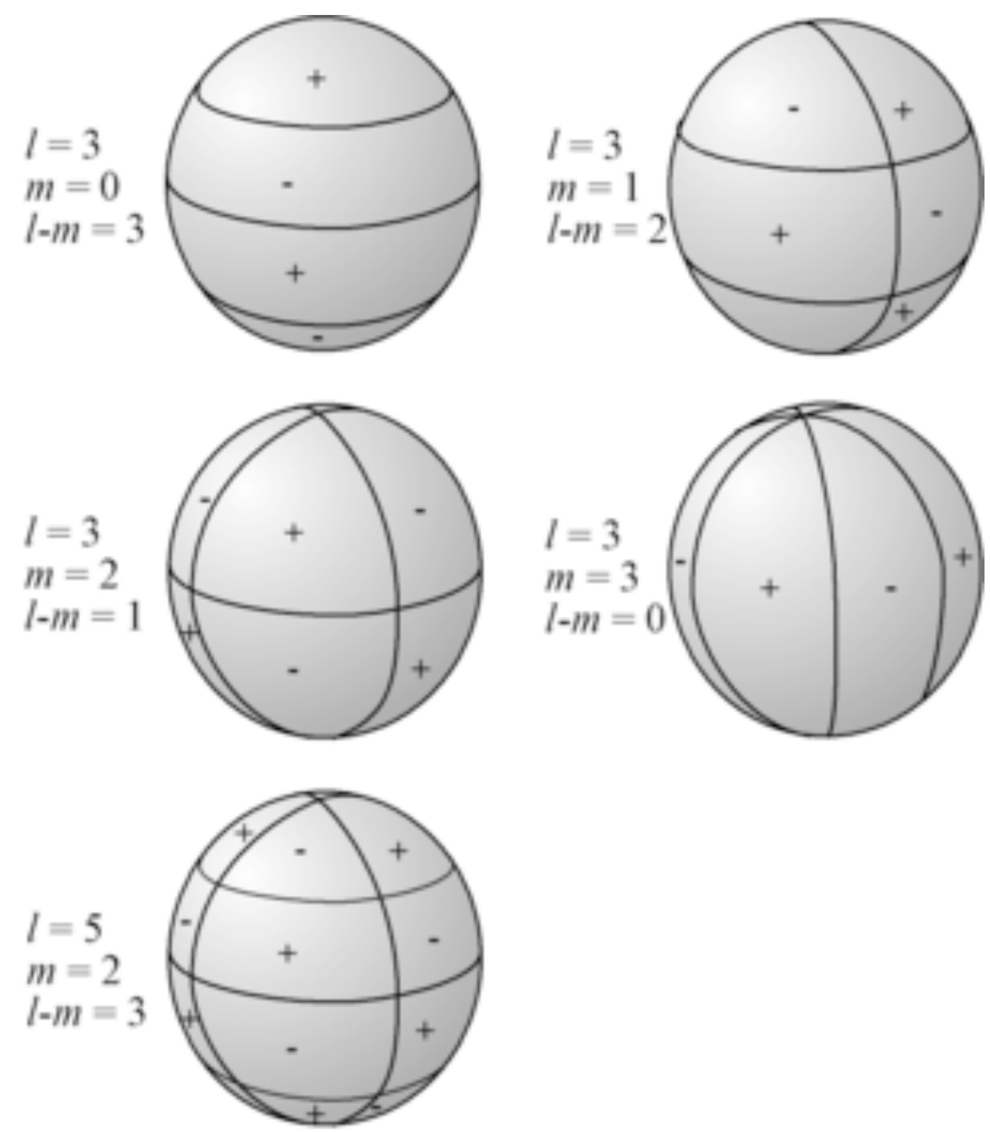
$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \; Y_{l'}^{m'*}(\theta,\phi)\; Y_l^m(\theta,\phi)\; d\theta d\phi = \delta_{l'l}\delta_{m'm}$$

The spherical harmonics are easily visualized by counting the number of zero crossings they possess in both the latitudinal and longitudinal directions. For the latitudinal direction, the associated Legendre functions possess l – |m| zeros, whereas for the longitudinal direction, the trigonomentric sin and cos functions possess 2 |m| zeros.

When the spherical harmonic order m is zero, the spherical harmonic functions do not depend upon longitude, and are referred to as **zonal**.

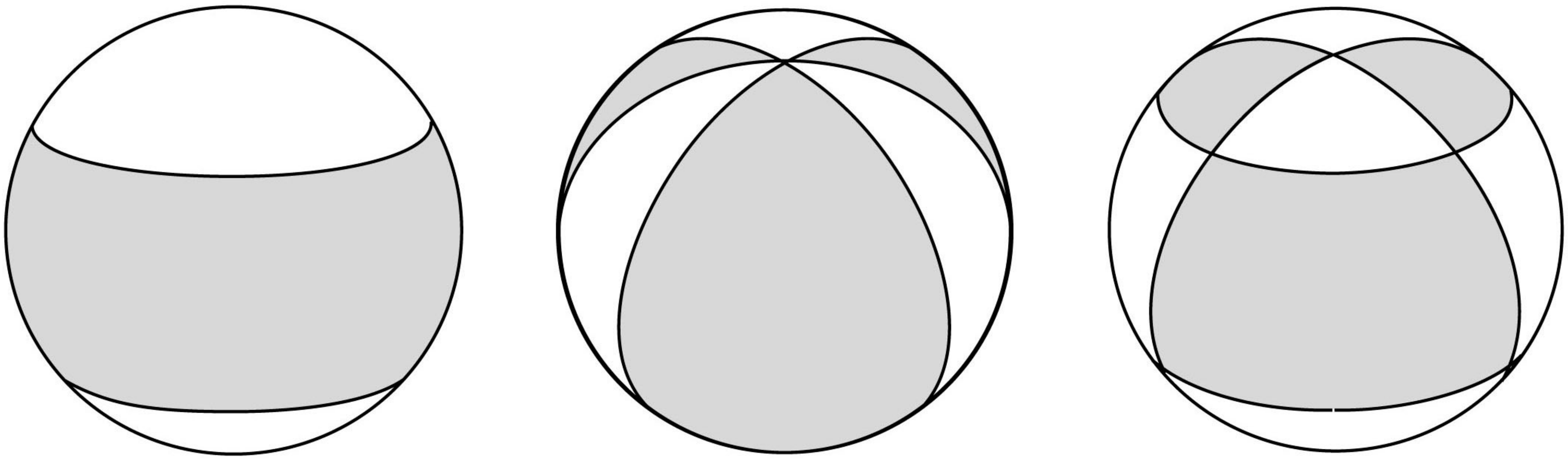
When l = |m| , there are no zero crossings in latitude, and the functions are referred to as **sectoral**.

For the other cases, the functions checker the sphere, and they are referred to as **tesseral**.



Spherical harmonics

Figure 2.9-4: Examples of spherical harmonics.



Y_2^0 $\quad$ $\mathrm{Re}(Y_3^3)$ $\quad$ $\mathrm{Re}(Y_4^2)$

$$Y_l^m(\theta, \phi) = (-1)^m \left[\left(\frac{2l+1}{4\pi} \right) \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(cos\theta) e^{im\phi}$$

The angular order, l, gives the number of nodal lines on the surface.

If the azimuthal order m is zero, the nodal lines are small circles about the pole. These are called *zonal* harmonics, and do not depend on ϕ

If $m = l$, then all of the surface nodal lines are great circles through the pole. These are called *sectoral* harmonics.

When $0 < |m| < l$, there are combined angular and azimuthal (colatitudinal and longitudinal) nodal patterns called *tesseral* harmonics.

Torsional & Spheroidal modes

Torsional modes ${}_nT^m_l$:

- No radial component: tangential only, normal to the radius: motion confined to the surface of n concentric spheres inside the Earth (**SH, Love waves**).
- Changes in the shape, not of volume
- Do not exist in a fluid: so only in the mantle (and the inner core?)

n - radial : nodal planes with depth

l - polar : # nodal planes in latitude
Max nodal planes = l - 1

m - azimuthal : # nodal planes in longitude

Spheroidal modes ${}_nS^m_l$:

- Horizontal component and vertical (radial) (**P-SV, Rayleigh waves**). No simple relationship between n and nodal spheres
- ${}_0S_2$ is the longest "fundamental"
- Affect the whole Earth (even into the fluid outer core !)

n : no direct relationship with nodes with depth

l : # nodal planes in latitude
Max nodal planes = l

m : # nodal planes in longitude

Torsional modes

Torsional (toroidal) modes:
(analogous to *SH* waves)

Surface eignefunctions given by vector spherical harmonics:

$$\mathbf{T}_l^m\,(r, \theta, \phi) = \left(0, \frac{1}{\sin\theta}\,\frac{\partial Y_l^m(\theta, \phi)}{\partial \phi}, -\frac{\partial Y_l^m(\theta, \phi)}{\partial \theta}\right)$$

The displacements are given by:

$$\mathbf{u}^T(r, \theta, \phi) = \sum_n \sum_l \sum_{m=-l}^{l} {}_nA_l^m\ {}_nW_l(r)\ \ \mathbf{T}_l^m(\theta, \phi)\ e^{i\,{}_n\omega_l^m t}$$

${}_nW_l(r)$ - The radial eigenfunction (varies with depth)

Torsional modes

For ${}_nT_l^m$:

n = radial order, l = angular order, m = azimuthal order.

The $2l + 1$ modes of different azimuthal orders $-l \le m \le l$ are called *singlets*, and the group of singlets is called a *multiplet*.

If earth were perfectly spherically symmetric and non-rotating, all singlets in a multiplet would have the same eigenfrequency (called *degeneracy*).

For example, the period of ${}_nT_l^0$ would be the same for ${}_nT_l^{\pm 1}$, ${}_nT_l^{\pm 2}$, ${}_nT_l^{\pm 3}$, etc. In the real earth, singlet frequencies vary (called *splitting*).

The splitting is usually small enough to ignore, so we drop the m superscript and refer to the entire ${}_nT_l^m$ multiplet as ${}_nT_l$, with eigenfrequency ${}_n\omega_l$.

Torsional modes

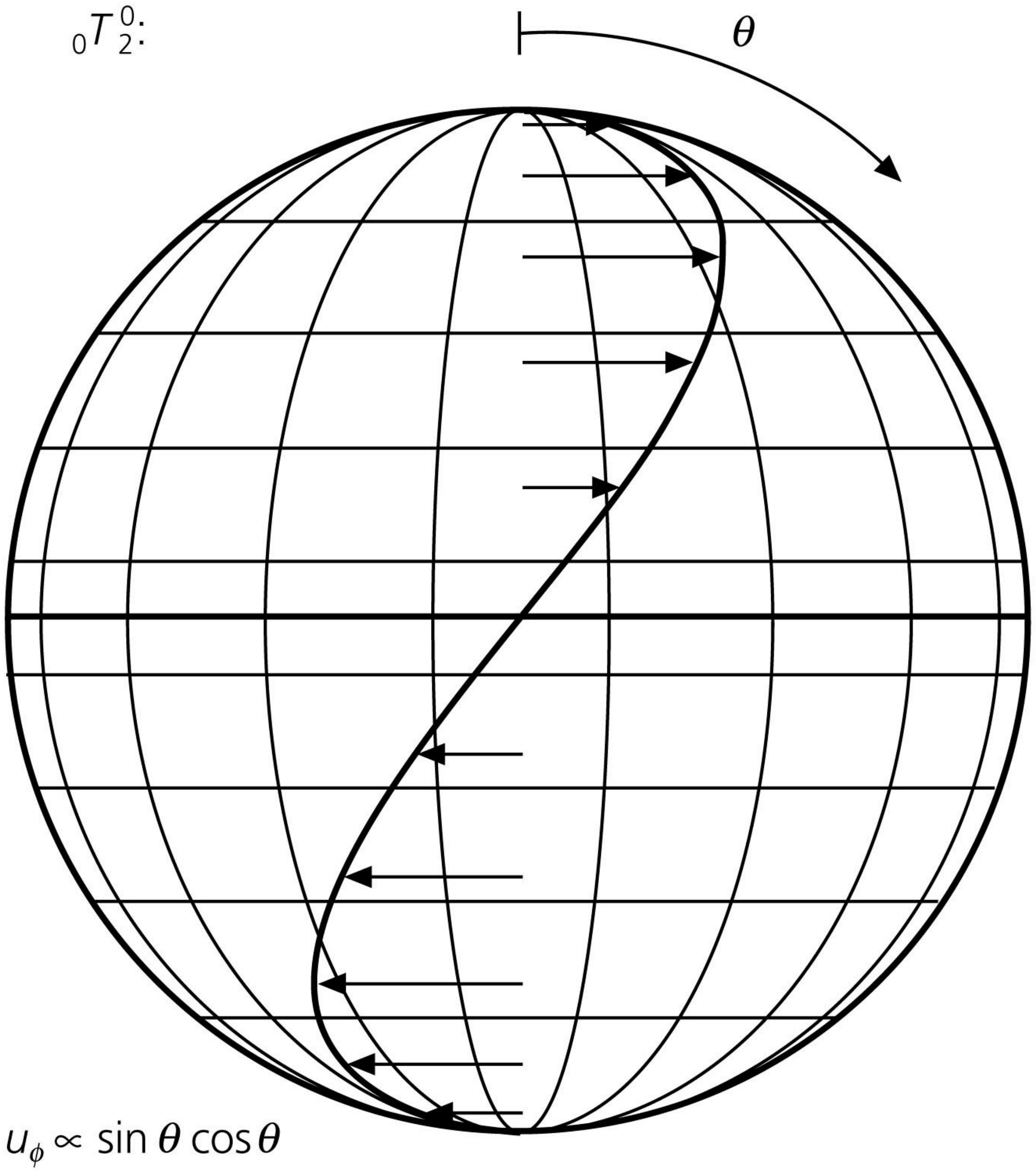
Example:

$$Y_l^m(\theta,\phi) = (-1)^m\left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(cos\theta)e^{im\phi}$$

$$\mathbf{T}_l^m = \left(0,\ \frac{1}{\sin\theta}\,\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi},\ -\frac{\partial Y_l^m(\theta,\phi)}{\partial\theta}\right)$$

$$e^{im\phi}\frac{\partial}{\partial\theta}P_2^0(\cos\theta) = 3\sin\theta\cos\theta$$

Figure 2.9-5: Displacement associated with torsional mode $_0T_2$.



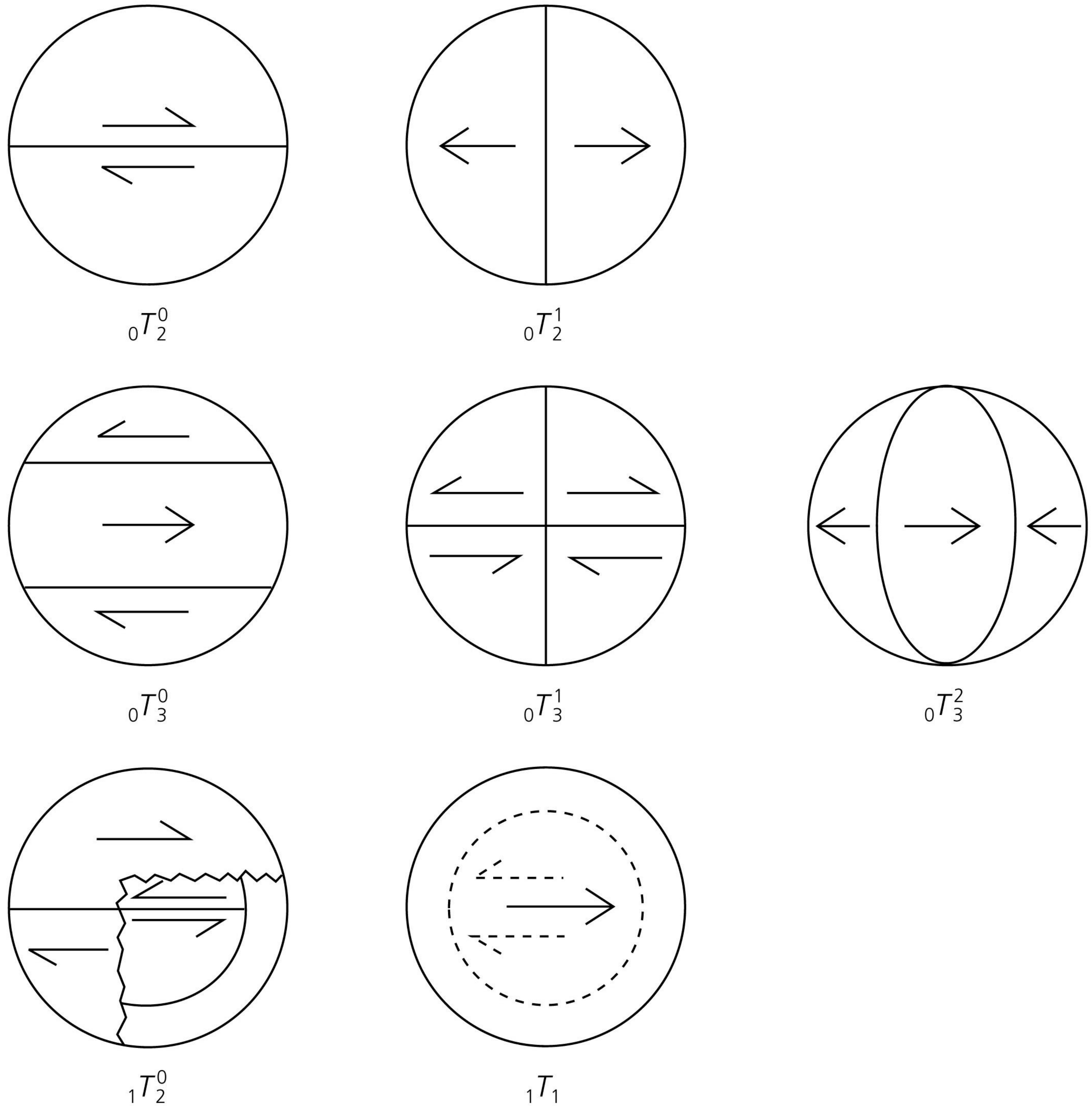
Torsional modes

Torsional modes with $n = 0$ (${}_0T_l^m$) are called *fundamental modes*. (motions at depth in the same direction as at the surface).

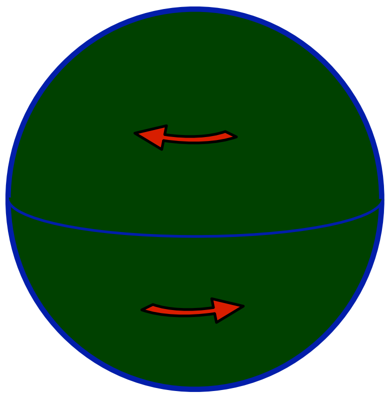
Modes with $n > 0$ are called *overtones*. (motions reverse directions at different depths)

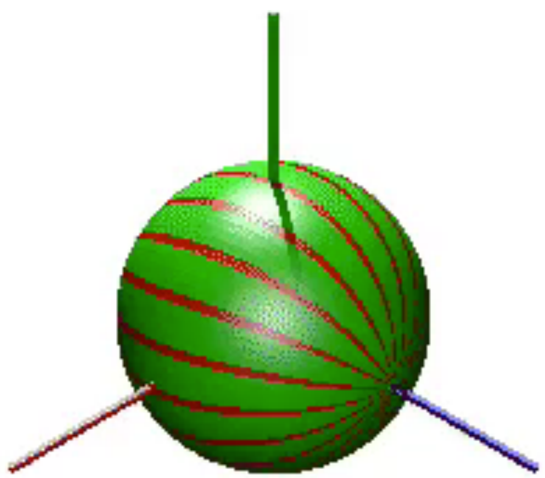
What happened to ${}_0T_1$ and ${}_0T_0$?

Figure 2.9-6: Examples of the displacements for several torsional modes.



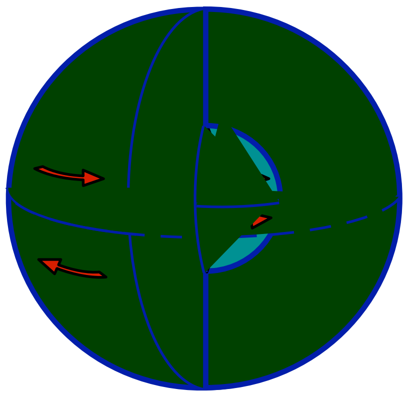
Toroidal normal modes: examples





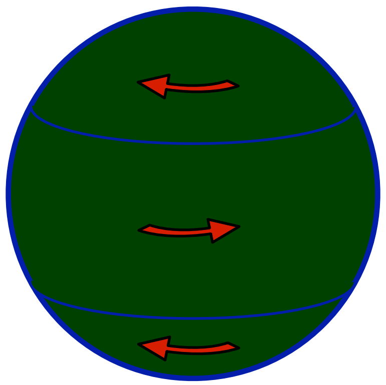
$_0T_2$: «twisting» mode

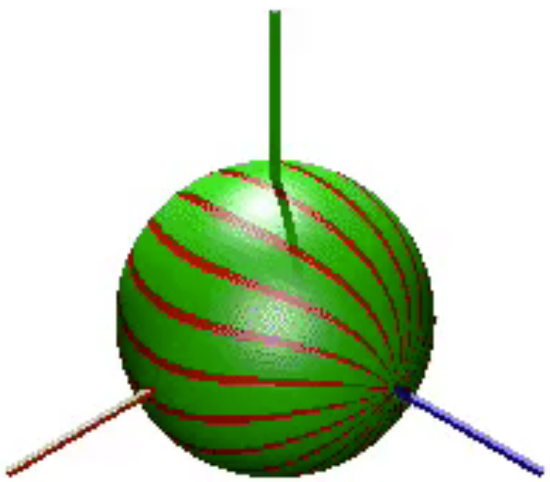
(44.2 minutes, observed in 1989 with an extensometer)



$_1T_2$

(12.6 minutes)





$_0T_3$

(28.4 minutes)

Animations from Lucien Saviot

http://lucien.saviot.free.fr/terre/index.en.html

Spheroidal modes

Spheroidal (poloidal) modes (involving *P*-*SV* motions):

The surface eigenfunctions are given by two other *vector spherical harmonics* with (r, θ, ϕ) components

$$\mathbf{R}_l^m = (Y_l^m, 0, 0)$$

$$\mathbf{S}_l^m = \left(0, \frac{\partial Y_l^m(\theta, \phi)}{\partial \theta}, \frac{1}{\sin\theta} \frac{\partial Y_l^m(\theta, \phi)}{\partial \phi}\right)$$

Each corresponds to a different radial eigenfunction, ${}_nU_l(r)$ and ${}_nV_l(r)$, so the displacement for spheroidal modes is

$$\mathbf{u}^S(r, \theta, \phi) = \sum_n \sum_l \sum_{m=-l}^{l} {}_nA_l^m \left[{}_nU_l(r)\, \mathbf{R}_l^m(\theta, \phi) + {}_nV_l(r)\, \mathbf{S}_l^m(\theta, \phi) \right] e^{i\, {}_n\omega_l^m t}$$

The radial eigenfunction ${}_nU_l(r)$ corresponds to radial motion and ${}_nV_l(r)$ corresponds to horizontal motion.

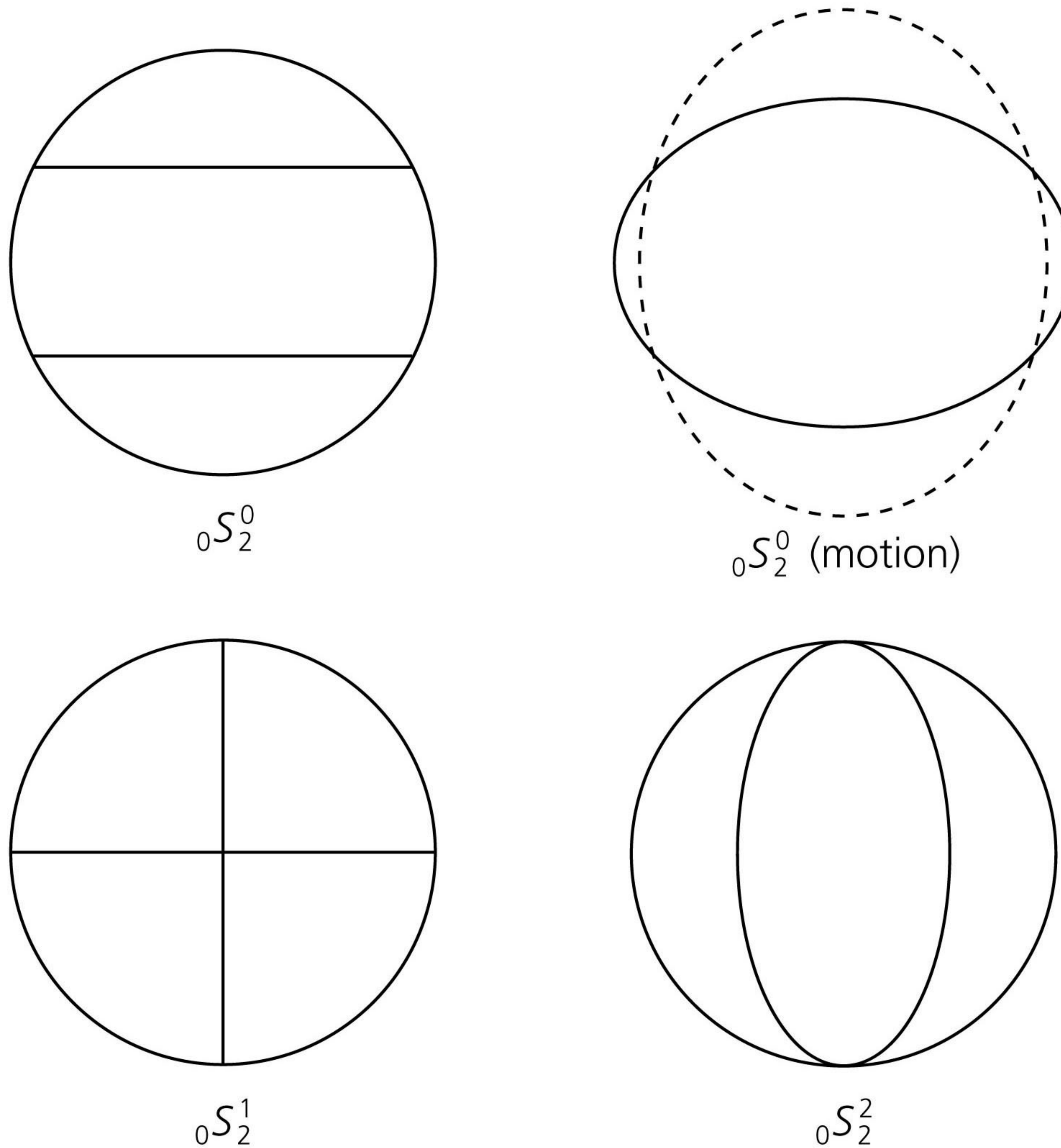
Spheroidal modes

$_0S_2$ (football mode) is the gravest (lowest frequency or longest period) of earth's modes, with a period of 3233 s, or 54 minutes.

There is no $_0S_1$ mode, which would correspond to a lateral translation of the planet.

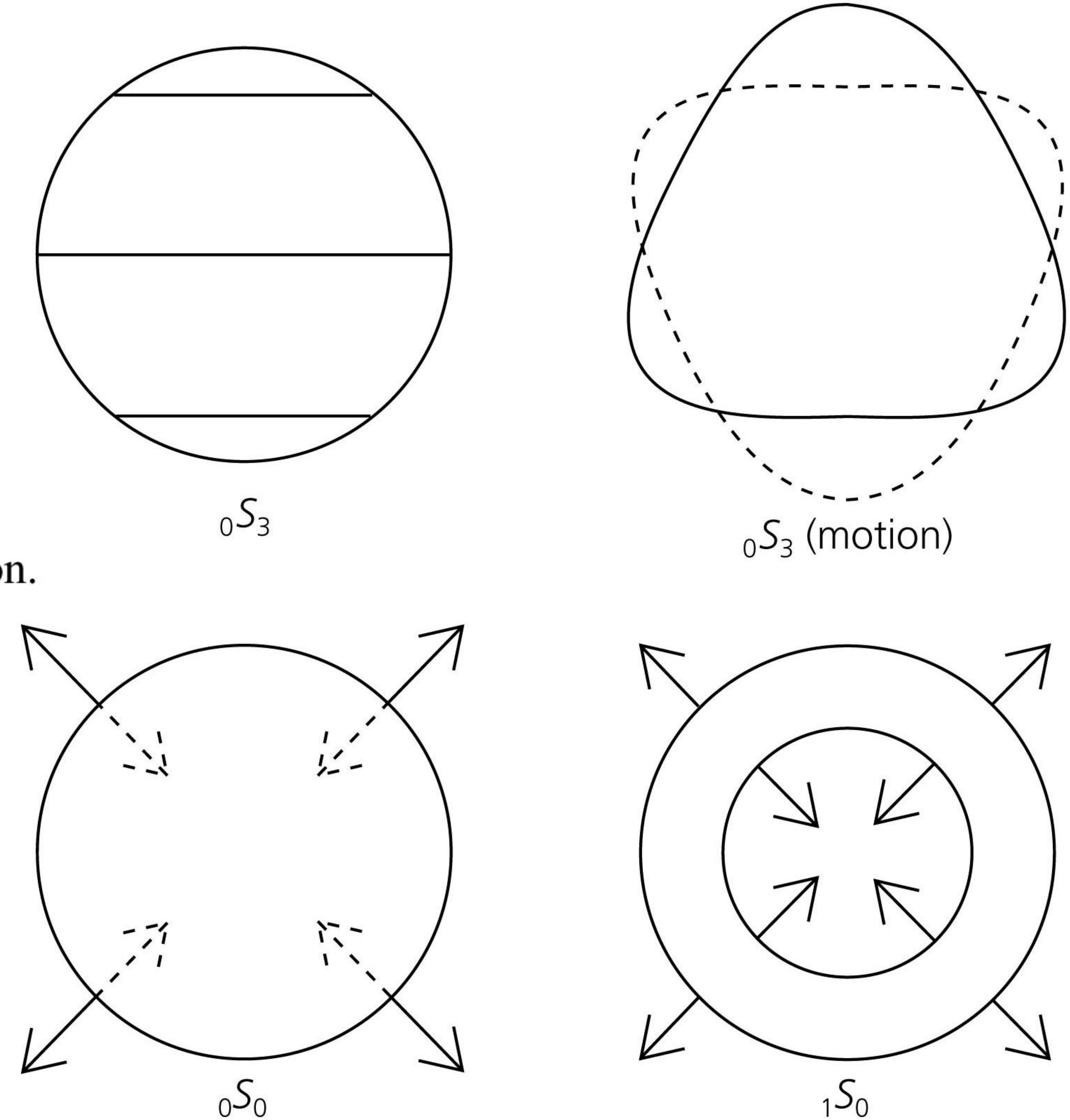
The $_1S_1$ Slichter mode due to lateral sloshing of the inner core through the liquid iron outer core, which has yet to be observed, should in theory have a period of about 5 1/2 hours.

Figure 2.9-7: Examples of the displacements for several spheroidal modes.

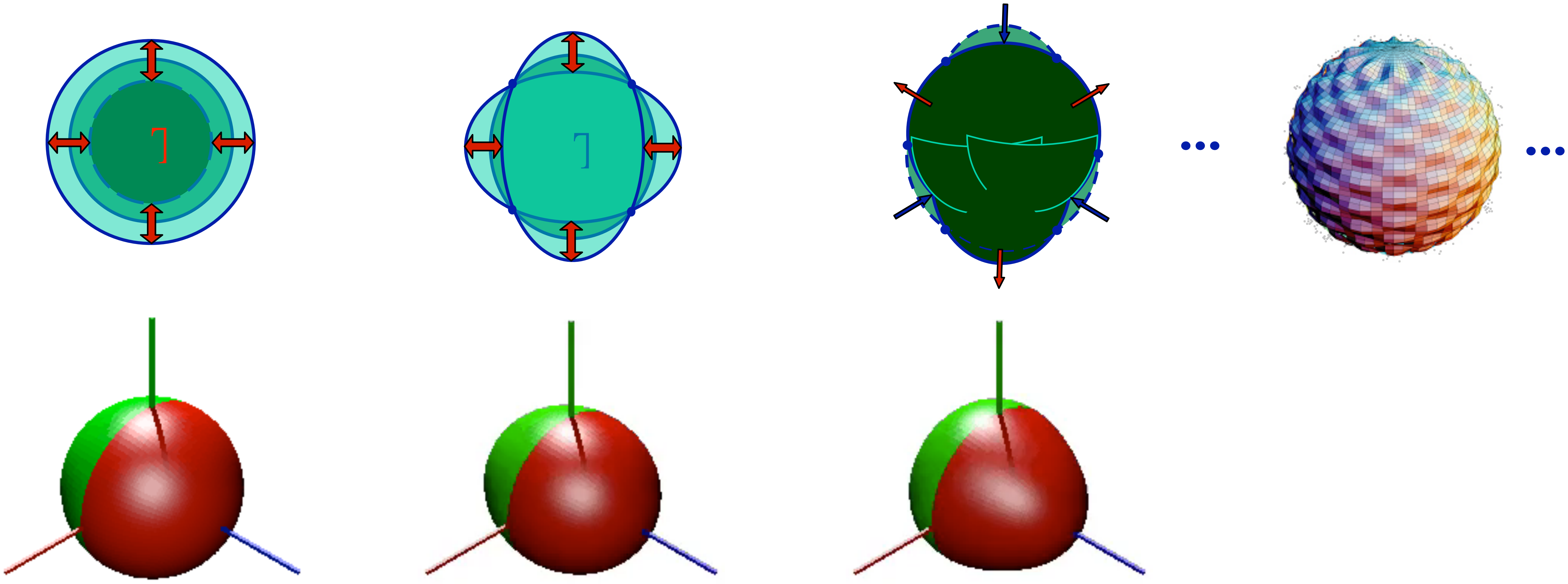


Spheroidal modes

The "breathing" mode ${}_0S_0$ involves radial motions of the entire earth that alternate between expansion and contraction.



Spheroidal normal modes: examples



${}_0S_0$: « balloon » or « breathing » : radial only (20.5 minutes)

${}_0S_2$: « football » mode (Fundamental, 53.9 minutes)

${}_0S_3$: (25.7 minutes)

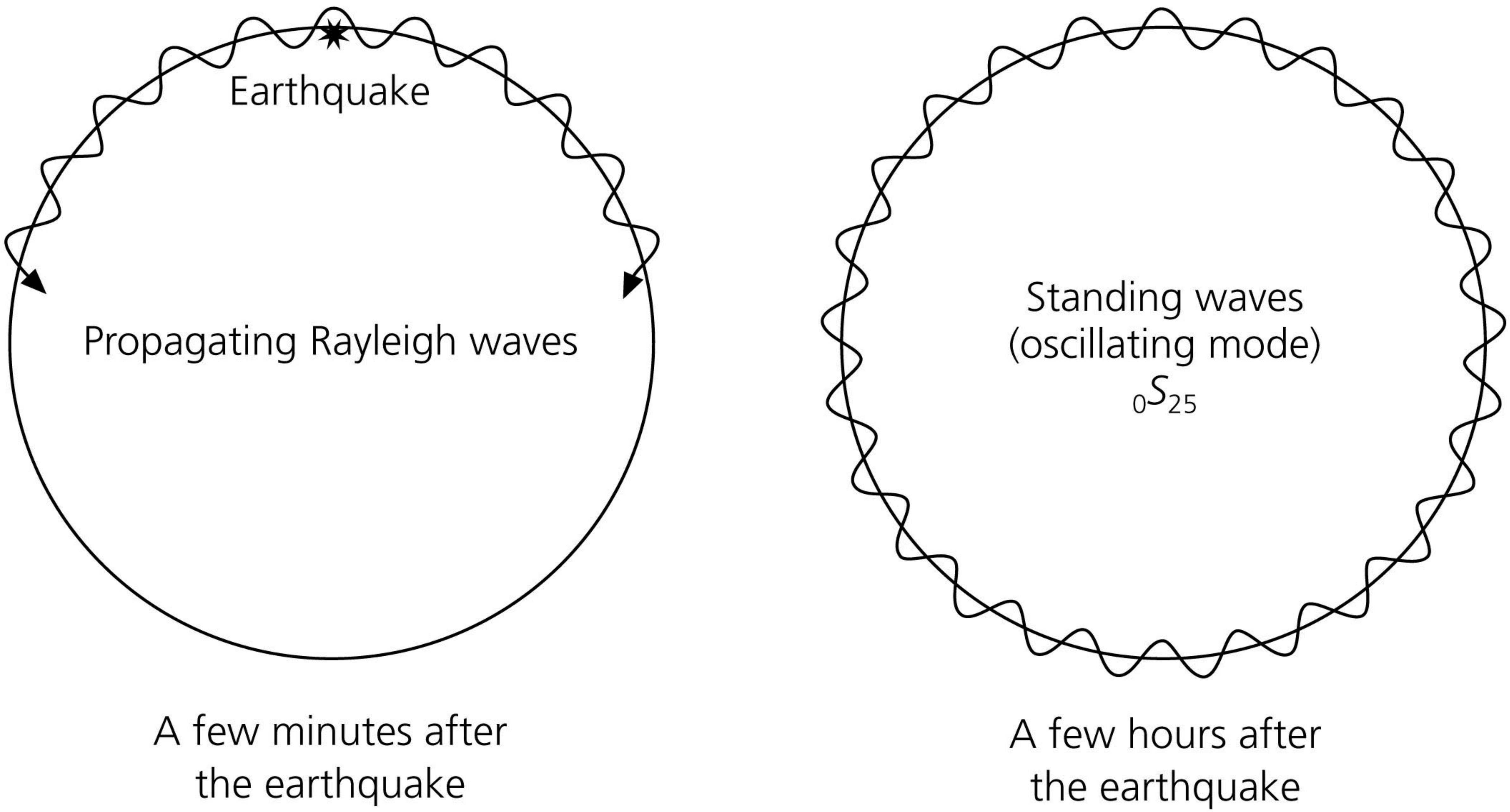
…

${}_0S_{29}$: (4.5 minutes)

See also Animations from Hein Haak

http://www.knmi.nl/cms/content/64722/eigentrillingen_van_de_sumatra_aardbeving

Figure 2.9-8: Cartoon of the equivalence of surface waves and normal modes.

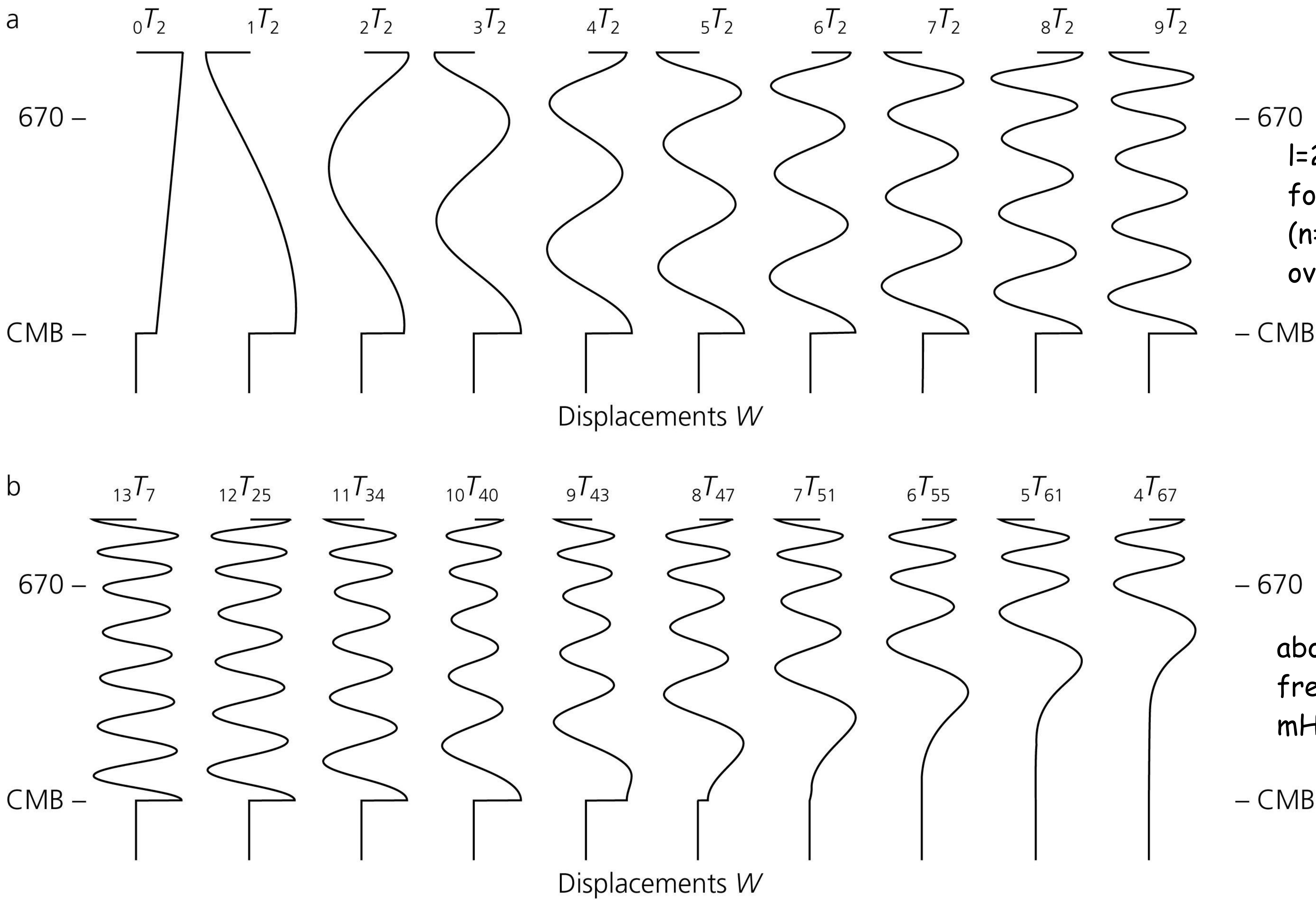


The mode with angular order l and frequency $_n\omega_l$ corresponds to a traveling wave with horizontal wavelength $\lambda_x = 2\pi/|\mathbf{k}_x| = 2\pi a/(l + 1/2)$ that has $l + 1/2$ wavelengths around the earth.

These waves travel at a horizontal phase velocity $c_x = {}_n\omega_l/|\mathbf{k}_x| = {}_n\omega_l a/(l + 1/2)$

Torsional eigenfunctions

Figure 2.9-9: Radial eigenfunctions for various modes.

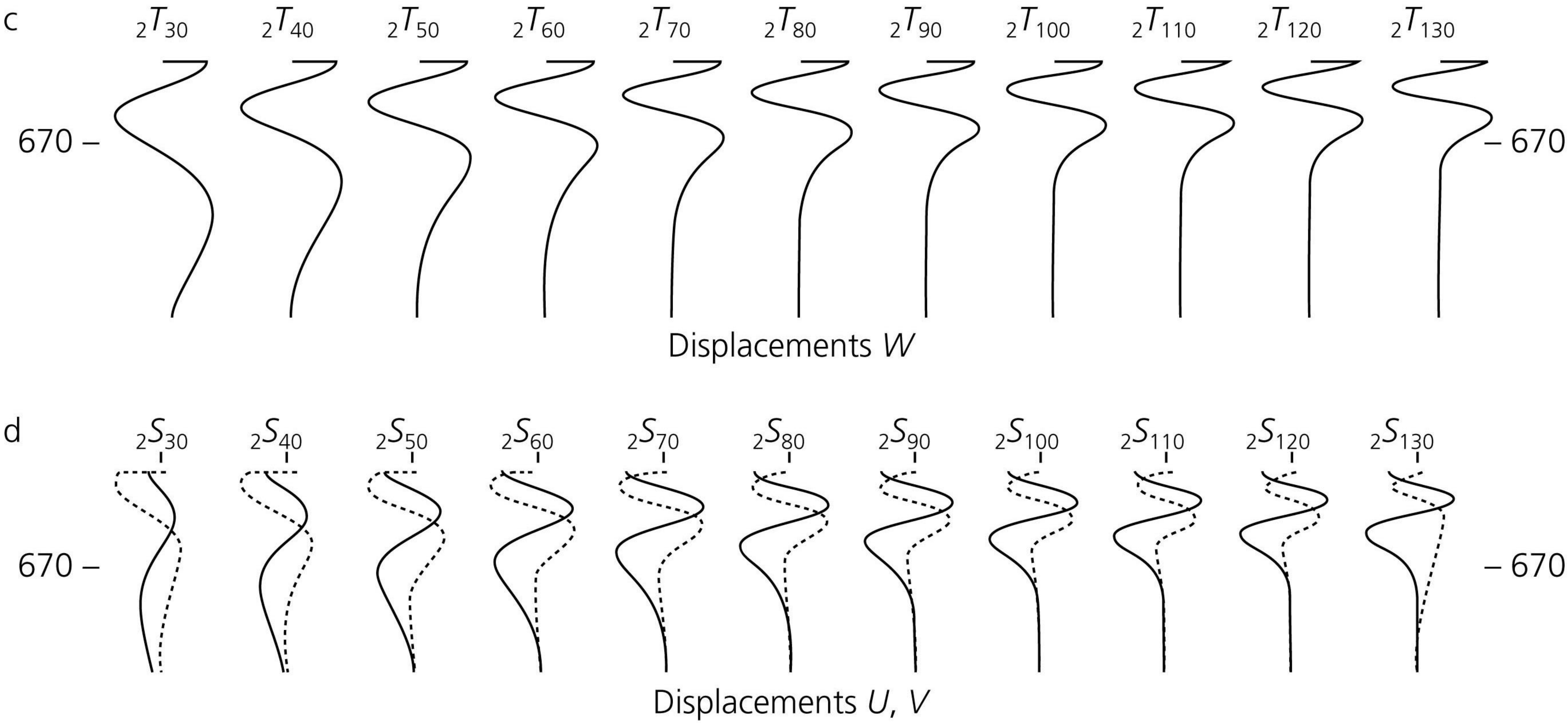


l=2
for fundamental (n=0) and higher overtones

about same frequency (14 mHz)

Spheroidal & Torsional eigenfunctions

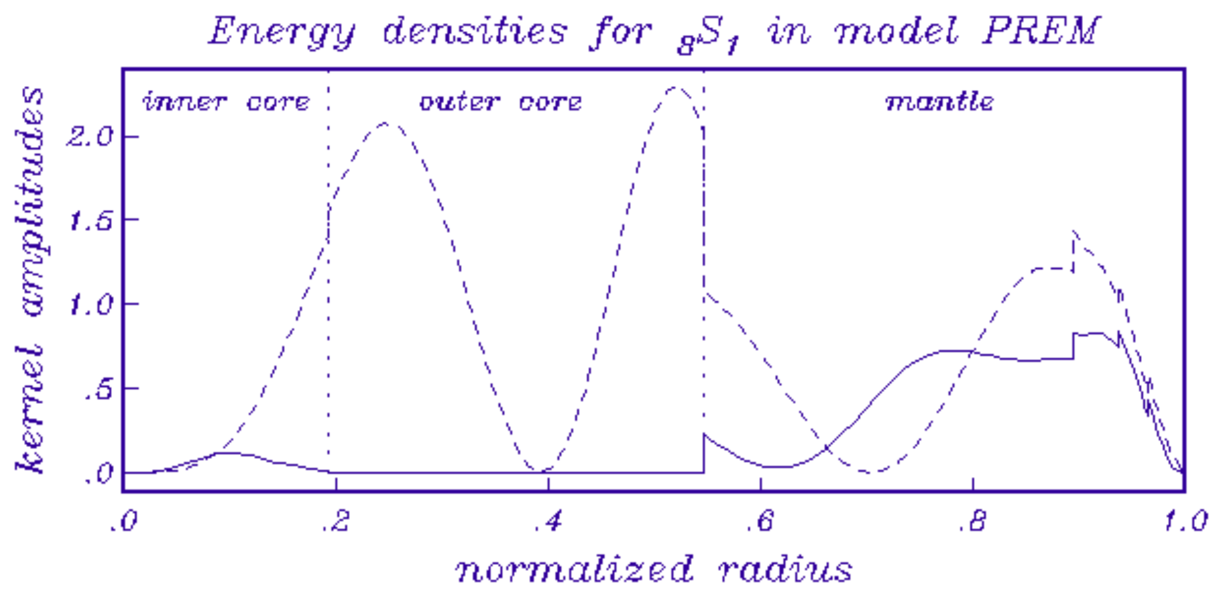
second overtone branch



second overtone branch

Modes energy

One of the modes used in 1971 to infer the solidity of the inner core:
Part of the shear and compressional energy in the inner core



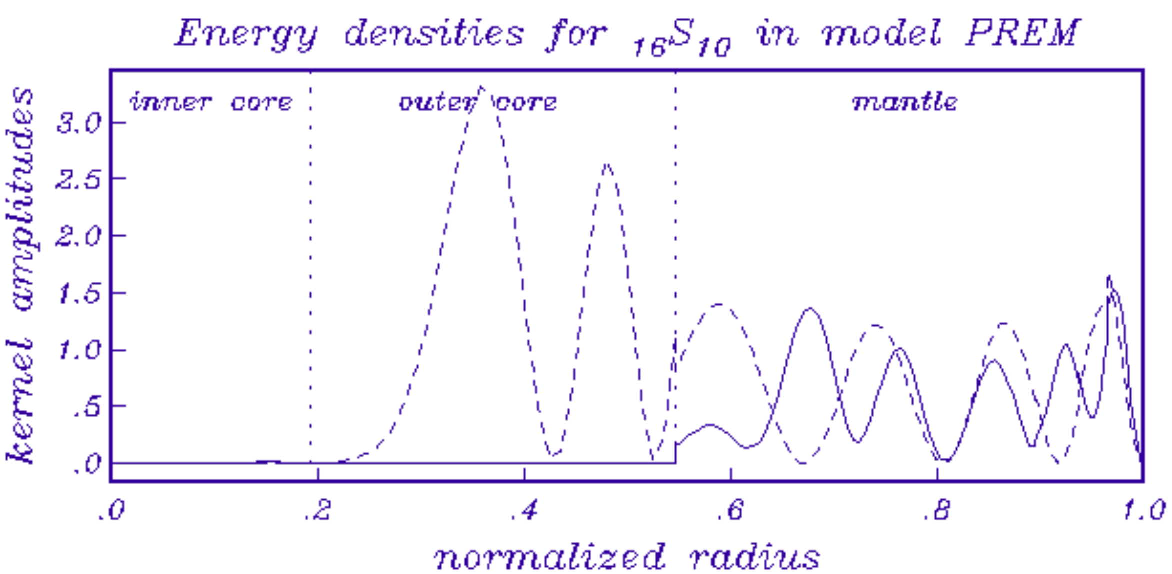
——— **shear energy density**

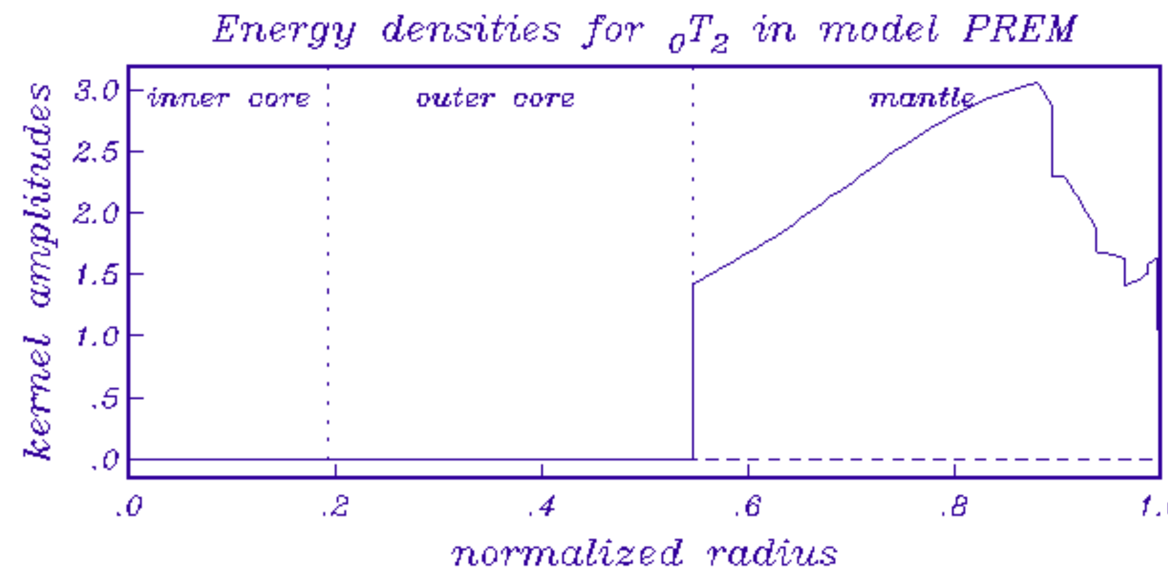
- - - - - - - - **compressional energy density**

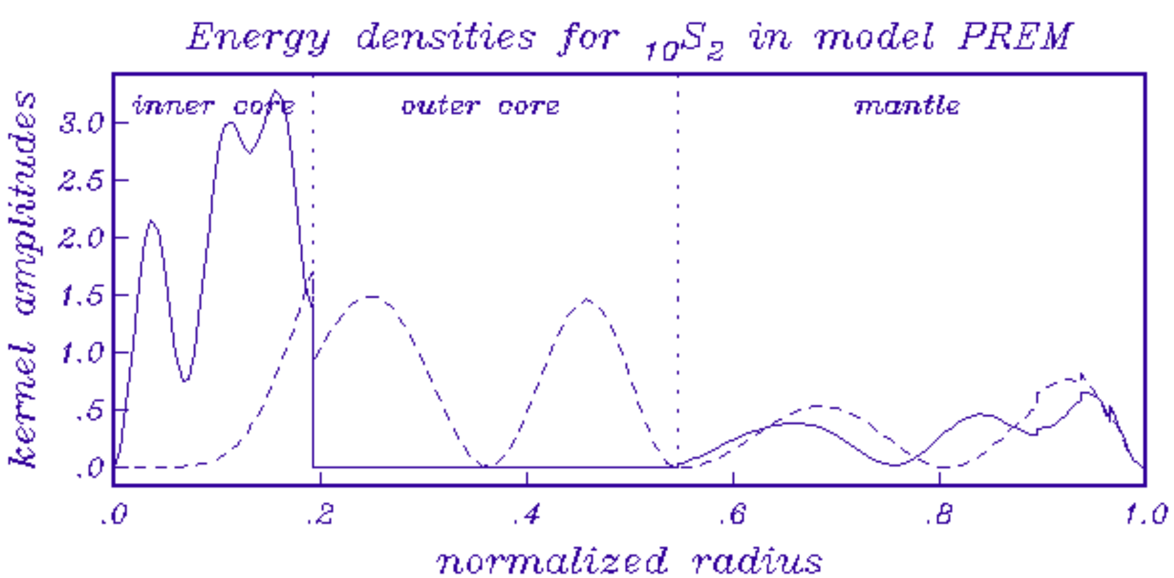
Modes energy

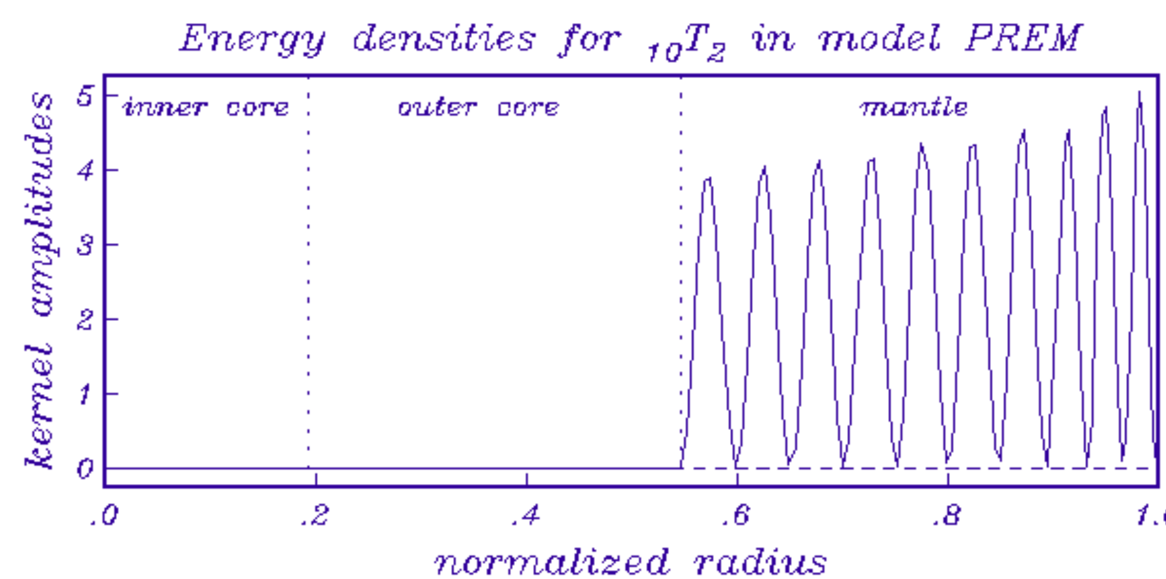
S can affect the whole Earth (esp. overtones)

T in the mantle only !









——— **shear energy density**

- - - - - **compressional energy density**

Eigenvalues

Some toroidal and spheroidal modes.

Mode	Period (s)	Description or associated phase
${}_0T_2$	2639.4	fundamental toroidal
${}_0T_3$	1707.6	fundamental toroidal
${}_1T_1$	808.4	radial overtone
${}_1T_2$	757.5	radial overtone
${}_9T_2$	104.4	radial overtone
${}_0T_{30}$	259.5	fundamental Love
${}_0T_{130}$	68.9	fundamental Love
${}_2T_{30}$	151.3	second-overtone Love
${}_4T_{67}$	71.3	*SH*
${}_{10}T_{40}$	71.4	SH_{diff}
${}_{13}T_7$	71.6	ScS_{SH}
${}_0S_0$	1228.1	fundamental radial
${}_1S_0$	613.0	radial overtone
${}_0S_2$	3233.5	football
${}_0S_3$	2134.4	pear-shaped
${}_0S_{30}$	262.1	fundamental Rayleigh
${}_0S_{130}$	75.8	fundamental Rayleigh
${}_1S_{30}$	160.9	second-overtone Rayleigh
${}_{10}S_6$	203.5	inner core *PKJKP*
${}_{11}S_5$	197.1	inner core *PKIKP*
${}_{14}S_3$	184.9	mantle ScS_{SV}
${}_1S_1$	19500	Slichter

Mid-1800’s – music of the spheres – Earth’s revolution is a C#, 33 octaves below middle C#
(breathing mode is an E, 20 octaves below middle E)

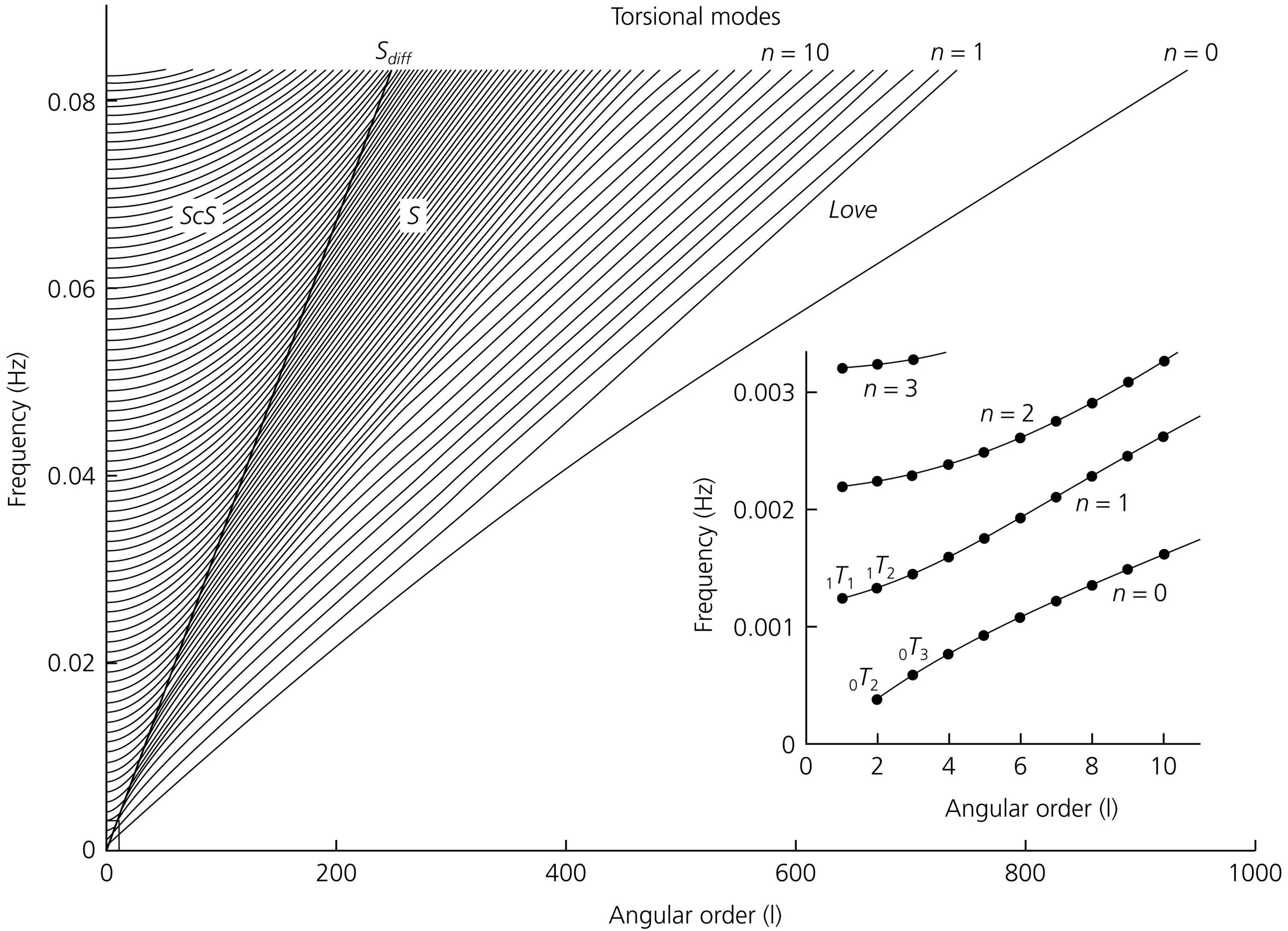
1882 – Lamb – fundamental mode of Earth (as steel ball), 78 minutes

1911 – Love – included self-gravitation – fundamental mode period of 60 minutes

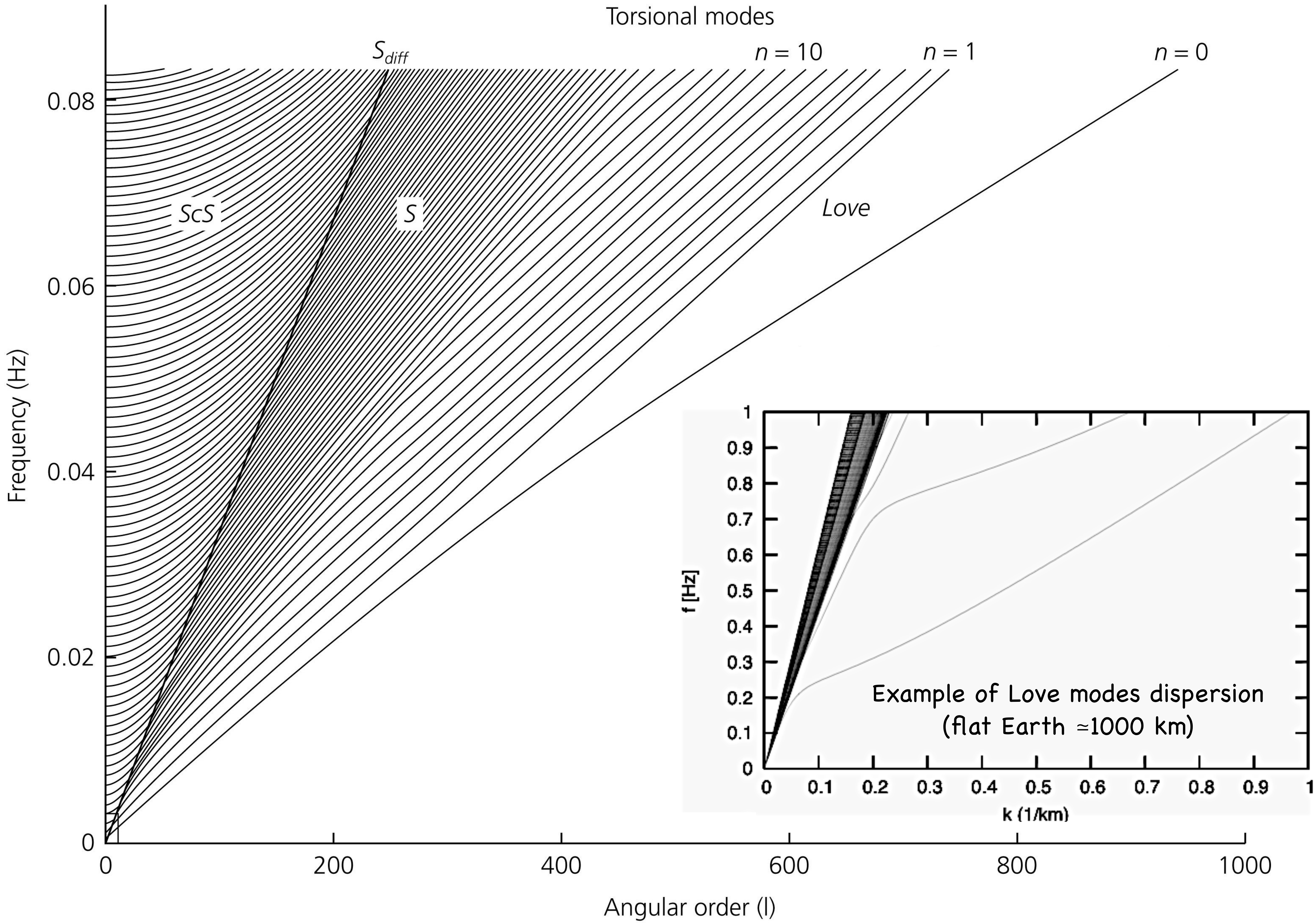
1952 – Kamchatka EQ is first to reveal Earth’s normal modes

1960 – Chile earthquake reveals over 40 modes

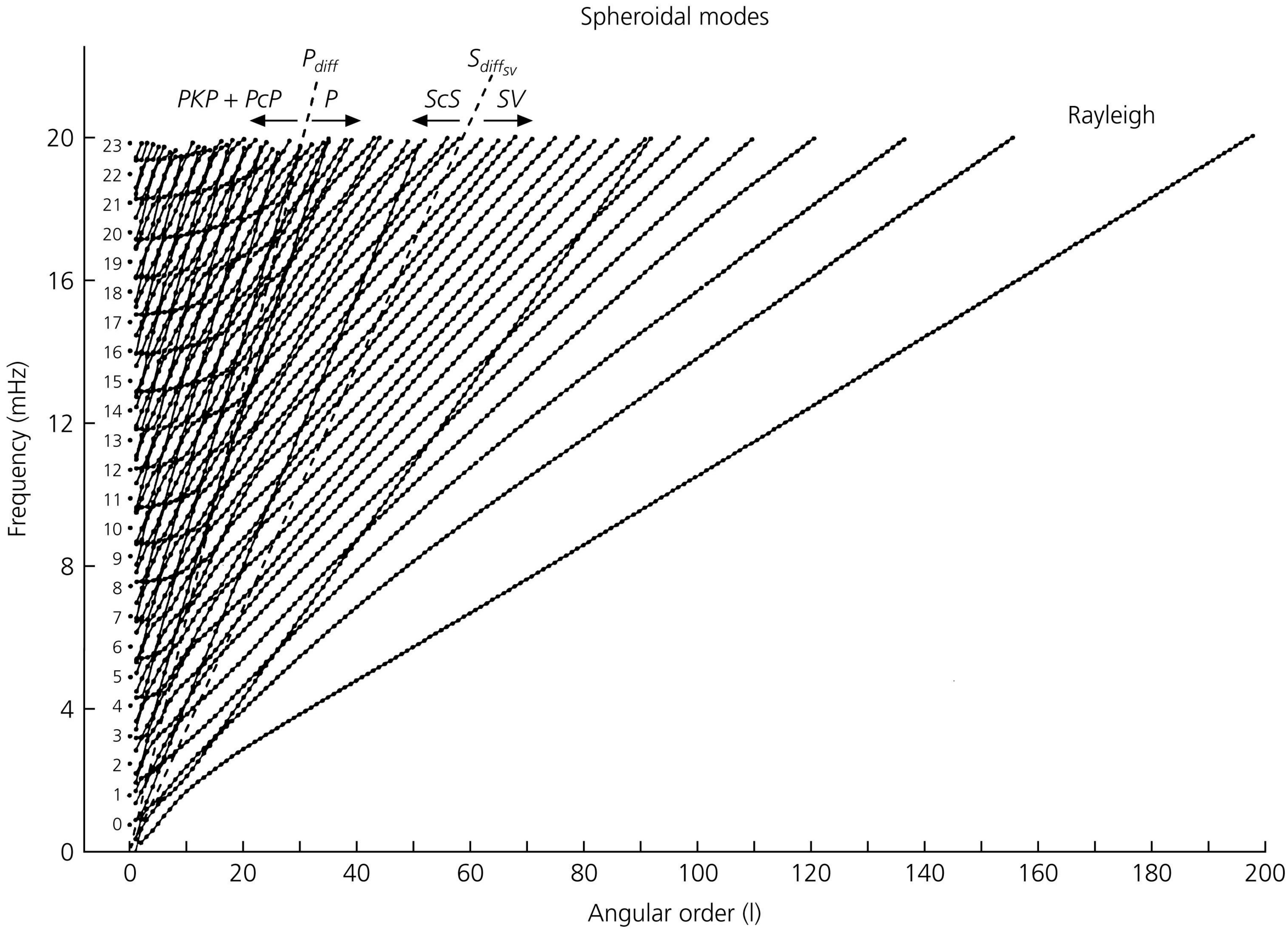
Torsional modes dispersion



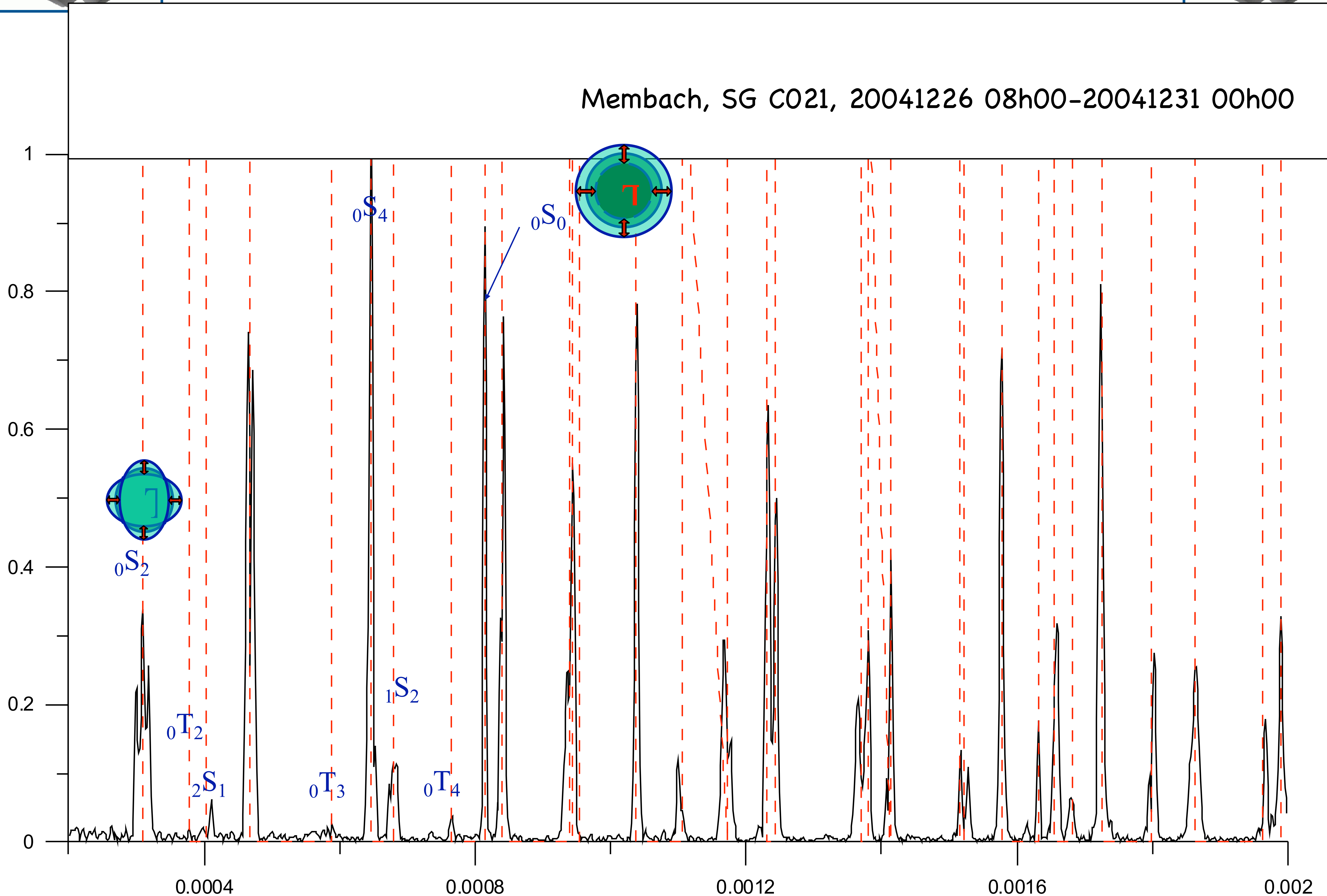
Torsional modes dispersion



Spheroidal modes dispersion



Sumatra: spectrum



Splitting

If SNREI (Solid Not Rotating Earth Isotropic) Earth :
Degeneracy:
for n and l, same frequency for $-l < m < l$

For each m = one singlet.
The 2m+1 group of singlets = multiplet

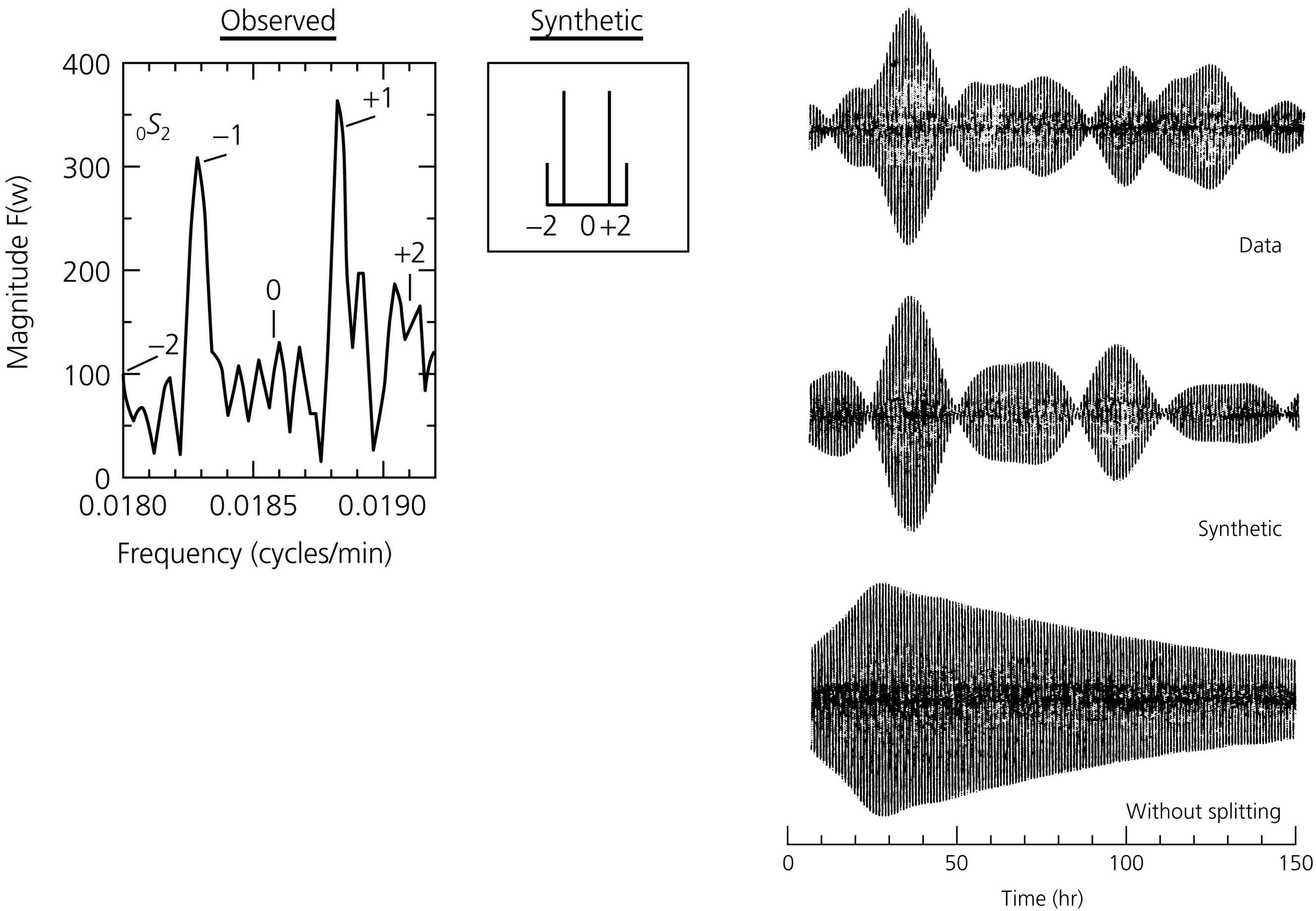
No more degeneracy if no more spherical symmetry :

- **Coriolis**
- **Ellipticity**
- **3D**

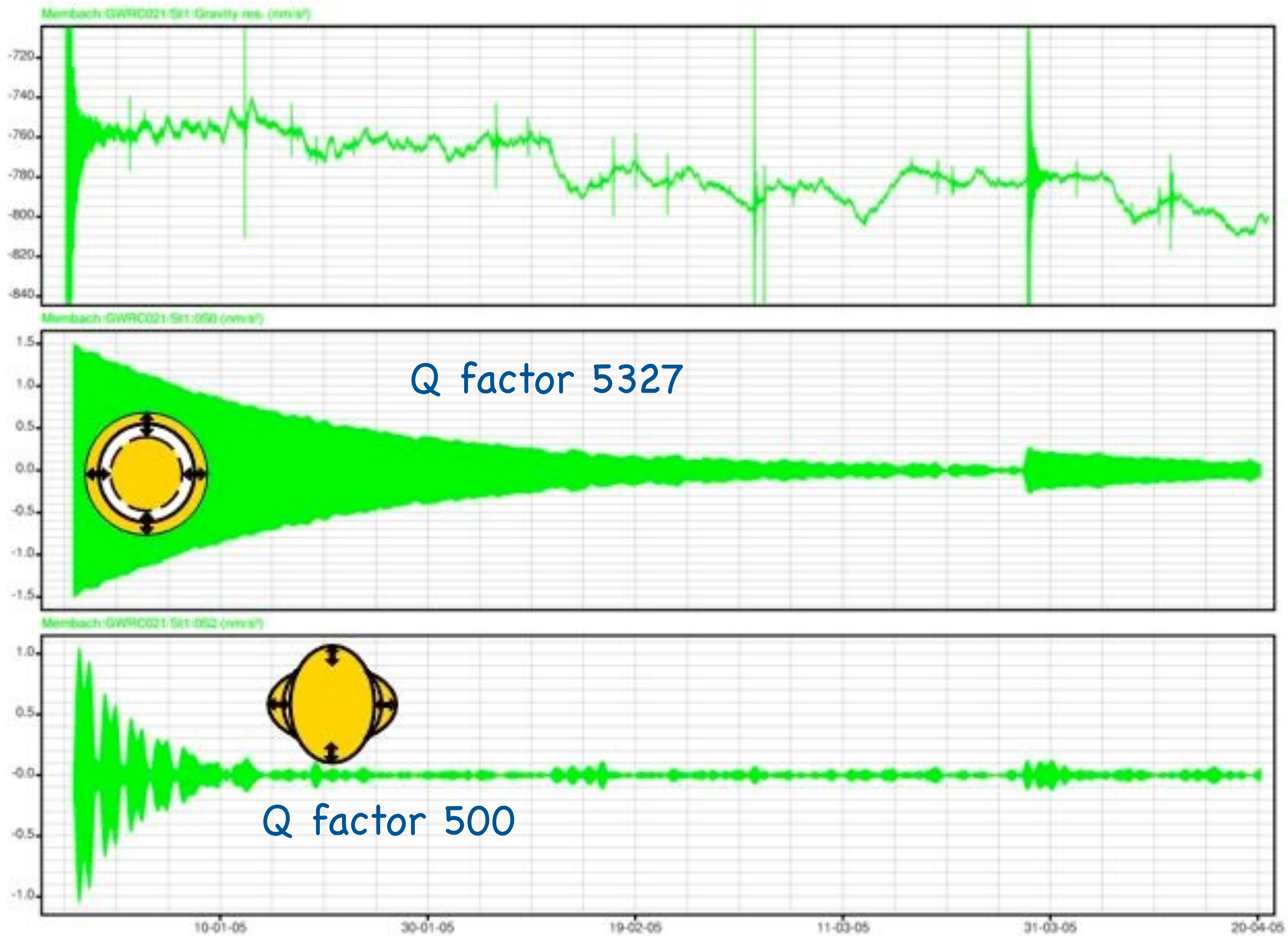
Different frequencies and eigenfunctions for each l,m

Splitting

Figure 2.9-16: Splitting of the $_0S_2$ mode for wave from the 1960 Chile earthquake.



Sumatra: time and Q

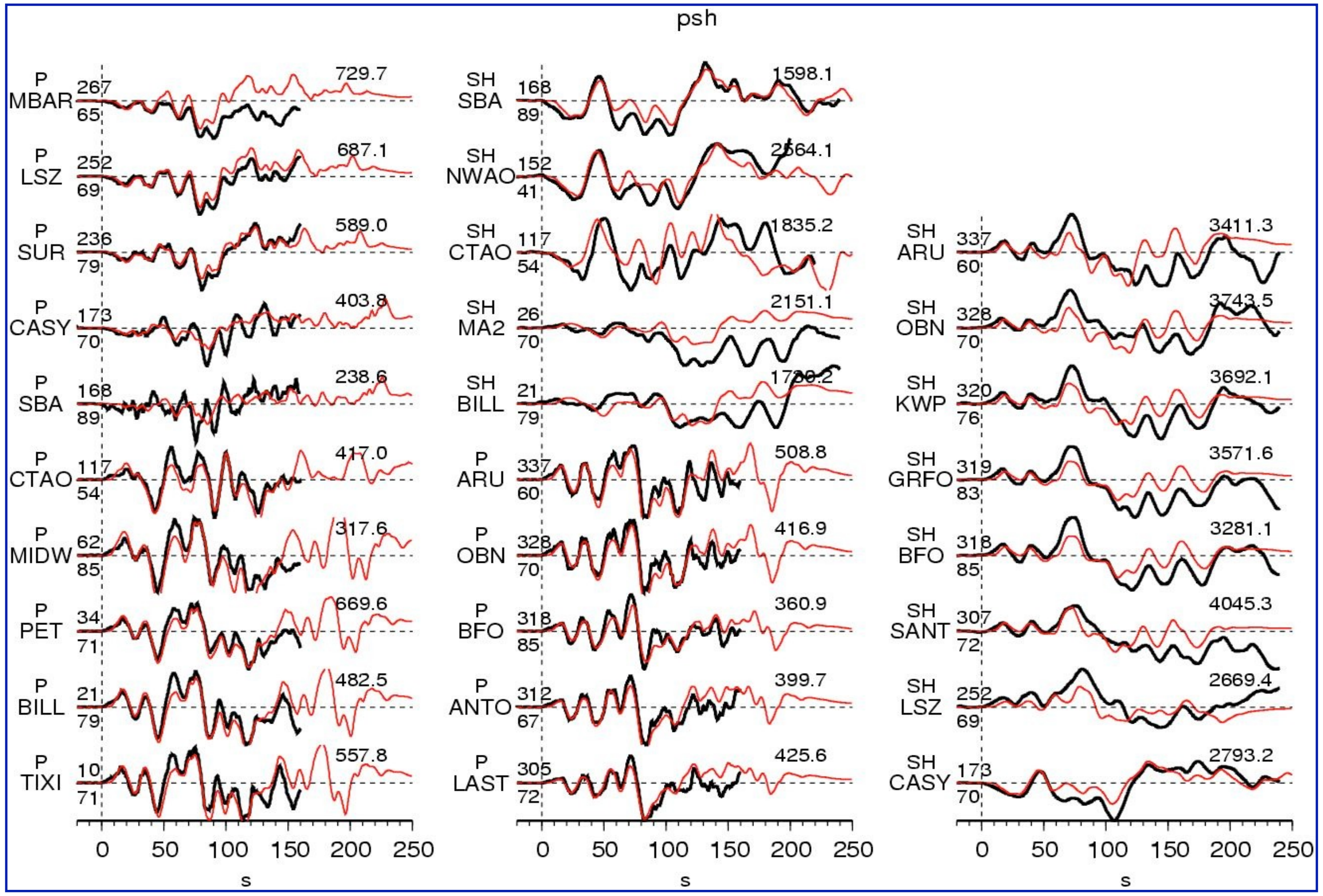


Magnitude

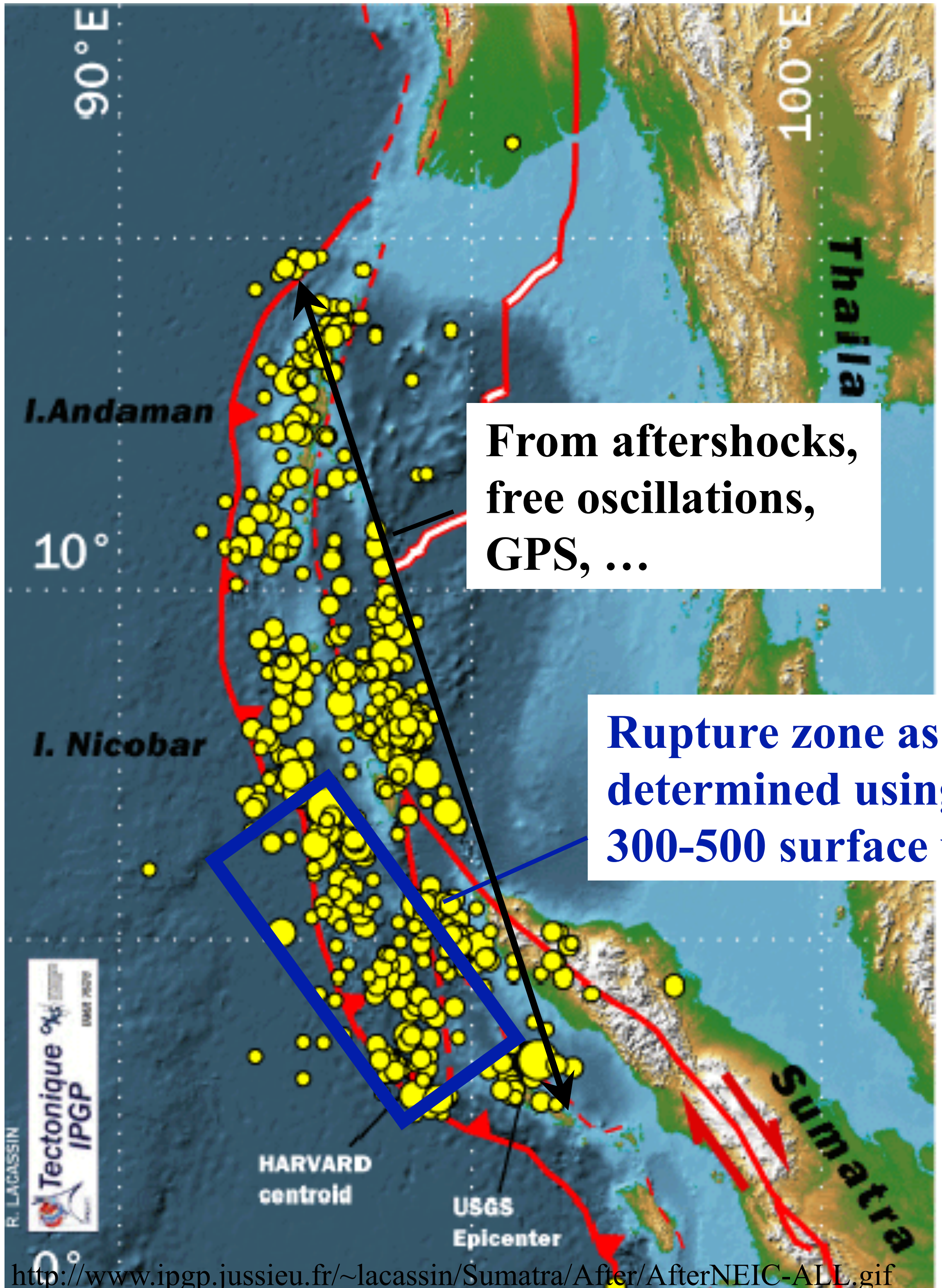
Time after beginning of the rupture:

00:11	8.0 (M_W)	P-waves 7 stations
00:45	8.5 (M_W)	P-waves 25 stations
01:15	8.5 (M_W)	Surface waves 157 stations
04:20	8.9 (M_W)	Surface waves (automatic)
19:03	9.0 (M_W)	Surface waves (revised)
Jan. 2005	9.3 (M_W)	Free oscillations
April 2005	9.2 (M_W)	GPS displacements

300-500 s surface waves

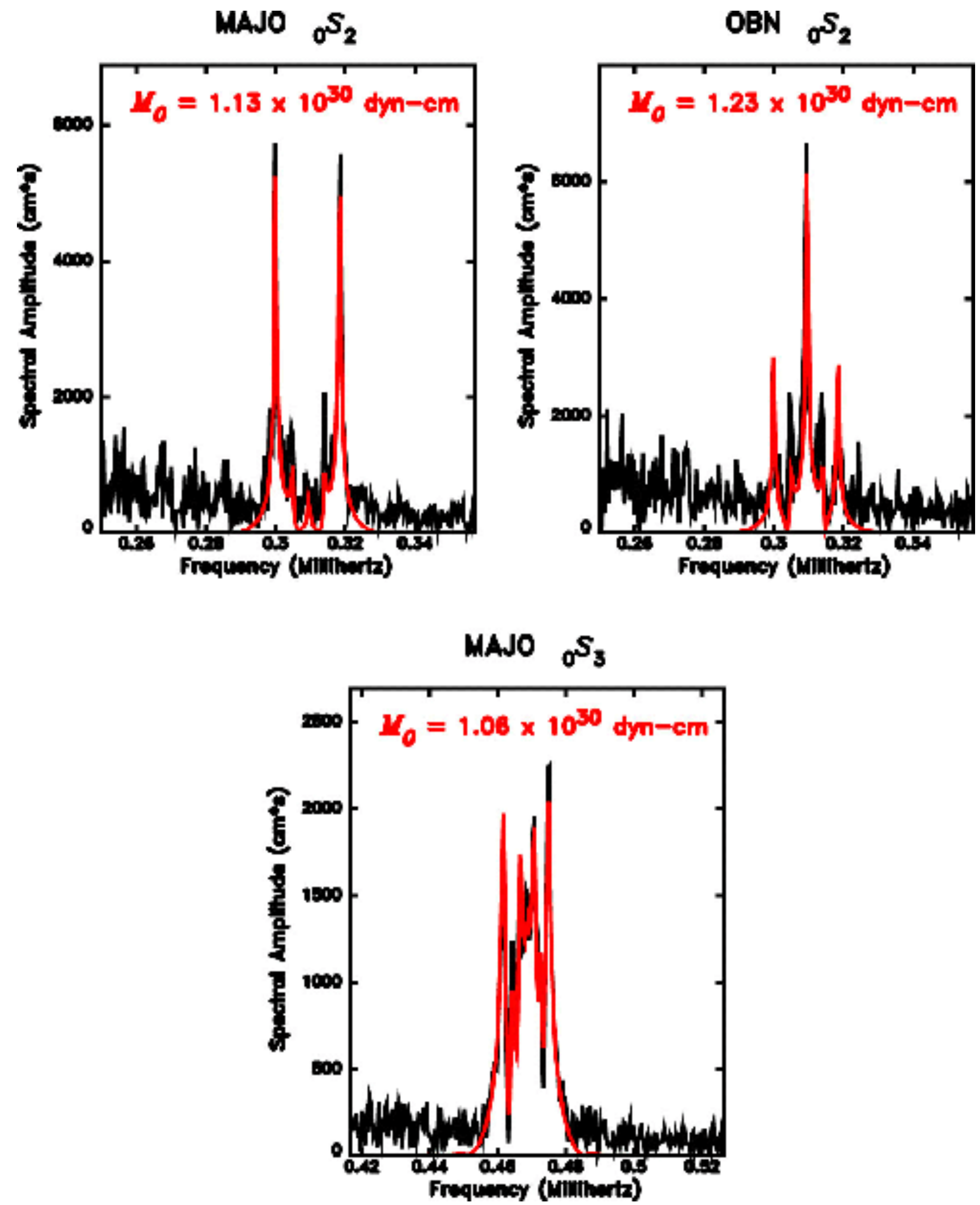


Magnitude



From aftershocks, free oscillations, GPS, …

Rupture zone as determined using 300-500 surface waves



Seth Stein and Emile Okal

Calculated vs. observed

https://cpb-us-e1.wpmucdn.com/sites.northwestern.edu/dist/8/1676/files/2017/05/nestasumatra-1cpxsmc.pdf

Modal summation on a sphere

For displacements in 3-D:

$$\mathbf{u}(r, \theta, \phi) = \sum_n \sum_l \sum_m {}_nA_l^m \; {}_ny_l(r) \; \mathbf{x}_l^m(\theta, \phi) \; e^{i\,{}_n\omega_l^m t}$$

n, l, m - radial, angular, and azimuthal orders

${}_ny_l(r)$ - scalar radial eigenfunction

$\mathbf{x}_l^m(\theta, \phi)$ - vector surface eigenfunction

${}_nA_l^m$ - excitation amplitudes (weights for eigenfunctions) that depend on the seismic source.

Modal summation (anelastic)

Normal mode synthetic seismograms:

$$\mathbf{u}^T(r_r, \theta_r, \phi_r) = \sum_n \sum_l \sum_{m=-l}^{l} {}_nA_l^m(r_s, r_r)\; {}_nW_l(r_r)\; \mathbf{T}_l^m(\theta_r, \phi_r)\; e^{i_n\omega_l^m t}\; e^{-\frac{{}_n\omega_l^m t}{2\,{}_nQ_l}}$$

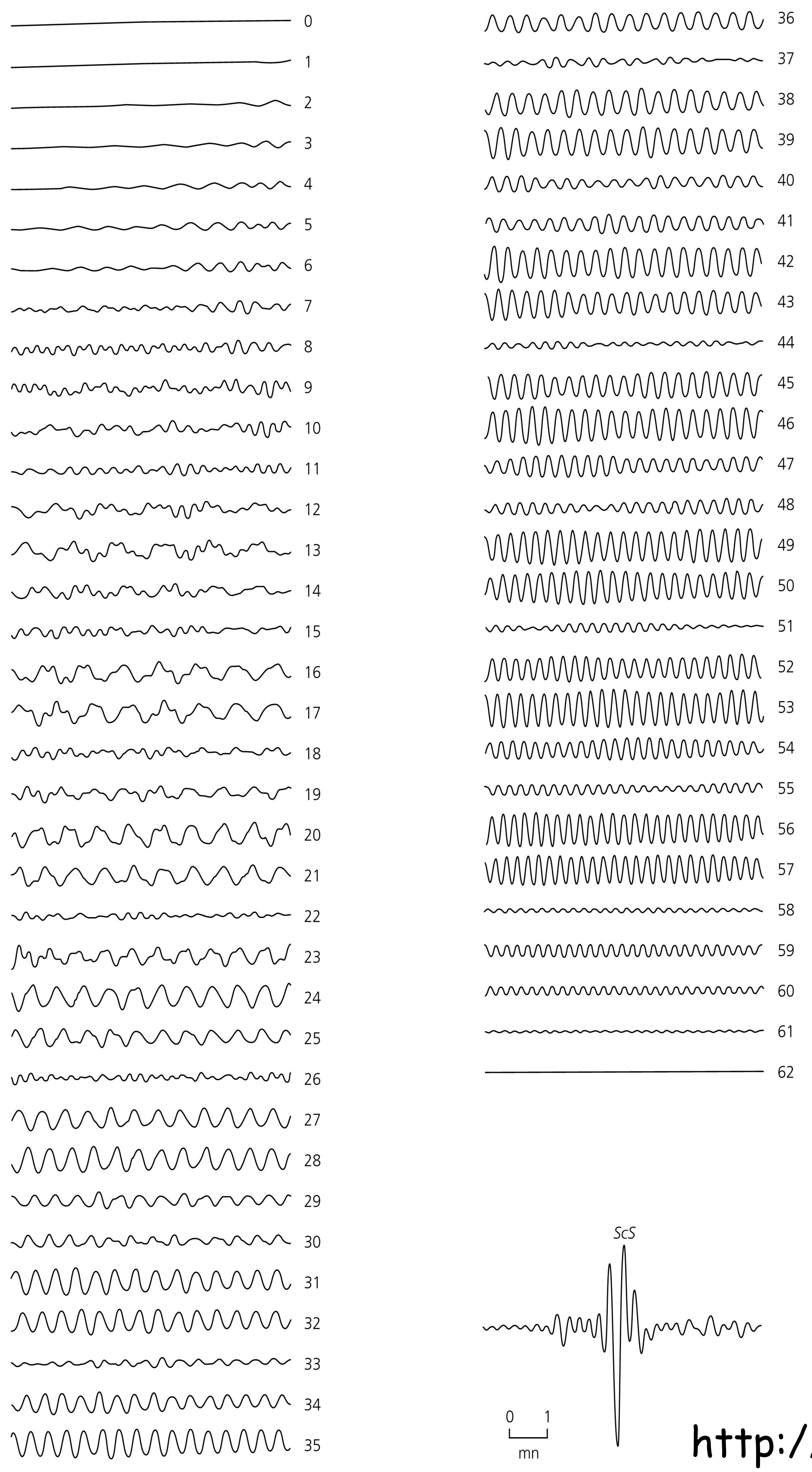
$e^{-\frac{{}_n\omega_l^m t}{2\,{}_nQ_l}}$ - the attenuation of the mode

${}_nQ_l$ - quality factor of the mode

After Q cycles of oscillation, the amplitude of a mode has fallen to a level of $e^{-\pi}$ or 4% of the original amplitude.

Modal summation: ScS

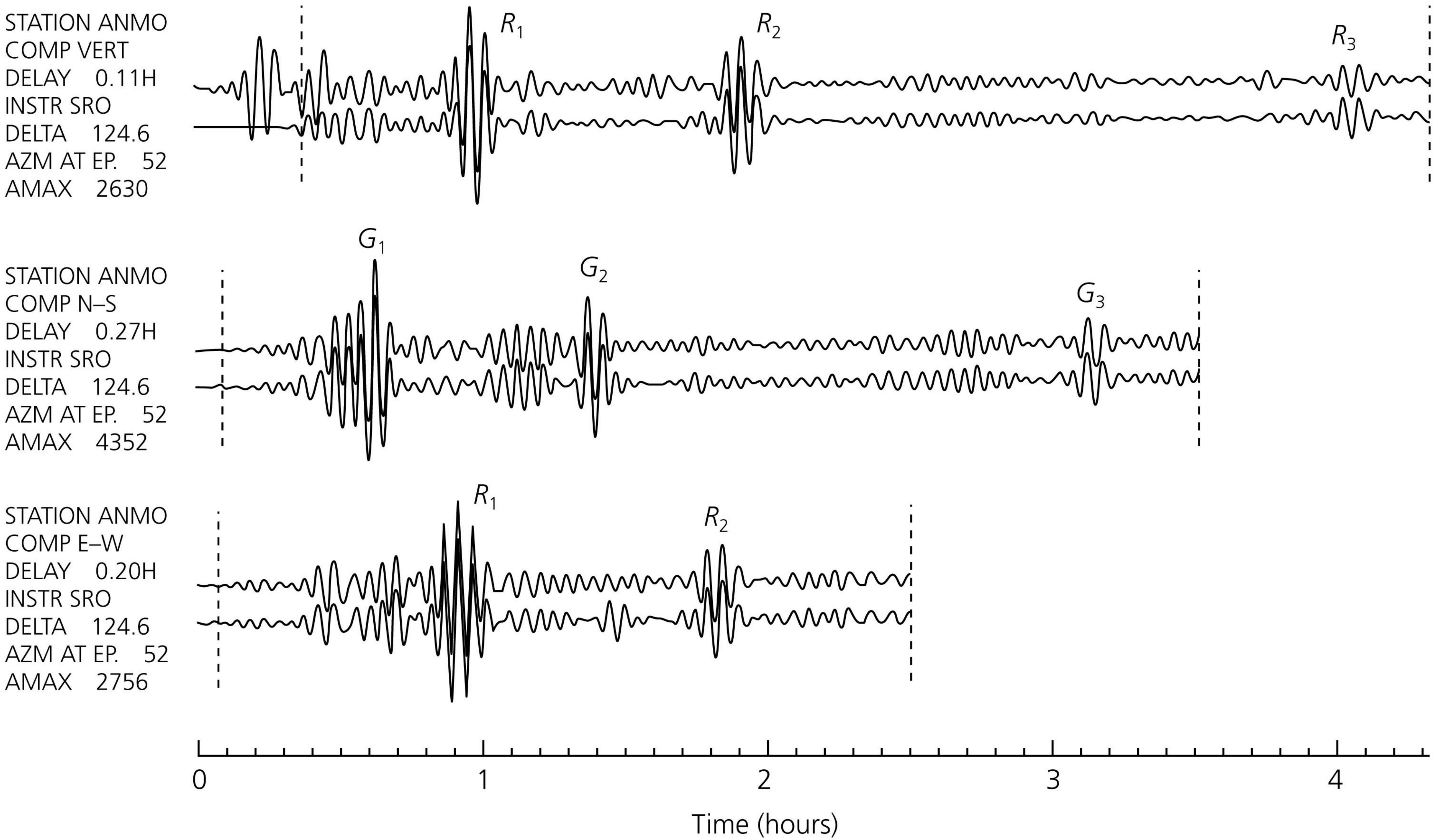
Figure 2.9-12: Synthesis of a body wave from normal mode summation.



http://epsc.wustl.edu/~saadia/page2.html

Modal summation: Rayleigh

Figure 2.9-13: Example of modeling data with normal mode synthetic seismograms.



Modal summation: S

Figure 2.9-14: Shear wave synthetic seismograms computed at a series of depths.