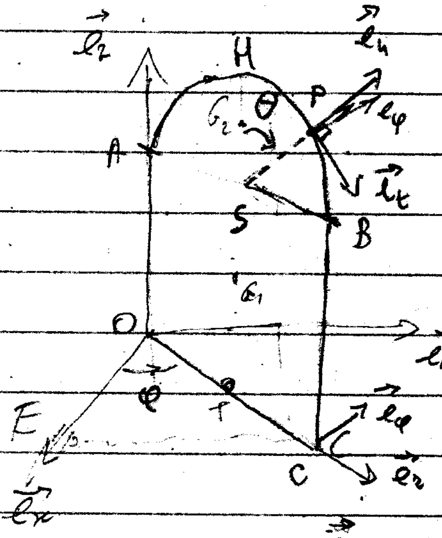
# Tema del 19/02/2018

Il modello è formato da 1 rigido più un punto materiale vincolato al rigido. Con il metodo dei congelamenti successivi si deduce che il modello ha 2 g.l.. Quindi può essere descritto dalle coordinate lagrangiane della figura

$$0 \le \varphi < 2\pi \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

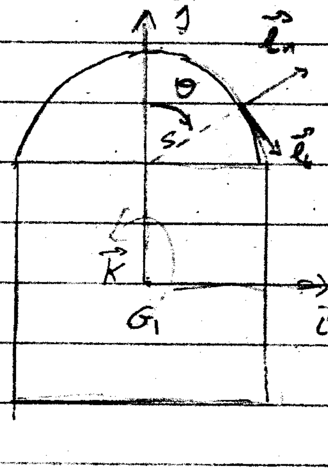



La cinematica del modello può essere descritta tramite le 3 basi

$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ : "fissa"

$(\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$ : "intermedia"

$(\vec{i}, \vec{j}, \vec{k})$ : "solidale al telaio"




$$(1.1)\ \begin{cases} \vec{e}_r = \cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y \\ \vec{e}_z = \vec{e}_z \end{cases} \qquad \begin{cases} \vec{e}_x = \cos\varphi\, \vec{e}_r - \sin\varphi\, \vec{e}_\varphi \\ \vec{e}_y = \sin\varphi\, \vec{e}_r + \cos\varphi\, \vec{e}_\varphi \\ \vec{e}_z = \vec{e}_z \end{cases}$$

$$(1.2)\ \begin{cases} \vec{i} = \vec{e}_r \\ \vec{j} = \vec{e}_z \\ \vec{k} = -\vec{e}_\varphi \end{cases} \qquad \begin{cases} \vec{e}_r = \vec{i} \\ \vec{e}_z = \vec{j} \\ \vec{e}_\varphi = -\vec{k} \end{cases}$$

Quindi:

$$C-O = 2a\,\vec{e}_r = 2a(\cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y) \quad , \quad C-E = 2a \sin\varphi\, \vec{e}_y$$

$$(1.3)\ P-S = a(\sin\theta\, \vec{e}_r + \cos\theta\, \vec{e}_z) = a(\sin\theta\cos\varphi\, \vec{e}_x + \sin\theta\sin\varphi\, \vec{e}_y + \cos\theta\, \vec{e}_z)$$

$$P-H = (P-S) + (S-H) = a(\sin\theta\, \vec{e}_r + \cos\theta\, \vec{e}_z) - a\,\vec{e}_z = a\left[\sin\theta\, \vec{e}_r + (\cos\theta - 1)\vec{e}_z\right]$$

$$P-O = (P-S) + (S-O) = a\sin\theta\, \vec{e}_r + a\cos\theta\, \vec{e}_z + a(\vec{e}_r + 4\vec{e}_z) =$$

$$= a\left[(\sin\theta + 1)\vec{e}_r + (\cos\theta + 4)\vec{e}_z\right]$$

Poiché la sollecitazione $\vec{F}_P^{(foll)}$ non è conservativa, conviene usare le eq. pure di equilibrio. Tuttavia, per calcolare le forze generalizzate dovute a $\vec{F}_C, \vec{F}_P^{(el)}$ e al peso proprio, che sono sollecitazioni conservative, utilizziamo la loro energia potenziale

$$V^{(el)} = \frac{1}{2} c\, \overline{CE}^2 + \frac{1}{2} c\, \overline{PH}^2 = \frac{1}{2} c \left[ 4a^2 \sin^2\varphi + a^2 (2 - 2\cos\theta) \right]$$
$$= c \left( 2a^2 \sin^2\varphi - a^2 \cos\theta + a^2 \right)$$

Quindi

$$Q_\varphi^{(el)} = -\frac{\partial V^{(el)}}{\partial \varphi} = -4a^2 c \sin\varphi \cos\varphi = -2a^2 c \sin 2\varphi$$
$$Q_\theta^{(el)} = -\frac{\partial V^{(el)}}{\partial \theta} = -a^2 c \sin\theta$$

$$V^{(peso)} = \cancel{-M\vec{g}\cdot\vec{x}_G} - m\vec{g}\cdot\vec{x}_P$$

G: baricentro del telaio

$$= mg\,\vec{e}_z \cdot (P-O) = mga(\cos\theta + 4)$$

Dunque

$$Q_\varphi^{(peso)} = -\frac{\partial V^{(peso)}}{\partial \varphi} = 0$$
$$Q_\theta^{(peso)} = -\frac{\partial V^{(peso)}}{\partial \theta} = + mga \sin\theta$$

Per calcolare la forza generalizzata dovuta al carico follower, utilizziamo la definizione 

$$Q_\varphi^{(foll)} = \vec F_P^{(foll)} \cdot \frac{\partial \vec x_P}{\partial \varphi} \qquad \frac{\partial \vec x_P}{\partial \varphi} = a\left[(\sin\theta+1)\frac{\partial \vec e_r}{\partial \varphi} + (\cos\theta+4)\cancel{\frac{\partial \vec e_z}{\partial \varphi}}\right]$$

$$Q_\theta^{(foll)} = \vec F_P^{(foll)} \cdot \frac{\partial \vec x_P}{\partial \theta} \qquad \frac{\partial \vec x_P}{\partial \theta} = a\left[\cos\theta\, \vec e_r + (\sin\theta+1)\cancel{\frac{\partial \vec e_r}{\partial \theta}} - \sin\theta\, \vec e_z + (\cos\theta+4)\cancel{\frac{\partial \vec e_z}{\partial \theta}}\right]$$

Allora

$$\frac{\partial \vec x_P}{\partial \varphi} = a(\sin\theta+1)\,\vec e_\varphi \quad , \quad \frac{\partial \vec x_P}{\partial \theta} = a(\cos\theta\, \vec e_r - \sin\theta\, \vec e_z)$$

Poiché $\vec F = F\,\vec e_\varphi \qquad F>0$

$$Q_\varphi^{(foll)} = Fa(\sin\theta+1)$$

$$Q_\theta^{(foll)} = 0 \qquad \Rightarrow \frac{\partial Q_\varphi}{\partial \theta} \neq \frac{\partial Q_\theta}{\partial \varphi} \Rightarrow \text{NON conservativa}$$

Dunque, le forze generalizzate sono

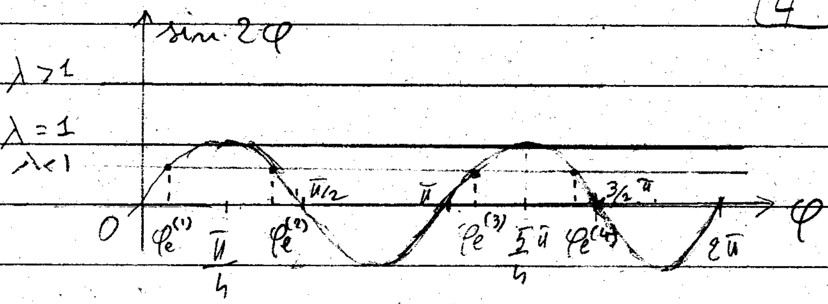
(3.1)
$$Q_\varphi = Q_\varphi^{(el)} + Q_\varphi^{(peso)} + Q_\varphi^{(foll)} = -4a^2c\sin\varphi\cos\varphi + Fa(\sin\theta+1)$$

$$Q_\theta = Q_\theta^{(el)} + Q_\theta^{(peso)} + Q_\theta^{(foll)} = -a^2c\sin\theta + mga\sin\theta = a(mg-ac)\sin\theta$$

Allora, le eq. pure di equilibrio si scrivono

(3.2)
$$\begin{cases} -2a^2c\sin2\varphi + F\cancel{a}(\sin\theta+1) = 0 \\ a(mg-ac)\sin\theta = 0 \end{cases} \qquad \begin{cases} \sin2\varphi = \dfrac{F}{2ac} = \lambda \\ \theta = 0 \end{cases}$$

se $\lambda > 1$ $\nexists$ soluzione




se $\lambda = 1$ $\vec{q}_e^{(5)} = \left(\frac{\pi}{4}, 0\right)$, $\vec{q}_e^{(6)} = \left(\frac{5}{4}\pi, 0\right)$

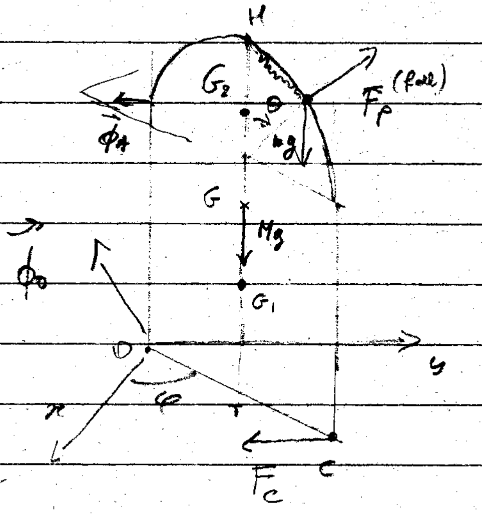
se $\lambda < 1$

$$(4.1) \quad \vec{q}_e^{(1)} = \left(\varphi_e^{(1)}, 0\right) = \left(\frac{1}{2}\arcsin\lambda, 0\right)$$

$$\vec{q}_e^{(2)} = \left(\frac{\pi}{2} - \varphi_e^{(1)}, 0\right), \quad \vec{q}_e^{(3)} = \left(\pi + \varphi_e^{(1)}, 0\right), \quad \vec{q}_e^{(4)} = \left(\frac{3}{2}\pi - \varphi_e^{(1)}, 0\right)$$

2) Reazioni e momento delle reazioni negli equilibri in O

Utilizziamo le ECS applicate a tutto il modello.




$$\vec{R}^{(att,est)} + \vec{\Phi}_O + \vec{\Phi}_A = \vec{0}$$

$$\vec{M}_O^{(att,est)} + (A-O) \times \vec{\Phi}_A = \vec{0}$$

dove

$$(4.2) \quad \vec{R}^{att,est} = \vec{F}_c + M\vec{g} + m\vec{g} + \vec{F}_p^{(fol)}$$

$$\vec{F}_c = -c(C-E) = -2ca\sin\varphi\,\vec{e}_y = -2ca\sin\varphi\left(\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_\varphi\right)$$

$$\vec{F}_p = F\,\vec{e}_\varphi$$

Quindi

$$(4.3) \quad \vec{R}^{att,est} = -2ca\sin^2\varphi\,\vec{e}_x - 2ac\sin\varphi\cos\varphi\,\vec{e}_\varphi - (M+m)g\vec{e}_z + F\,\vec{e}_\varphi$$
$$= -2ac\sin^2\varphi\,\vec{e}_x + (F - ac\sin 2\varphi)\,\vec{e}_\varphi - (M+m)g\,\vec{e}_z$$

Dunque,

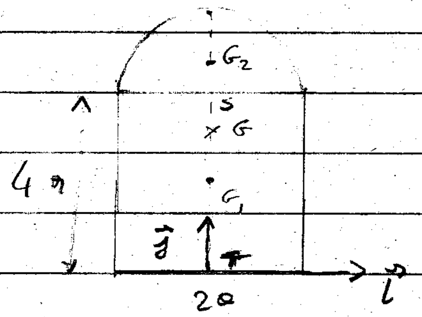
$$(4.4) \quad \vec{\Phi}_O = 2ca\sin^2\varphi_e\,\vec{e}_x - (F - ac\sin 2\varphi_e)\,\vec{e}_{\varphi_e} + (M+m)g\,\vec{e}_3 - \vec{\Phi}_A|_{\vec{q}_e}$$

Inoltre,

(5.1) $\vec{M}_O^{(st, est)} = (C-O) \times \vec{F}_c + (G-O) \times M\vec{g} + (P-O) \times m\vec{g} + (P-O) \times \vec{F}_p^{(koll)}$

Calcolo di G del telaio

Dalla definizione di baricentro



$$G-T = \frac{M_1(G_1-T) + M_2(G_2-T)}{M}$$

dove $M_1$ e $M_2$ sono, rispettivamente, la massa della parte rettangolare e di quella semicircolare. Quindi, rispetto alla coppia di assi $(T; \vec{i}, \vec{j})$

$$G_1 - T = 2a\,\vec{j}, \quad G_2 - T = (4a + \overline{G_2 S})\,\vec{j}$$

$\overline{G_2 S} = \frac{2}{\pi} a$ dal Teo di Guldino applicato alla semicirconferenza

(5.2) $M_1 = \frac{M}{L} L_1 = M \frac{12}{12+\pi}$ $L = a(12+\pi)$: lunghezza totale del telaio

$L_1 = 12a$ : " della parte rettangolare

(5.3) $M_2 = \frac{M}{L} L_2 = M \frac{\pi}{12+\pi}$ $L_2 = \pi a$ : " della parte semicircolare

$\sigma = M/L = \frac{M}{a(12+\pi)}$ : densità di massa.

N.B. Il baricentro G $\notin$ all'asse di rotazione, quindi la lamina NON è bilanciata staticamente.

$$G-T = \frac{\cancel{M} L_1}{L \cancel{M}} 2a\vec{j} + \frac{\cancel{M} L_2}{L\cancel{M}}\left(4a + \frac{2}{\pi}a\right)\vec{j}$$

(5.4)
$$= \frac{a}{L}\left(2L_1 + L_2\left(4 + \frac{2}{\pi}\right)\right)\vec{j}$$

$$= \frac{\cancel{a}}{\cancel{a}(12+\pi)}\left(24a + \cancel{\pi} a \frac{4\pi+2}{\cancel{\pi}}\right)\vec{j}$$

$$= \frac{a}{(12+\pi)}(26+4\pi)\,\vec{j} = 2a\,\frac{13+2\pi}{12+\pi}\,\vec{j} = a\,h\,\vec{j} \qquad h := 2\,\frac{13+2\pi}{12+\pi}$$

$$\vec{M}_O^{(att,est)} = 2a\,\vec{e}_r \times \left(-c\,2a \sin\varphi\, \vec{e}_\varphi\right) + \left(a\,\vec{e}_r + 2h\,\vec{e}_z\right) \times \left(-Mg\,\vec{e}_z\right) +$$
$$+ a\left[(\sin\theta+1)\,\vec{e}_r + (\cos\theta+4)\,\vec{e}_z\right] \times \left(-mg\,\vec{e}_z + F\,\vec{e}_\varphi\right)$$

$$= -4a^2 c \sin\varphi\cos\varphi\, \vec{e}_r\times\vec{e}_\varphi - Mga\,\vec{e}_r \times \vec{e}_z +$$
$$+ a(\sin\theta+1)\left(-mg\,\vec{e}_r \times \vec{e}_z + F\,\vec{e}_r \times \vec{e}_\varphi\right) +$$
(6.1) $$+ a(\cos\theta+4)\,F\,\vec{e}_z \times \vec{e}_\varphi$$

$$= -4a^2 c \sin\varphi\cos\varphi\,\vec{e}_z + Mga\,\vec{e}_\varphi + a(\sin\theta+1)\,mg\,\vec{e}_\varphi +$$
$$+ a(\sin\theta+1)\,F\,\vec{e}_z - a(\cos\theta+4)\,F\,\vec{e}_r$$

$$= -(\cos\theta+4)\,Fa\,\vec{e}_r + \left(M + m(1+\sin\theta)\right) ag\,\vec{e}_\varphi +$$
$$+ a\left[(1+\sin\theta)F - 4ac\sin\varphi\cos\varphi\right]\vec{e}_z$$

Dalla II ECS otteniamo

(6.2) $$\vec{\phi}_A = \frac{(A-O)\times \vec{M}_O^{(att,est)}}{|A-O|^2} + \cancel{\mu(A-O)} = \frac{1}{4a}\,\vec{e}_z \times \left[-(\cos\theta+4)\,Fa\,\vec{e}_r + \left(M+m(1+\sin\theta)\right)ag\,\vec{e}_\varphi\right]$$
$$= -\frac{1}{4}\left[(\cos\theta_e+4)\,F\,\vec{e}_\varphi + \left(M+m(1+\sin\theta_e)\right) g\,\vec{e}_r\right]$$

$$\vec{\phi}_{A|\vec{q}_e} = -\frac{1}{4}\left[5F\,\vec{e}_{\varphi|\vec{q}_e} + (M+m)\,g\,\vec{e}_{r|\vec{q}_e}\right]$$

Dunque, le componenti scalari di $\vec{\phi}_A$ ——— sono uguali in tutte le configurazioni di equilibrio. Invece, per la reazione vincolare $\vec{\phi}_O$ vale

$2\sin^2\varphi_e = 1-\cos 2\varphi_e =$

| | | |
|---|---|---|
| $1-\sqrt{1-\lambda^2}$ | $\varphi_e = \varphi_e^{(1)}$ | $\vec{\phi}_O\cdot\vec{e}_r = ac\left(1-\sqrt{1-\lambda^2}\right)+\frac{(M+m)g}{4}$ |
| $1$ | $\varphi_e = \frac{\pi}{4}$ | $\vec{\phi}_O\cdot\vec{e}_r = ac + \frac{M+m}{4}g$ |
| $1+\sqrt{1-\lambda^2}$ | $\varphi_e = \varphi_e^{(2)}$ | $\vec{\phi}_O\cdot\vec{e}_r = ac\left(1+\sqrt{1-\lambda^2}\right) + \frac{M+m}{4}g$ |
| $1-\sqrt{1-\lambda^2}$ | $\varphi_e = \varphi_e^{(3)}$ | $\vec{\phi}_O\cdot\vec{e}_r = ac\left(1-\sqrt{1-\lambda^2}\right) + \frac{M+m}{4}g$ |
| $1$ | $\varphi_e = \frac{5}{4}\pi$ | $\vec{\phi}_O\cdot\vec{e}_r = ac + \frac{(M+m)}{4}g$ |
| $1+\sqrt{1-\lambda^2}$ | $\varphi_e = \varphi_e^{(4)}$ | $\vec{\phi}_O\cdot\vec{e}_r = ac\left(1+\sqrt{1-\lambda^2}\right) + \frac{(M+m)}{4}g$ |

mentre

$$\vec{\phi}_O\cdot\vec{e}_{\varphi_e} = -\frac{F}{2}+\frac{5F}{4} = \frac{3F}{4},\quad \vec{\phi}_O\cdot\vec{e}_z = (M+m)\,g$$

3) Reazioni della Domina nel punto P 

Applicando l'eq. della statica nel punto P

$$\vec{F}_{P|eq} + \vec{\Phi}_{P|eq} = \vec{0} ,$$

poiché

$$\vec{F}_{P|eq} = m\vec{g} + \vec{F}^{(pell)}_{P|eq} + \cancel{\vec{F}^{(el)}_{P|eq}} = -mg\,\vec{e}_z + F\,\vec{e}_{\varphi|\vec{q}_e}$$

si ottiene

$$\vec{\Phi}_{P|eq} = mg\,\vec{e}_z - F\,\vec{e}_{\varphi|\vec{q}_e}$$

4) Scriviamo le eq. di Lagrange. A tale scopo, dobbiamo calcolare l'energia cinetica

$$(8.1)\qquad K = K^{(telaio)} + K^{(punto)}$$

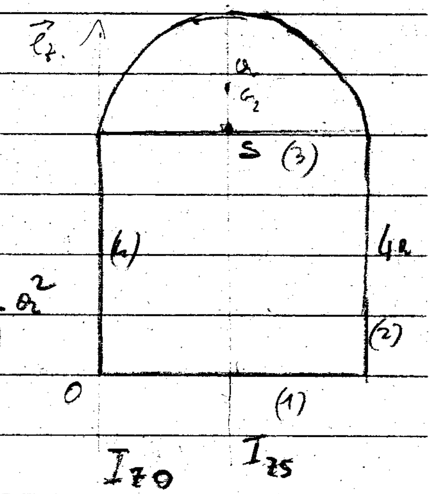
Calcolo di $K^{(telaio)}$

Poichè il telaio è un rigido con asse fisso, vale che

$$(8.2)\qquad K^{(telaio)} = \frac{1}{2}\vec{\omega}\cdot \mathbb{I}_O(\vec{\omega}) = \frac{1}{2}\dot{\varphi}\vec{e}_z\cdot \mathbb{I}_O(\dot{\varphi}\vec{e}_z) = \frac{1}{2}\dot{\varphi}^2\vec{e}_z\cdot \mathbb{I}_O(\vec{e}_z) = \frac{1}{2}\dot{\varphi}^2 I_{zO}$$

dove $I_{zO}$ denota il momento d'inerzia rispetto all'asse $(O;\vec{e}_z)$. Quindi

$$I_{zO} = I_{zO}^{(rett)} + I_{zO}^{(sc)}$$




$$(8.3)\qquad I_{zO}^{(rett)} = 2I_{zO}^{(1)} + I_{zO}^{(2)} = 2\,\frac{1}{3}M_{11}(2a)^2 + M_{12}(2a)^2 = \frac{2}{3}\frac{M}{(12+\pi)a}2a\,4a^2 + \frac{M}{(12+\pi)a}4a\,4a^2 = \frac{16+48}{3(12+\pi)} = \frac{64}{3(12+\pi)}a^2$$

$$(8.4)\qquad I_{zO}^{(sc)} = I_{zS}^{(sc)} + M_2 a^2 = \frac{1}{2}M_2 a^2 + M_2 a^2 = \frac{3}{2}M_2 a^2 \overset{(5.3)}{=} \frac{3}{2}a^2\frac{M\pi}{12+\pi}$$

Dunque

$$(8.5)\qquad I_{zO} = \frac{M}{12+\pi}a^2\left(\frac{64}{3}+\frac{3}{2}\pi\right) = \frac{Ma^2}{12+\pi}\,\frac{128+9\pi}{6} = Ma^2 b$$

$$(8.6)\qquad K^{(telaio)} = \frac{1}{2}Ma^2 b\,\dot{\varphi}^2 \qquad\qquad b := \frac{128+9\pi}{6(12+\pi)}$$

$$K^{(punto)} = \frac{1}{2} m |\vec{v}_P|^2$$

$$\vec{v}_P = \dot{\vec{x}}_P \overset{(1.3)}{=} a \left[ \cos\theta\,\dot\theta\,\vec{e}_z + (1+\sin\theta)\,\dot{\vec{e}}_2 - \sin\theta\,\dot\theta\,\vec{e}_7 \right] \qquad \dot{\vec{e}}_2 = \dot\varphi\,\vec{e}_\varphi$$

(9.1) $$= a \left[ \cos\theta\,\dot\theta\,\vec{e}_z + (1+\sin\theta)\,\dot\varphi\,\vec{e}_\varphi - \sin\theta\,\dot\theta\,\vec{e}_7 \right] \qquad \dot{\vec{e}}_\varphi = -\dot\varphi\,\vec{e}_2$$

$$|\vec{v}_P|^2 = \vec{v}_P \cdot \vec{v}_P = a^2 \left[ \cos^2\theta\,\dot\theta^2 + (1+\sin\theta)^2\dot\varphi^2 + \sin^2\theta\,\dot\theta^2 \right]$$

$$= a^2 \left[ \dot\theta^2 + (1+\sin\theta)^2\dot\varphi^2 \right]$$

Quindi

(9.2) $$K^{(punto)} = \frac{1}{2} m a^2 \left[ \dot\theta^2 + (1+\sin\theta)^2\dot\varphi^2 \right]$$

Dunque

(9.3) $$K = \frac{1}{2} M a^2 b\,\dot\varphi^2 + \frac{1}{2} m a^2 \left[ \dot\theta^2 + (1+\sin\theta)^2\dot\varphi^2 \right] =$$

$$= \frac{1}{2} a^2 \left[ M b + m(1+\sin\theta)^2 \right] \dot\varphi^2 + \frac{1}{2} m a^2 \dot\theta^2$$

$$\frac{\partial K}{\partial\dot\varphi} = a^2 \left[ M b + m(1+\sin\theta)^2 \right] \dot\varphi \qquad \frac{\partial K}{\partial\varphi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot\varphi}\right) = a^2 \left[ M b + m(1+\sin\theta)^2 \right] \ddot\varphi + 2a^2 m (1+\sin\theta)\cos\theta\,\dot\theta\,\dot\varphi$$

(9.4) $$EL_\varphi: \quad a^2 \left[ M b + m(1+\sin\theta)^2 \right] \ddot\varphi + 2a^2 m (1+\sin\theta)\cos\theta\,\dot\theta\,\dot\varphi = -2a^2 c \sin 2\varphi + F a (1+\sin\theta)$$

$$\frac{\partial K}{\partial\dot\theta} = m a^2 \dot\theta, \quad \frac{\partial K}{\partial\theta} = a^2 m (1+\sin\theta)\cos\theta\,\dot\varphi^2, \quad \frac{d}{dt}\left(\frac{\partial K}{\partial\dot\theta}\right) = m a^2 \ddot\theta$$

(9.5) $$EL_\theta: \quad m a^2 \ddot\theta - m a^2 (1+\sin\theta)\cos\theta\,\dot\varphi^2 = a(mg - ac)\sin\theta$$

5) Linearizzazione intorno agli equilibri

Introdotto il vettore degli scarti $\vec{x} = \underset{\mathcal{E}}{\vec{q}(t) - \vec{q}_e}$, il sistema delle EL linearizzate è dato da

$$(10.1) \quad A\ddot{\vec{x}} + B\dot{\vec{x}} + C\vec{x} = 0,$$

dove

$$(10.2) \quad A_{ij} = \left.\frac{\partial^2 K}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{eq}, \quad \cancel{B_{ij} = -\left.\frac{\partial Q_i}{\partial \dot{q}_j}\right|_{eq} = 0}, \quad C_{ij} = -\left.\frac{\partial Q_i}{\partial q_j}\right|_{eq}$$

Allora

$$(10.3) \quad A = a^2 \begin{bmatrix} Mb + m(1+\sin\vartheta_e)^2 & 0 \\ 0 & m \end{bmatrix} = a^2 \begin{bmatrix} Mb+m & 0 \\ 0 & m \end{bmatrix}$$

$$(10.4) \quad C = -\begin{bmatrix} \frac{\partial Q_\varphi}{\partial \varphi} & \frac{\partial Q_\varphi}{\partial \vartheta} \\ \frac{\partial Q_\vartheta}{\partial \varphi} & \frac{\partial Q_\vartheta}{\partial \vartheta} \end{bmatrix}_{eq} = -\begin{bmatrix} -4a^2c\cos 2\varphi_e & Fa\cos\vartheta_e \\ 0 & a(mg - ac)\cos\vartheta_e \end{bmatrix}$$

$$= +\begin{bmatrix} 4a^2c\cos 2\varphi_e & -Fa \\ 0 & a(ac-mg) \end{bmatrix}$$

Allora, il sistema (10.1) si scrive

$$(10.5) \quad \begin{cases} a^2(Mb+m)\ddot{x}_1 + 4a^2c\cos 2\varphi_e\, x_1 - F\cancel{a}\, x_2 = 0 \\ a^2 m\ddot{x}_2 + \cancel{a}(ac-mg)\, x_2 = 0 \end{cases}$$

Quindi, intorno a 

$\vec{q}_e^{\,(1)}$ e $\vec{q}_e^{\,(3)}$:

$\cos 2\varphi_e = \sqrt{1-\lambda^2}$

$$\begin{cases} (Mb+m)\ddot{x}_1 + 4ae\sqrt{1-\lambda^2}\,x_1 - F x_2 = 0 \\ am\,\ddot{x}_2 + (ae-mg)\,x_2 = 0 \end{cases}$$

$\vec{q}_e^{\,(2)}$ e $\vec{q}_e^{\,(4)}$:

$\cos 2\varphi_e = -\sqrt{1-\lambda^2}$

$$\begin{cases} (Mb+m)\ddot{x}_1 - 4ae\sqrt{1-\lambda^2}\,x_1 - F x_2 = 0 \\ am\,\ddot{x}_2 + (ae-mg)\,x_2 = 0 \end{cases}$$

$\vec{q}_e^{\,(5)}$ e $\vec{q}_e^{\,(6)}$:

$\cos 2\varphi_e = 0$

$$\begin{cases} (Mb+m)\ddot{x}_1 - F x_2 = 0 \\ am\,\ddot{x}_2 + (ae-mg)\,x_2 = 0 \end{cases}$$

6) Reazioni dinamiche in O sulla lamina

Usiamo le ECD su tutto il modello

$$(12.1)\begin{cases} \vec{R}^{\,est,\,att} + \vec{\phi}_O' + \vec{\phi}_A' = M\,\vec{a}_G + m\,\vec{a}_P \\ \vec{M}_O^{\,est,\,att} + (A-O)\times\vec{\phi}_A' = \dfrac{d\vec{L}_O}{dt} \end{cases}$$

$$\dot{\vec{e}}_r = \dot{\varphi}\,\vec{e}_\varphi \qquad \dot{\vec{e}}_\varphi = -\dot{\varphi}\,\vec{e}_r$$

$$\vec{a}_G = \dot{\vec{v}}_G = \ddot{\vec{x}}_G = \frac{d^2}{dt^2}\left(a\,\vec{e}_r + a\,h\,\vec{e}_z\right) = a\frac{d^2}{dt^2}\vec{e}_r =$$

$$(12.2)\qquad = a\frac{d}{dt}\left(\dot{\varphi}\,\vec{e}_\varphi\right) = a\left(\ddot{\varphi}\,\vec{e}_\varphi + \dot{\varphi}\,\dot{\vec{e}}_\varphi\right) = a\left(\ddot{\varphi}\,\vec{e}_\varphi - \dot{\varphi}^2\,\vec{e}_r\right)$$

$$\vec{a}_P = \dot{\vec{v}}_P \overset{(9.1)}{=} a\Big[\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right)\vec{e}_r + \cos\theta\,\dot{\theta}\,\dot{\vec{e}}_r +$$

$$\left(\cos\theta\,\dot{\theta}\,\dot{\varphi} + (1+\sin\theta)\,\ddot{\varphi}\right)\vec{e}_\varphi + (1+\sin\theta)\,\dot{\varphi}\,\dot{\vec{e}}_\varphi +$$

$$-\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right)\vec{e}_z\Big]$$

$$(12.3)\qquad = a\Big[\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta} - (1+\sin\theta)\,\dot{\varphi}^2\right)\vec{e}_r +$$

$$+\left(2\cos\theta\,\dot{\theta}\,\dot{\varphi} + (1+\sin\theta)\,\ddot{\varphi}\right)\vec{e}_\varphi - \left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right)\vec{e}_z\Big]$$

Quindi, tenendo conto della (4.3), (12.2) e (12.3), si ottiene

$$\vec{\phi}_O' = +2ae\sin^2\varphi\,\vec{e}_r - \left(F - ae\sin^2\varphi\right)\vec{e}_\varphi + (M+m)\,g\,\vec{e}_z$$

$$+ Ma\left(\ddot{\varphi}\,\vec{e}_\varphi - a\dot{\varphi}^2\,\vec{e}_r\right) +$$

$$+ ma\Big[\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta} - (1+\sin\theta)\,\dot{\varphi}^2\right)\vec{e}_r +$$

$$+\left(2\cos\theta\,\dot{\theta}\,\dot{\varphi} + (1+\sin\theta)\,\ddot{\varphi}\right)\vec{e}_\varphi - \left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right)\vec{e}_z\Big] +$$

$$-\vec{\phi}_A'$$

$$\vec{\phi}_O' = \left[2ac\sin^2\varphi - Ma^2\dot\varphi^2 + ma\left(-\sin\theta\,\dot\theta^2+\cos\theta\,\ddot\theta - (1+\sin\theta)\dot\varphi^2\right)\right]\vec{e}_r +$$

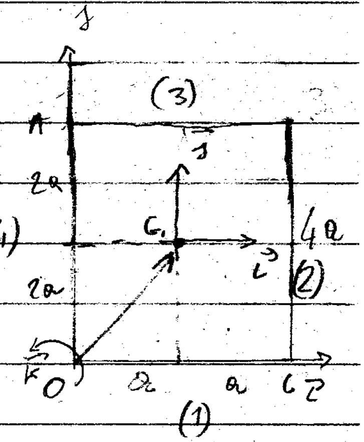
(13.1) $$+\left[-F + ac\sin 2\varphi + Ma\ddot\varphi + ma\left(2\cos\theta\,\dot\theta\dot\varphi + (1+\sin\theta)\ddot\varphi\right)\right]\vec{e}_\varphi +$$

$$+\left[(M+m)g - ma\left(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta\right)\right]\vec{e}_z - \vec{\phi}_A'$$

Calcolo di $\vec L_O$

$$\vec L_O = \vec L_O^{(telaio)} + \vec L_O^{(punto)}$$

$$\vec L_O^{(telaio)} = \vec L_O^{(rett)} + \vec L_O^{(sc)} = \mathbb{I}_O^{(rett)}(\vec\omega) + \mathbb{I}_O^{(sc)}(\vec\omega)$$




Dal Teo di Huygens-Steiner per l'op. d'inerzia

$$I_O^{(rett)} = I_{G_1}^{rett} + M_1 \begin{bmatrix}(2a)^2 & -2a^2 & 0\\ -2a^2 & a^2 & 0\\ 0&0&5a^2\end{bmatrix}$$

$$I_{G_1}^{(rett)} = \begin{bmatrix} I_{11}^{(rett)} &&\\ & I_{22}^{(rett)} & \\ && I_{11}^{(rett)}+I_{22}^{(rett)}\end{bmatrix} = \frac{Ma^2\,2}{3(12+\pi)}\begin{bmatrix}40&&\\&14&\\&&54\end{bmatrix}$$

$$I_{11}^{(rett)} = 2\left(I_{11}^{(2)}+I_{11}^{(3)}\right) = 2\left(\frac{16}{3(12+\pi)}+\frac{8}{12+\pi}\right)Ma^2$$

poiché

$$I_{11}^{(2)} = \frac{1}{12}M_{12}(4a)^2 = \frac{1\,\cancel{4}\,M\,16a^2}{\cancel{12}_3(12+\pi)}$$

$$M_{12} = \rho L_{12} = \frac{M}{(12+\pi)\cancel{a}}4\cancel{a}$$

$$I_{11}^{(3)} = M_{13}(2a)^2 = \frac{2M}{12+\pi}4a^2 = \frac{8Ma^2}{12+\pi}$$

$$M_{13} = \rho L_{13} = \frac{M}{(12+\pi)\cancel{a}}2\cancel{a}$$

$$I_{22}^{(rett)} = 2\left(I_{22}^{(2)}+I_{22}^{(3)}\right) = 2\left(\frac{4}{12+\pi}+\frac{2}{3(12+\pi)}\right)$$

poiché

$$I_{22}^{(2)} = M_{12}a^2 = \frac{4}{12+\pi}Ma^2$$

$$I_{22}^{(3)} = \frac{1}{12}M_{13}(2a)^2 = \frac{1}{\cancel{12}}\frac{2}{3(12+\pi)}\cancel{4}a^2$$

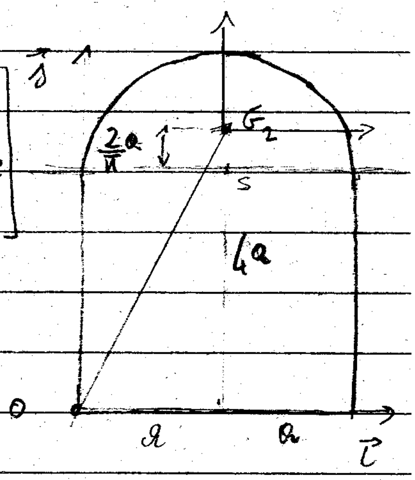
$$I_O^{(rett)} = \frac{Ma^2}{12+\pi}\begin{bmatrix}\frac{224}{3}&-24&0\\-24&\frac{64}{3}&0\\0&0&96\end{bmatrix}$$

$$\mathbb I_O^{(rett)}(\vec\omega) = \dot\varphi\,\mathbb I_O^{(rett)}(\vec e_z) = \dot\varphi\,\mathbb I_O^{(rett)}(\vec j) = \frac{Ma^2\dot\varphi}{12+\pi}\left(-24\vec i + \frac{64}{3}\vec j\right) =$$

$$= \frac{8}{12+\pi}Ma^2\dot\varphi\left(3\vec e_r + \frac{8}{3}\vec e_z\right)$$

$$\vec{L}_O^{(x)} = \mathbb{I}_O^{(x)}(\vec{\omega})$$

$$I_O^{(x)} = I_{G_2}^{(x)} + M_2 \begin{bmatrix} (4a+\overline{G_2S})^2 & -a(4a+\overline{G_2S}) & 0 \\ -a(4a+\overline{G_2S}) & a^2 & 0 \\ 0 & 0 & a^2+(4a+\overline{G_2S})^2 \end{bmatrix}$$




$$I_{G_2}^{(x)} = \begin{bmatrix} I_{11}^{(x)} & 0 & 0 \\ 0 & I_{22}^{(x)} & 0 \\ 0 & 0 & I_{11}^{(x)}+I_{22}^{(x)} \end{bmatrix}$$

dove
$$I_{11}^{(x)} = I_{1S} - M_2 \overline{G_2S}^2 = \frac{1}{2} M_2 a^2 - M_2 \overline{G_2S}^2 = M_2 a^2 \left(\frac{1}{2} - \left(\frac{2}{\pi}\right)^2\right)$$
$$I_{22}^{(x)} = I_{2S} = \frac{1}{2} M_2 a^2, \quad I_{11}^{(x)}+I_{22}^{(x)} = M_2 a^2 \left(1 - \left(\frac{2}{\pi}\right)^2\right)$$

Quindi

$$I_{G_2}^{(x)} = M_2 a^2 \begin{bmatrix} \frac{1}{2} - \left(\frac{2}{\pi}\right)^2 & & \\ & \frac{1}{2} & \\ & & 1 - \left(\frac{2}{\pi}\right)^2 \end{bmatrix}$$

Dunque

$$I_O^{(x)} = M_2 a^2 \begin{bmatrix} \frac{1}{2} - \left(\frac{2}{\pi}\right)^2 & & \\ & \frac{1}{2} & \\ & & 1 - \left(\frac{2}{\pi}\right)^2 \end{bmatrix} + M_2 \begin{bmatrix} \left(4a+\frac{2a}{\pi}\right)^2 & -a\left(4a+\frac{2a}{\pi}\right) & 0 \\ -a\left(4a+\frac{2a}{\pi}\right) & a^2 & 0 \\ 0 & 0 & a^2+\left(4a+\frac{2a}{\pi}\right)^2 \end{bmatrix}$$

Allora

$$(15.1)\quad I_0^{(sc)} = M_2 a^2 \begin{bmatrix} \frac{33}{2}+\frac{16}{\pi} & -\left(4+\frac{2}{\pi}\right) & 0 \\ -\left(4+\frac{2}{\pi}\right) & \frac{3}{2} & 0 \\ 0 & 0 & 18+\frac{16}{\pi} \end{bmatrix}$$

Dunque, il momento angolare della parte semicircolare è

$$\vec{L}_0^{(sc)} = \mathbb{I}_0^{(sc)}\left(\dot\varphi \vec{e}_r\right) = \dot\varphi\, \mathbb{I}_0^{(sc)}\left(\vec{j}\right) = M_2 a^2 \dot\varphi\left[-\left(4+\frac{2}{\pi}\right)\vec{i} + \frac{3}{2}\vec{j}\right]$$

$$= \frac{\pi}{12+\pi} M a^2 \dot\varphi\left[-\left(4+\frac{2}{\pi}\right)\vec{e}_r + \frac{3}{2}\vec{e}_z\right] =$$

Quindi, il momento angolare di tutto il telaio

$$\vec{L}_0^{(tel)} = \frac{12}{12+\pi} M a^2 \dot\varphi\left(-\frac{3}{2}\vec{e}_r + \frac{4}{3}\vec{e}_z\right) + \frac{\pi}{12+\pi} M a^2 \dot\varphi\left[-\left(4+\frac{2}{\pi}\right)\vec{e}_r + \frac{3}{2}\vec{e}_z\right]$$

$$(15.2)\quad = \frac{M}{12+\pi} a^2 \dot\varphi\left[\left(-18-\pi\left(4+\frac{2}{\pi}\right)\right)\vec{e}_r + \left(16+\frac{3\pi}{2}\right)\vec{e}_z\right]$$

$$= \frac{M}{12+\pi} a^2 \dot\varphi\left[-\left(20+4\pi\right)\vec{e}_r + \left(16+\frac{3}{2}\pi\right)\vec{e}_z\right]$$

N.B. Dato che $\vec{L}_0 = I_0(\dot\varphi \vec{e}_r)$, si noti che $\vec{e}_r$ NON è $API(O)$ per il telaio quindi esso NON è bilanciato dinamicamente.

$$\frac{d\vec{L}_0^{(tel)}}{dt} = \frac{M}{12+\pi} a^2 \ddot\varphi\left[-\left(20+4\pi\right)\vec{e}_r + \left(16+\frac{3}{2}\pi\right)\vec{e}_z\right] +$$

$$(15.3)\quad + \frac{M}{12+\pi} a^2 \dot\varphi\left[-\left(20+4\pi\right)\dot\varphi\,\vec{e}_\varphi\right] =$$

$$= \frac{M}{12+\pi} a^2 \left[-\left(20+4\pi\right)\left(\ddot\varphi\,\vec{e}_r + \dot\varphi^2\,\vec{e}_\varphi\right) + \left(16+\frac{3}{2}\pi\right)\ddot\varphi\,\vec{e}_z\right]$$

Calcoliamo, ora, il momento angolare del punto P.

$$\vec{L}_O^{(P)} = (P-O) \times m\vec{v}_P = \begin{vmatrix} \vec{e}_r & \vec{e}_\varphi & \vec{e}_z \\ a(\sin\theta+1) & 0 & a(\cos\theta+4) \\ ma(\cos\theta\,\dot{\theta}) & ma(1+\sin\theta)\dot{\varphi} & -ma\sin\theta\,\dot{\theta} \end{vmatrix} =$$

$$= -\vec{e}_r \; m a^2(\cos\theta+4)(1+\sin\theta)\,\dot{\varphi} +$$

$$(16.1) \quad -\vec{e}_\varphi\left[-ma^2(\sin\theta+1)\sin\theta\,\dot{\theta} - m a^2\cos\theta\,\dot{\theta}\,(\cos\theta+4)\right] +$$

$$+\vec{e}_z \; m a^2(\sin\theta+1)^2\,\dot{\varphi}$$

$$= m a^2\left[(\cos\theta+4)(\sin\theta+1)\,\dot{\varphi}\,\vec{e}_r + (1+\sin\theta+4\cos\theta)\,\dot{\theta}\,\vec{e}_\varphi + (\sin\theta+1)^2\,\dot{\varphi}\,\vec{e}_z\right]$$

La sua derivata risp. al tempo è:

$$\frac{d\vec{L}_O^{(P)}}{dt} = (\vec{v}_P - \vec{v}_O)\times m\vec{v}_P + (P-O)\times m\vec{a}_P =$$

$$\overset{(12.3)}{=} \begin{vmatrix} \vec{e}_r & \vec{e}_\varphi & \vec{e}_z \\ a(\sin\theta+1) & 0 & a(\cos\theta+4) \\ ma(-\sin\theta\,\dot{\theta}^2+\cos\theta\,\ddot{\theta}-(1+\sin\theta)\dot{\varphi}^2) & ma(2\cos\theta\,\dot{\theta}\dot{\varphi}+(1+\sin\theta)\ddot{\varphi}) & -ma(\cos\theta\,\dot{\theta}^2+\sin\theta\,\ddot{\theta}) \end{vmatrix}$$

$$= -m\vec{e}_r\, a^2(\cos\theta+4)(2\cos\theta\,\dot{\theta}\dot{\varphi}+(1+\sin\theta)\,\ddot{\varphi}) +$$

$$+ m\vec{e}_\varphi\left[+a^2(\sin\theta+1)(\cos\theta\,\dot{\theta}^2+\sin\theta\,\ddot{\theta}) + a^2(\cos\theta+4)(-\sin\theta\,\dot{\theta}^2+\cos\theta\,\ddot{\theta}-(1+\sin\theta)\dot{\varphi}^2)\right]$$

$$(16.2) \quad + m\vec{e}_z\, a^2(\sin\theta+1)(2\cos\theta\,\dot{\theta}\dot{\varphi}+(1+\sin\theta)\,\ddot{\varphi})$$

$$= -\vec{e}_r\, m a^2(\cos\theta+4)(2\cos\theta\,\dot{\theta}\dot{\varphi}+(1+\sin\theta)\,\ddot{\varphi}) +$$
$$+\vec{e}_\varphi\, a^2 m\left[(\cos\theta-4\sin\theta)\dot{\theta}^2+(\sin\theta+4\cos\theta+1)\ddot{\theta}-(\cos\theta+4)(\sin\theta+1)\dot{\varphi}^2\right]+$$
$$+\vec{e}_z\, a^2 m(\sin\theta+1)(2\cos\theta\,\dot{\theta}\dot{\varphi}+(1+\sin\theta)\,\ddot{\varphi})$$

Dunque, dalla (15.3) e dalla (16.2) si ottiene

$$\frac{d\vec{L}_0}{dt} = a^2\left[\frac{-M}{12+\pi}(20+4\pi) - m a^2(\cos\theta+4)\big(2\cos\theta\,\dot\theta\dot\varphi + (1+\sin\theta)\big)\right]\ddot\varphi\,\vec{e}_z +$$

$$(17.1)\qquad + a^2\left[\left(\frac{-M}{12+\pi}(20+4\pi) - m(\cos\theta+4)(\sin\theta+1)\right)\dot\varphi^2 + m\left[(\cos\theta-4\sin\theta)\dot\theta^2 + (\sin\theta+4\cos\theta+1)\ddot\theta\right]\right]\vec{e}_\varphi$$

$$+ a^2\left[\left(\frac{M}{12+\pi}\left(16+\frac{3}{2}\pi\right) + m(\sin\theta+1)^2\right)\ddot\varphi + m(\sin\theta+1)2\cos\theta\,\dot\theta\dot\varphi\right]\vec{e}_z$$

Allora, tenendo conto della (6.1), si ottiene

$$(17.2)\quad \vec{\phi}'_A\cdot\vec{e}_z = -\frac{g}{4}\big(M+m(1+\sin\theta)\big) + a^2\left[-\frac{M}{12+\pi}(20+4\pi) - m a^2(\cos\theta+4)\big(2\cos\theta\,\dot\theta\dot\varphi + (1+\sin\theta)\big)\right]\ddot\varphi$$

$$(17.3)\quad \vec{\phi}'_A\cdot\vec{e}_\varphi = -\frac{1}{4}(\cos\theta+4)F +$$

$$+ a^2\left[\frac{-M}{12+\pi}(20+4\pi) - m(\cos\theta+4)(\sin\theta+1)\dot\varphi^2 + m\left[(\cos\theta-4\sin\theta)\dot\theta^2 + (\sin\theta+4\cos\theta+4)\ddot\theta\right]\right]$$

$$(17.4)\quad \vec{\phi}'_A\cdot\vec{e}_z = a^2\left[\left(\frac{M}{12+\pi}\left(16+\frac{3}{2}\pi\right) + m(\sin\theta+1)^2\right)\ddot\varphi + m(\sin\theta+1)2\cos\theta\,\dot\theta\dot\varphi\right]$$

Si osservi che il secondo termine della (17.4) equivale all'eq. di Lagrange $EL_\varphi$, quindi è nullo, come ci si aspetta, dato che l'anellino liscio in A NON può esercitare una reazione lungo il suo asse.

Quiz: qual è il significato meccanico della $EL_\theta$ (9.5)?