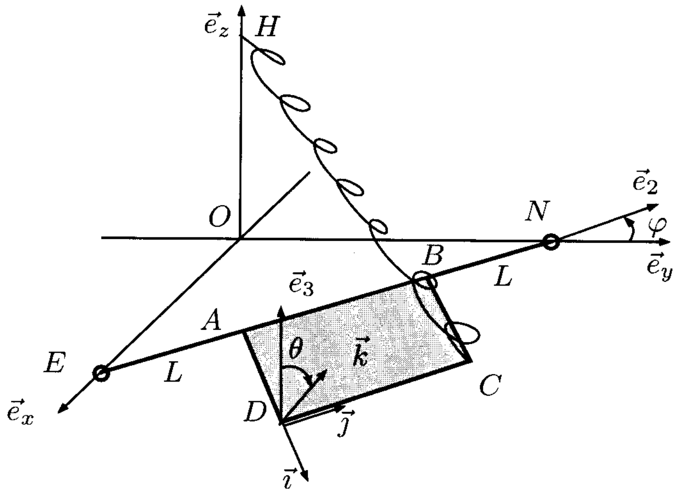
Compito di Meccanica Razionale (9 CFU)

Trieste, 26 giugno 2017. (G. Tondo)



Una lamina omogenea rettangolare di lati L e $2L$ e massa m è saldata, lungo il lato più lungo, ad un'asta EN di lunghezza $4L$ e di massa trascurabile. Gli estremi dell'asta sono vincolati, come in figura, a scorrere senza attrito lungo due guide fisse ortogonali, tramite due cerniere sferiche "bucate". Sul rigido agisce il peso proprio opposto ad $\vec{e}_z$, la forza di richiamo di una molla lineare, di costante elastica c, fissata al vertice C e nel punto fisso H, di quota $2L$.

STATICA

Siano φ l'angolo tra i versori $\vec{e}_y$ ed $\vec{e}_2$, e θ l'angolo di rotazione della lamina intorno all'asse EN orientato come $\vec{e}_2$, misurato tra i versori $\vec{e}_3$ e $\vec{k}$: si considerino entrambi gli angoli appartenenti all'intervallo $]-\pi, \pi]$.

1) Verificare che le configurazioni $\vec{q} = (\varphi = 0, \theta = \pm\frac{\pi}{2})$ sono di equilibrio e determinare e la loro stabilità in funzione dei parametri del modello;

2) le reazioni vincolari esterne sul rigido in E, nelle configurazioni di equilibrio suddette;

3) le reazioni vincolari esterne sul rigido in N, nelle configurazioni di equilibrio suddette.

DINAMICA

4) Scrivere le equazione differenziali pure di moto;

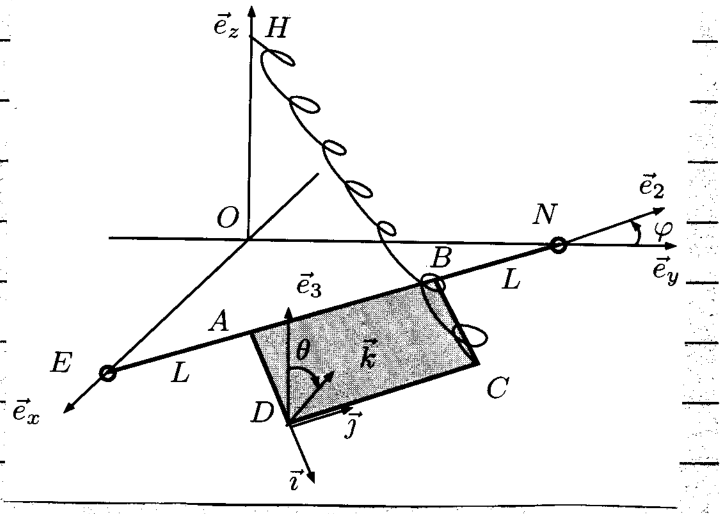
5) linearizzare le equazioni di moto intorno alle configurazioni di equilibrio del punto **1**;

6) calcolare le reazioni vincolare esterne nei punti E e N, durante il moto.

Nota Bene. Si suggerisce di usare, oltre alla base fissa $\mathcal{B} = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$ anche una base intermedia formata dai versori $\mathcal{B}' = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$, con $\vec{e}_2$ diretto come l'asta, $\vec{e}_3 = \vec{e}_z$ ed $\vec{e}_1 = \vec{e}_2 \times \vec{e}_3$. Inoltre, si consiglia di prendere una base $(\vec{\imath}, \vec{\jmath}, \vec{k})$, solidale alla lamina, con $\vec{\jmath} = \vec{e}_2$, il versore $\vec{\imath}$ lungo il lato AD e $\vec{k} = \vec{\imath} \times \vec{\jmath}$.

Tema del 26/6/2017

Il modello è un rigido $\mathcal{R}$ vincolato con 2 cerniere sferiche "bucate" alle guide $(O, \vec{e}_x)$ e $(O, \vec{e}_y)$. Con il metodo dei congelamenti successivi si può osservare che, una volta bloccato lo spostamento virtuale di E, non rimane che lo spostamento virtuale rototorio intorno all'asse EN. Dunque il rigido ha 2 g. l. Coordinate libere: $-\pi < \varphi < \pi$, $-\pi < \theta \leq \pi$.



Conviene usare le seguenti basi ON:

$$\mathcal{B} = (\vec{e}_x, \vec{e}_y, \vec{e}_z), \quad \mathcal{B}' = (\vec{e}_1, \vec{e}_2, \vec{e}_3), \quad \mathcal{B}'' = (\vec{i}, \vec{j}, \vec{k})$$

$$(1.1)\begin{cases} \vec{e}_1 = \vec{e}_2 \times \vec{e}_3 = \cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y \\ \vec{e}_2 = \cos\varphi\, \vec{e}_y - \sin\varphi\, \vec{e}_x \\ \vec{e}_3 = \vec{e}_z \end{cases}$$

$$[\vec{e}_1, \vec{e}_2, \vec{e}_3] = [\vec{e}_x, \vec{e}_y, \vec{e}_z] \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\varphi}$$

$$(1.2)\begin{cases} \vec{i} = \vec{j} \times \vec{k} = \cos\theta\, \vec{e}_1 - \sin\theta\, \vec{e}_3 \\ \vec{j} = \vec{e}_2 \\ \vec{k} = \cos\theta\, \vec{e}_3 + \sin\theta\, \vec{e}_1 \end{cases}$$

$$[\vec{i}, \vec{j}, \vec{k}] = [\vec{e}_1, \vec{e}_2, \vec{e}_3] \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\vec{i}, \vec{j}, \vec{k}] = [\vec{e}_x, \vec{e}_y, \vec{e}_z] R_\varphi R_\theta = [\vec{e}_x, \vec{e}_y, \vec{e}_z] \begin{bmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad (1.3)$$

$$E-O = 4L\sin\varphi\,\vec{e}_x$$

$$G-E = L\left(\frac{1}{2}\vec{i} + 2\,\vec{j}\right)$$

$$G-O = (G-E)+(E-O) = 4L\sin\varphi\,\vec{e}_x + L\left(\frac{1}{2}\vec{i}+2\vec{j}\right) =$$
$$= 4L\sin\varphi\,\vec{e}_x + \frac{L}{2}\cos\varphi\cos\theta\,\vec{e}_x + \frac{L}{2}\sin\varphi\cos\theta\,\vec{e}_y - \frac{L}{2}\sin\theta\,\vec{e}_z + 2L\left(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\right)$$
$$= \left(2L\sin\varphi + \frac{L}{2}\cos\varphi\cos\theta\right)\vec{e}_x + \left(\frac{L}{2}\sin\varphi\cos\theta + 2L\cos\varphi\right)\vec{e}_y - \frac{L}{2}\sin\theta\,\vec{e}_z$$

$$C-B = L\vec{i}, \quad B-E = 3L\vec{j}$$
$$C-O = (C-B)+(B-E)+(E-O) = L\left(\vec{i}+3\vec{j}\right) + 4L\sin\varphi\,\vec{e}_x$$

$$H-O = 2L\,\vec{e}_z$$

$$C-H = (C-O)+(O-H) = L\left(\vec{i}+3\vec{j}\right) + 4L\sin\varphi\,\vec{e}_x - 2L\,\vec{e}_z$$

$$|\overline{CH}|^2 = (C-H)\cdot(C-H) = L^2\left(\vec{i}+3\vec{j}+4\sin\varphi\,\vec{e}_x - 2\,\vec{e}_z\right)\cdot\left(\vec{i}+3\vec{j}+4\sin\varphi\,\vec{e}_x - 2\vec{e}_z\right)$$
$$= L^2\Big[1 + 9 + 16\sin^2\varphi + 4 + \cancel{6\,\vec{i}\cdot\vec{j}} + 8\sin\varphi\,\vec{i}\cdot\vec{e}_x - 4\,\vec{i}\cdot\vec{e}_z +$$
$$+ 24\sin\varphi\,\vec{j}\cdot\vec{e}_x - \cancel{12\,\vec{j}\cdot\vec{e}_z} - \cancel{16\sin\varphi\,\vec{e}_x\cdot\vec{e}_z}\Big]$$
$$= L^2\left(14 + 16\sin^2\varphi + 8\sin\varphi\cos\varphi\cos\theta + 4\sin\theta + 24\sin\varphi(-\sin\varphi)\right)$$
$$= L^2\left(14 - 8\sin^2\varphi + 4\sin 2\varphi\cos\theta + 4\sin\theta\right)$$

Statica

Dato che la sollecitazione è conservativa, cerchiamo i punti stazionari dell'energia potenziale

$$V(\varphi,\theta) = V^{(peso)} + V^{(molla)}$$

$$V^{(peso)} = -m\vec{g}\cdot(G-O) = mg\,\vec{e}_2 \cdot \left(4L\sin\varphi\,\vec{e}_x + L\left(\frac{1}{2}\vec{\imath} + 2\vec{\jmath}\right)\right)$$

$$= mgL\,\frac{1}{2}\left(\vec{e}_2\cdot\vec{\imath} + 2\,\cancel{\vec{e}_2\cdot\vec{\jmath}}\right) =$$

$$= mg\frac{L}{2}(-\sin\theta)$$

$$V^{(molla)} = \frac{1}{2}c\overline{CH}^2 = \frac{1}{2}cL^2\left(-8\sin^2\varphi + 4\sin 2\varphi\cos\theta + 4\sin\theta\right)$$

$$V = -mg\frac{L}{2}\sin\theta + \frac{1}{2}cL^2\left(-8\sin^2\varphi + 4\sin 2\varphi\cos\theta + 4\sin\theta\right)$$

$$= \left(2cL^2 - mg\frac{L}{2}\right)\sin\theta - 4cL^2\sin^2\varphi + 2cL^2\sin 2\varphi\cos\theta$$

$$-Q_\varphi = \frac{\partial V}{\partial\varphi} = -8cL^2\sin\varphi\cos\varphi + 4cL^2\cos 2\varphi\cos\theta$$

$$= 4cL^2\left(-\sin 2\varphi + \cos 2\varphi\cdot\cos\theta\right)$$

$$-Q_\theta = \frac{\partial V}{\partial\theta} = \left(2cL - \frac{mg}{2}\right)L\cos\theta - 2cL^2\sin 2\varphi\sin\theta$$

$$\left.\frac{\partial V}{\partial\varphi}\right|_{\substack{\varphi=0\\ \theta=\pm\pi/2}} = 0 \quad , \quad \left.\frac{\partial V}{\partial\theta}\right|_{\substack{\varphi=0\\ \theta=\pm\pi/2}} = 0 \Rightarrow \vec{q} = \left(0, \pm\frac{\pi}{2}\right) \text{ sono}$$

configurazioni di equilibrio.

Stabilità

$\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right)$, $\vec{q}_e^{(2)} = \left(0, -\frac{\pi}{2}\right)$

Determiniamo la matrice Hessiana di $V(\varphi, \theta)$ e calcoliamola in $\vec{q}_e^{(1)}$ e $\vec{q}_e^{(2)}$.

$$\frac{\partial^2 V}{\partial \varphi^2} = 4cl^2\left(-2\cos 2\varphi - 2\sin 2\varphi \cos\theta\right)$$

$$\frac{\partial^2 V}{\partial \varphi \partial \theta} = -4cl^2 \cos 2\varphi \sin\theta$$

$$\frac{\partial^2 V}{\partial \theta^2} = -\left(2cl - \frac{mg}{2}\right) l \sin\theta - 2cl^2 \sin 2\varphi \cos\theta$$

$$\mathcal{H} = \begin{bmatrix} -8cl^2\left(\cos 2\varphi + \sin 2\varphi \cos\theta\right) & -4cl^2 \cos 2\varphi \sin\theta \\ -4cl^2 \cos 2\varphi \sin\theta & -\left(2cl - \frac{mg}{2}\right) l \sin\theta - 2cl^2 \sin 2\varphi \cos\theta \end{bmatrix}$$

$$\mathcal{H}_{11}\Big|_{\substack{\varphi = 0 \\ \theta = \pm\frac{\pi}{2}}} = -8cl^2 < 0 \Rightarrow \text{instabilità}$$

2) e 3) Reazioni in E e N agli equilibri

Poiché i vincoli nei punti in E e N sono cerniere sferiche "bucate" liscie, si ha che

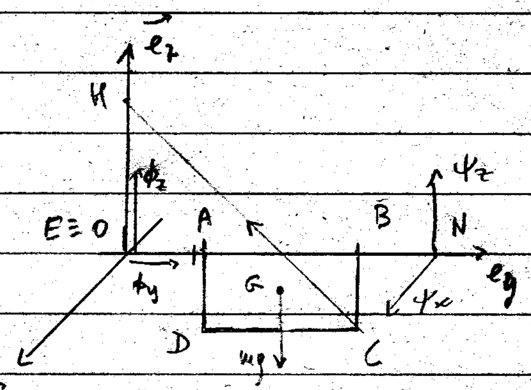
(5.1) $$\mathcal{L}^{reatt} = \left\{ \left(E, \vec{\phi} = \phi_y \vec{e}_y + \phi_z \vec{e}_z\right), \left(N, \vec{\psi} = \psi_x \vec{e}_x + \psi_z \vec{e}_z\right) \right\}$$

Dunque, abbiamo 4 incognite

(5.2) $$\left(\phi_y, \phi_z, \psi_x, \psi_z\right)$$

Per determinarle, scriviamo le ECS che devono essere soddisfatte nelle configurazioni di equilibrio $\vec{q}_e^{(1)}, \vec{q}_e^{(2)}$:

(5.3) $$\begin{cases} \vec{R}^{(est,att)} + \vec{\phi}_E + \vec{\psi}_N = \vec{0} \\ \vec{M}_E^{(est,att)} + (N-E) \times \vec{\psi}_N = \vec{0} \end{cases}$$



In $\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right)$

(5.4) $$\vec{F}_c = -c\,(C-H) = -c\left(3L\,\vec{e}_y - 3L\,\vec{e}_z\right)$$

(5.5) $$\vec{M}_G^{(peso)} = (G-E) \times m\vec{g} = -2mgL\,\vec{e}_x$$

(5.6) $$\vec{M}_E^{(molle)} = (C-E) \times \vec{F}_c = \left(3L\,\vec{e}_y - L\,\vec{e}_z\right) \times \left[-c\left(3L\,\vec{e}_y - 3L\,\vec{e}_z\right)\right]$$
$$= -c\left(-9L^2\vec{e}_x + 3L^2\vec{e}_x\right) = 6cL^2\,\vec{e}_x$$

(5.7) $$\vec{M}_E^{(est,reatt)} = (N-E) \times \vec{\psi}_N = 4L\,\vec{e}_y \times \left(\psi_x \vec{e}_x + \psi_z \vec{e}_z\right) = 4L\left(-\psi_x \vec{e}_z + \psi_z \vec{e}_x\right)$$

(5.8) $\vec{e}_x$: $\quad \psi_x = 0$

(5.9) $\vec{e}_y$: $\quad -3cL + \phi_y = 0 \iff \phi_y = 3cL$

(5.10) $\vec{e}_z$: $\quad 3cL - mg + \phi_z + \psi_z = 0 \iff \phi_z = mg - 3cL - \psi_z$

Resta da determinare l'incognita ψ_z. A tale scopo, utilizziamo la II ECS (5.3) proiettata lungo $\vec{e}_x$:

$$(6.1)\quad \vec{e}_x:\ -2mgL + 6cL^2 + 4L\psi_z = 0$$

Quindi,

$$(6.2)\quad \psi_z = \frac{2mgL - 6cL^2}{4L} = \frac{mg-3cL}{2}$$

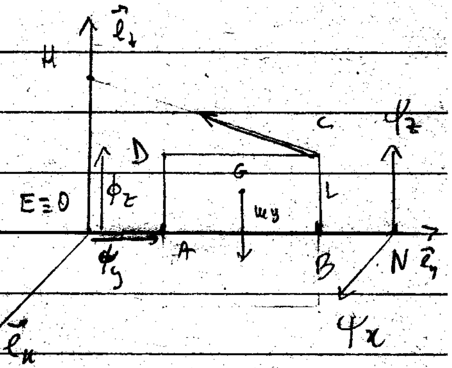
che, sostituito nella (5.10) fornisce

$$(6.3)\quad \phi_z = mg - 3cL - \frac{mg-3cL}{2} = \frac{mg-3cL}{2}$$

Dunque, in $\vec{q_e}^{(1)}$

$$(6.4)\quad \vec{\phi}_E\Big|_{\vec{q_e}^{(1)}} = 3cL\,\vec{e}_y + \frac{mg-3cL}{2}\,\vec{e}_z,\quad \vec{\psi}_N\Big|_{\vec{q_e}^{(1)}} = \frac{mg-3cL}{2}\,\vec{e}_z$$

Analogamente, in $\vec{q_e}^{(2)} = \left(0, -\frac{\pi}{2}\right)$



$$(6.5)\quad \vec{F}_c = -c(C-H) = -c\left(3L\,\vec{e}_y - L\,\vec{e}_z\right)$$

$$(6.6)\quad \vec{M}_E^{(peso)} = (G-E)\times m\vec{g} = -2mgL\,\vec{e}_x$$

$$(6.7)\quad \vec{M}_E^{(molla)} = (C-E)\times\vec{F}_c = \left(3L\,\vec{e}_y + L\,\vec{e}_z\right)\times\left[-c\left(3L\,\vec{e}_y - L\,\vec{e}_z\right)\right] =$$

$$= -c\left(-3L^2\,\vec{e}_x + 3L^2(-\vec{e}_x)\right) = 6cL^2\,\vec{e}_x$$

$$(6.8)\quad \vec{M}_E^{(est,reaz)} = (N-E)\times\vec{\psi}_N = 4L\left(-\psi_x\,\vec{e}_z + \psi_z\,\vec{e}_x\right)$$

$$(6.9)\quad \vec{e}_x:\quad \psi_x = 0$$

$$(6.10)\quad \vec{e}_y:\quad -3cL + \phi_y = 0 \iff \phi_y = 3cL$$

$$(6.11)\quad \vec{e}_z:\quad cL - mg + \phi_z + \psi_z = 0 \iff \phi_z = mg - cL - \psi_z$$

Proiettando la II ECS (5.3) lungo $\vec{e}_x$, si trova

(7.1) $\vec{e}_x: \; -2mgL + 6cL^2 + 4L\psi_z = 0 \iff \psi_z = \dfrac{mg - 3cL}{2}$

che, sostituita nella (6.11) fornisce

(7.2) $\phi_z = mg - cL - \dfrac{mg - 3cL}{2} = \dfrac{mg + cL}{2} > 0$

Dunque, in $\vec{q}_e^{\,(2)}$

(7.3) $\vec{\phi}_E\big|_{\vec{q}_e^{\,(2)}} = 3cL\,\vec{e}_y + \dfrac{mg + cL}{2}\,\vec{e}_z\,, \quad \psi_N\big|_{\vec{q}_e^{\,(2)}} = \dfrac{mg - 3cL}{2}\,\vec{e}_z$

N.B. Si osservi che, sia in $\vec{q}_e^{\,(1)}$, sia in $\vec{q}_e^{\,(2)}$, le componenti della II ECS lungo $\vec{e}_y$ ed $\vec{e}_z$ sono identicamente soddisfatte. Infatti,

$\vec{e}_y: \; 0 = 0$

$\vec{e}_z: \; -4L\psi_x = 0$

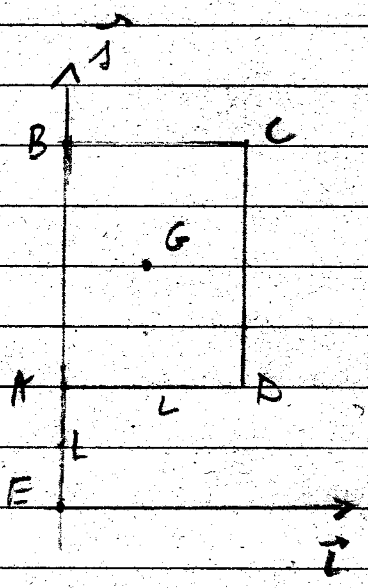
4) Scriviamo le EL. A tale scopo, calcoliamo l'energia cinetica del rigido

$$K = \frac{1}{2} m|\vec{V}_E|^2 + \frac{1}{2}\vec{\omega}\cdot I_E(\vec{\omega}) + m\,\vec{V}_E\cdot\vec{\omega}\times(G-E) \qquad (8.1)$$

$$\vec{v}_E = \frac{d}{dt}(E-O) = 4L\cos\varphi\,\dot{\varphi}\,\vec{e}_x \qquad (8.2)$$

$$|\vec{v}_E|^2 = 16L^2\cos^2\varphi\,\dot{\varphi}^2$$

$$\vec{\omega} = \dot{\varphi}\,\vec{e}_z + \dot{\theta}\,\vec{e}_2 = \dot{\varphi}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{k}) + \dot{\theta}\,\vec{j}$$



Calcolo di $[I_E]^{B''}$

$$[I_E]^{B''} = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

$$I_{11} = I_E(\vec{i})\cdot\vec{i} = I_G(\vec{i})\cdot\vec{i} + m(2L)^2 = \frac{1}{12}m(2L)^2 + 4mL^2 = \frac{13}{3}mL^2$$

$$I_{22} = I_E(\vec{j})\cdot\vec{j} = \frac{1}{3}mL^2, \quad I_{33} = I_{11} + I_{22} = \frac{14}{3}mL^2$$

$$I_{12} = I_E(\vec{i})\cdot\vec{j} = -\frac{m}{2L^2}\int_0^L x\,dx\int_L^{3L} y\,dy = -\frac{m}{2L^2}\left[\frac{x^2}{2}\right]_0^L\left[\frac{y^2}{2}\right]_L^{3L} =$$

$$= -\frac{m}{2L^2}\,\frac{L^2}{2}\,\frac{(3L)^2 - L^2}{2} = -\frac{m}{8}\,8L^2 = -mL^2$$

$$[I_E]^{B''} = mL^2\begin{bmatrix} \frac{13}{3} & -1 & 0 \\ -1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{14}{3} \end{bmatrix}$$

Termine polare di K:

$$\frac{1}{2}\vec{\omega}\cdot I_r(\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot mL^2 \begin{bmatrix} \frac{13}{3} & -1 & 0 \\ -1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{14}{3} \end{bmatrix} \begin{bmatrix} -\dot{\varphi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi}\cos\theta \end{bmatrix} =$$

$$= \frac{1}{2} mL^2 \left[-\dot{\varphi}\sin\theta, \dot{\theta}, \dot{\varphi}\cos\theta\right] \cdot \begin{bmatrix} -\frac{13}{3}\dot{\varphi}\sin\theta - \dot{\theta} \\ \dot{\varphi}\sin\theta + \frac{1}{3}\dot{\theta} \\ \frac{14}{3}\dot{\varphi}\cos\theta \end{bmatrix} =$$

$$= \frac{1}{2}\left(\frac{13}{3}\dot{\varphi}^2\sin^2\theta + \dot{\varphi}\dot{\theta}\sin\theta + \dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2 + \frac{14}{3}\dot{\varphi}^2\cos^2\theta\right)$$

$$= \frac{1}{2} mL^2 \left[\left(\frac{13}{3}\sin^2\theta + \frac{14}{3}\cos^2\theta\right)\dot{\varphi}^2 + 2\dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2\right]$$

$$= \frac{1}{2} mL^2 \left[\frac{1}{3}\left(13 + \cos^2\theta\right)\dot{\varphi}^2 + 2\dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2\right]$$

Termine misto di K:

$$m\vec{v}_r \cdot \vec{\omega} \times (G-E) = m4L\cos\varphi\,\dot\varphi\,\vec{e}_x \cdot \left(\dot\varphi\,\vec{e}_3 + \dot\theta\,\vec{e}_2\right) \times L\left(\frac{1}{2}\vec{i} + 2\vec{j}\right)$$

$$= m4L^2\cos\varphi\,\dot\varphi\,\vec{e}_x \cdot \left(\frac{\dot\varphi}{2}\,\vec{e}_3\times\vec{i} + 2\dot\varphi\,\vec{e}_3\times\vec{e}_2 + \frac{\dot\theta}{2}\,\vec{j}\times\vec{i}\right)$$

$$= m4L^2\cos\varphi\,\dot\varphi\,\vec{e}_x \cdot \left(\frac{\dot\varphi}{2}\cos\theta\,\vec{e}_2 - 2\dot\varphi\,\vec{e}_1 - \frac{\dot\theta}{2}\vec{k}\right)$$

$$= m4L^2\cos\varphi\,\dot\varphi\left(\frac{\dot\varphi}{2}\cos\theta\;\vec{e}_x\cdot\vec{e}_2 - 2\dot\varphi\,\vec{e}_x\cdot\vec{e}_1 - \frac{\dot\theta}{2}\,\vec{e}_x\cdot\vec{k}\right)$$

$$= m4L^2\cos\varphi\,\dot\varphi\left(\frac{\dot\varphi}{2}\cos\theta(-\sin\varphi) - 2\dot\varphi\cos\varphi - \frac{\dot\theta}{2}\cos\varphi\sin\theta\right)$$

$$= -m4L^2\cos\varphi\,\dot\varphi\left(+\frac{\dot\varphi}{2}\sin\varphi\cos\theta + 2\dot\varphi\cos\varphi + \frac{\dot\theta}{2}\cos\varphi\sin\theta\right) =$$

$$= -4mL^2\left[\cos\varphi\left(\frac{1}{2}\sin\varphi\cos\theta + 2\cos\varphi\right)\dot\varphi^2 + \cos^2\varphi\sin\theta\,\frac{\dot\theta}{2}\,\dot\varphi\right]$$

Dunque,

$$K = \frac{1}{2}m\,\overset{8\,2}{\cancel{16L^2}}\cos^2\varphi\,\dot\varphi^2_{\Delta} + \frac{1}{2}mL^2\left[\frac{1}{3}\left(13 + \cos^2\theta\right)\dot\varphi^2_{\Delta} + 2\dot\varphi\dot\theta\sin\theta_{\square} + \frac{1}{3}\dot\theta^2\right] +$$

$$-4mL^2\left[\cos\varphi\left(\frac{1}{2}\sin\varphi\cos\theta + \cancel{2\cos\varphi}\right)\dot\varphi^2_{\Delta} + \cos^2\varphi\sin\theta_{\square}\,\frac{\dot\varphi\dot\theta}{2}\right]$$

$$= mL^2\left[\left(\frac{13}{6} + \frac{1}{6}\cos^2\theta - \sin 2\varphi\cos\theta\right)\dot\varphi^2 + \frac{1}{6}\dot\theta^2 + \left(1-2\cos^2\varphi\right)\sin\theta\,\dot\varphi\dot\theta\right]$$

$$= \frac{1}{2}mL^2\left[\dot\varphi\,,\,\dot\theta\right]\begin{bmatrix} 2\left(\frac{13}{6}+\frac{1}{6}\cos^2\theta - \sin 2\varphi\cos\theta\right) & \left(1-2\cos^2\varphi\right)\sin\theta \\ \left(1-2\cos^2\varphi\right)\sin\theta & \frac{1}{3} \end{bmatrix}\begin{bmatrix}\dot\varphi \\ \dot\theta\end{bmatrix}$$

$$\frac{\partial K}{\partial \dot\varphi} = 2mL^2\left(\frac{13}{6}+\frac{1}{6}\cos^2\theta - \sin 2\varphi\cos\theta\right)\dot\varphi + mL^2\left(1-2\cos^2\varphi\right)\sin\theta\,\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\varphi}\right) = 2mL^2\left(\frac{13}{6}+\frac{1}{6}\cos^2\theta - \sin 2\varphi\cos\theta\right)\ddot\varphi +$$
$$+ 2mL^2\left(-\frac{1}{3}\cos\theta\sin\theta\,\dot\theta - 2\cos 2\varphi\cos\theta\,\dot\varphi + \sin 2\varphi\sin\theta\,\dot\theta\right)\dot\varphi$$
$$+ mL^2\left(1-2\cos^2\varphi\right)\left(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta\right) + mL^2 4\cos\varphi\sin\varphi\sin\theta\,\dot\varphi\,\dot\theta,$$

$$\frac{\partial K}{\partial \varphi} = mL^2\left(-2\cos 2\varphi\cos\theta\,\dot\varphi^2 + 4\cos\varphi\sin\varphi\sin\theta\,\dot\varphi\,\dot\theta\right)$$

Quindi

$$EL_\varphi:\ 2mL^2\left(\frac{13}{6}+\frac{1}{6}\cos^2\theta - \sin 2\varphi\cos\theta\right)\ddot\varphi - 2mL^2\cos 2\varphi\cos\theta\,\dot\varphi^2$$
$$+ 2mL^2\left(-\frac{1}{3}\cos\theta + \sin 2\varphi\right)\sin\theta\,\dot\theta\,\dot\varphi + mL^2\left(1-2\cos^2\varphi\right)\left(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta\right)$$
$$= 4cL^2\left(+\sin 2\varphi - \cos 2\varphi\cos\theta\right)$$

$$\frac{\partial K}{\partial \dot\theta} = \frac{1}{3}mL^2\dot\theta + mL^2\left(1-2\cos^2\varphi\right)\sin\theta\,\dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\theta}\right) = mL^2\left[\frac{1}{3}\ddot\theta + \left(1-2\cos^2\varphi\right)\left(\cos\theta\,\dot\theta\dot\varphi + \sin\theta\,\ddot\varphi\right) + 4\cos\varphi\sin\varphi\sin\theta\,\dot\varphi^2\right]$$

$$\frac{\partial K}{\partial \theta} = mL^2\left[\left(-\frac{1}{3}\cos\theta\sin\theta + \sin 2\varphi\sin\theta\right)\dot\varphi^2 + \left(1-2\cos^2\varphi\right)\cos\theta\,\dot\varphi\,\dot\theta\right]$$

$$EL_\theta:\ mL^2\left[\frac{1}{3}\ddot\theta + \left(1-2\cos^2\varphi\right)\sin\theta\,\ddot\varphi + \left(\sin 2\varphi + \frac{1}{3}\cos\theta\right)\sin\theta\,\dot\varphi^2\right]$$
$$= \left(\frac{mg}{2} - 2cL\right)L\cos\theta + 2cL^2\sin 2\varphi\sin\theta$$

:) Poiché la sollecitazione è conservativa, le EL linearizzate (12
intorno alle configurazioni di equilibrio si scrivono

$$\mathcal{A}\ddot{\vec{x}} + \mathcal{H}_{|\vec{q}_e}\vec{x} = \vec{0} \qquad \vec{x} = \frac{\vec{q}-\vec{q}_e}{\varepsilon}$$

dove

$$\mathcal{A} = mL^2\begin{bmatrix} 2\cdot\frac{13}{\cancel{6}\,3} & -\sin\theta_e \\ -\sin\theta_e & \frac{1}{3}\end{bmatrix}$$

$$\mathcal{H}_{|\vec{q}_e} = \begin{bmatrix} -8cL^2 & -4cL^2\sin\theta_e \\ -4cL^2\sin\theta_e & -\left(2cL-\frac{mg}{2}\right)L\sin\theta_e\end{bmatrix}$$

Dunque,

$$mL^2\begin{bmatrix}\frac{13}{3} & -\sin\theta_e \\ -\sin\theta_e & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} - \begin{bmatrix}8cL^2 & 4cL^2\sin\theta_e \\ 4cL^2\sin\theta_e & \left(2cL-\frac{mg}{2}\right)L\sin\theta_e\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

In $\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right)$, $\sin\theta_e = 1$, quindi

$$\begin{cases} mL^2\left(\frac{13}{3}\ddot{x}_1 - \ddot{x}_2\right) - 8cL^2x_1 - 4cL^2x_2 = 0 \\ mL^2\left(-\ddot{x}_1 + \frac{1}{3}\ddot{x}_2\right) - 4cL^2x_1 - \left(2cL - \frac{mg}{2}\right)L\,x_2 = 0\end{cases}$$

In $\vec{q}_e^{(1)} = \left(0, -\frac{\pi}{2}\right)$, si $\theta_e = -1$, quindi

$$\begin{cases} mL^2\left(\frac{13}{3}\ddot{x}_1 + \ddot{x}_2\right) - 8cL^2 x_1 + 4cL^2 x_2 = 0 \\ mL^2\left(+\ddot{x}_1 + \frac{1}{3}\ddot{x}_2\right) + 4cL^2 x_1 + \left(2cL - \frac{mg}{2}\right)L x_2 = 0 \end{cases}$$

6) <u>Sollecitazione reattiva dinamica in E e N</u>

Analogamente alla Statica, la reazione dinamica è data da

$$(14.1)\quad \mathcal{S}^{(reatt)'} = \left\{ \left(E, \vec{\phi}' = \phi'_y \vec{e}_y + \phi'_z \vec{e}_z\right), \left(N, \vec{\psi}' = \psi'_x \vec{e}_x + \psi'_z \vec{e}_z\right) \right\};$$

quindi abbiamo 4 incognite da trovare,

$$(14.2)\quad \left(\phi'_y, \phi'_z, \psi'_x, \psi'_z\right)$$

Per determinarle, usiamo le ECD

$$(14.3)\quad \begin{cases} \vec{R}^{(est,att)} + \vec{\phi}'_E + \vec{\psi}'_N = m\,\vec{a}_G \\ \vec{M}_E^{(est,att)} + (N-E)\times\vec{\psi}'_N = \dfrac{d\vec{L}_E}{dt} + \vec{v}_E \times m\vec{v}_G \end{cases}$$

Calcoliamo $\vec{a}_G$.

$$\vec{v}_G = \frac{d}{dt}(G-O) = \left[2L\cos\varphi\,\dot\varphi - \frac{L}{2}\left(\sin\varphi\cos\theta\,\dot\varphi + \cos\varphi\sin\theta\,\dot\theta\right)\right]\vec{e}_x +$$
$$+\left(\frac{L}{2}\cos\varphi\cos\theta\,\dot\varphi + \frac{L}{2}\sin\varphi\sin\theta\,\dot\theta - 2L\sin\varphi\,\dot\varphi\right)\vec{e}_y +$$
$$-\frac{L}{2}\cos\theta\,\dot\theta\,\vec{e}_z$$

$$\vec{a}_G = \dot{\vec{v}}_G = \Big(-2L\sin\varphi\,\dot\varphi^2 + 2L\cos\varphi\,\ddot\varphi - \frac{L}{2}\cos\varphi\cos\theta\,\dot\varphi^2 + \frac{L}{2}\sin\varphi\sin\theta\,\dot\theta\dot\varphi - \frac{L}{2}\sin\varphi\cos\theta\,\ddot\varphi +$$
$$+\frac{L}{2}\sin\varphi\sin\theta\,\dot\varphi\dot\theta - \frac{L}{2}\cos\varphi\cos\theta\,\dot\theta^2 - \frac{L}{2}\cos\varphi\sin\theta\,\ddot\theta\Big)\vec{e}_x +$$
$$+\Big(-\frac{L}{2}\sin\varphi\cos\theta\,\dot\varphi^2 - \frac{L}{2}\cos\varphi\sin\theta\,\dot\varphi\dot\theta + \frac{L}{2}\cos\varphi\cos\theta\,\ddot\varphi +$$
$$-\frac{L}{2}\cos\varphi\sin\theta\,\dot\varphi\dot\theta - \frac{L}{2}\sin\varphi\cos\theta\,\dot\theta^2 - \frac{L}{2}\sin\varphi\sin\theta\,\ddot\theta +$$
$$-2L\cos\varphi\,\dot\varphi^2 - 2L\sin\varphi\,\ddot\varphi\Big)\vec{e}_y +$$
$$-\frac{L}{2}\left(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\right)\vec{e}_z =$$

$$\vec{a}_G = \Big[\Big(-2L\sin\varphi - \frac{L}{2}\cos\varphi\cos\theta\Big)\dot\varphi^2 + \Big(2L\cos\varphi - \frac{L}{2}\sin\varphi\cos\theta\Big)\ddot\varphi +$$

$$+ L\sin\varphi\sin\theta\,\dot\varphi\,\dot\theta - \frac{L}{2}\cos\varphi\cos\theta\,\dot\theta^2 - \frac{L}{2}\cos\varphi\sin\theta\,\ddot\theta\Big]\vec{e}_x +$$

(15.1) $$+\Big[\Big(-\frac{L}{2}\sin\varphi\cos\theta - 2L\cos\varphi\Big)\dot\varphi^2 - L\cos\varphi\sin\theta\,\dot\varphi\,\dot\theta + \Big(\frac{L}{2}\cos\varphi\cos\theta - 2L\sin\varphi\Big)\ddot\varphi +$$

$$-\frac{L}{2}\sin\varphi\cos\theta\,\dot\theta^2 - \frac{L}{2}\sin\varphi\sin\theta\,\ddot\theta\Big]\vec{e}_y +$$

$$-\frac{L}{2}\Big(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\Big)\vec{e}_z$$

$$\vec{R}^{(at,est)} = \vec{F}_e - m\vec{g} = -c\big(G-H\big) - mg\,\vec{e}_z =$$

$$= -c\Big[4L\sin\varphi\,\vec{e}_x - 2L\,\vec{e}_z + L\big(\cos\varphi\cos\theta\,\vec{e}_x + \sin\varphi\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z\big) +$$

(15.2) $$+ 3L\big(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\big)\Big] - mg\,\vec{e}_z =$$

$$= -cL\big(\sin\varphi + \cos\varphi\cos\theta\big)\vec{e}_x - cL\big(\sin\varphi\cos\theta + 3\cos\varphi\big)\vec{e}_y +$$

$$+ \Big[cL\big(+2 + \sin\theta\big) - mg\Big]\,\vec{e}_z$$

Proiettando la I ECD (14.3) sulla base $\mathcal{B}$, si trova

(15.3) $\vec{e}_x$: $\Psi'_x = cL(\sin\varphi + \cos\varphi\cos\theta) + m\,\vec{a}_G\cdot\vec{e}_x$ (vedi 15.1)

(15.4) $\vec{e}_y$: $\Phi'_y = cL(\sin\varphi\cos\theta + 3\cos\varphi) + m\,\vec{a}_G\cdot\vec{e}_y$ (vedi 15.1)

(15.5) $\vec{e}_z$: $\Phi'_z + \Psi'_z = -cL(2 + \sin\theta) + mg + m\,\vec{a}_G\cdot\vec{e}_z$ (vedi 15.1)

Per determinare le 2 incognite (Φ'_z, Ψ'_z) utilizziamo la (15.5) e la II ECD (14.3), proiettata lungo il versore $\vec{e}_x$:

(15.6) $$\vec{M}_E^{(est,att)}\cdot\vec{e}_x + (N-E)\times\vec{\Psi}'_N\cdot\vec{e}_x = \frac{d\vec{L}_E}{dt}\cdot\vec{e}_x + \cancel{\vec{V}_E\times m\vec{V}_G\cdot\vec{e}_x} \Leftarrow \vec{V}_E \parallel \vec{e}_x$$

Calcoliamo i termini della (13.6)

$$\vec{M}_E^{(est,est)} \cdot \vec{e}_x = (C-E) \times [-c(C-H)] \cdot \vec{e}_x + (G-E) \times m\vec{g} \cdot \vec{e}_x$$

$$= -cL(\vec{i}+3\vec{j}) \times [L(\vec{i}+3\vec{j}) + 4L \sin\varphi\, \vec{e}_x - 2L\, \vec{e}_z] \cdot \vec{e}_x +$$

$$+ L\left(\frac{\vec{i}}{2}+2\vec{j}\right) \times (-mg\, \vec{e}_z) \cdot \vec{e}_x$$

$$= -cL^2 \left[\cos\varphi\cos\theta\, \vec{e}_x + \sin\varphi\cos\theta\, \vec{e}_y - \sin\theta\, \vec{e}_z + 3(-\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y) \times\right.$$

$$\left. \times (4\sin\varphi\, \vec{e}_x - 2\, \vec{e}_z)\right] \cdot \vec{e}_x +$$

$$- mgL \left[\frac{1}{2}(\cos\varphi\cos\theta\, \vec{e}_x + \sin\varphi\cos\theta\, \vec{e}_y - \sin\theta\, \vec{e}_z) + 2(-\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y) \times\right.$$

$$\left. \times \vec{e}_z \cdot \vec{e}_x\right]$$

$$= -cL^2 \left[\sin\varphi\cos\theta\, \vec{e}_y \times (-2\, \vec{e}_z) \cdot \vec{e}_x + 3\cos\varphi\, \vec{e}_y \times (-2\, \vec{e}_z) \cdot \vec{e}_x\right] +$$

$$- mgL \left[\frac{1}{2}\sin\varphi\cos\theta\, \vec{e}_y \times \vec{e}_z \cdot \vec{e}_x + 2\cos\varphi\, \vec{e}_y \times \vec{e}_z \cdot \vec{e}_x\right]$$

$$= -cL^2(-2\sin\varphi\cos\theta - 6\cos\varphi) - mgL\left(\frac{1}{2}\sin\varphi\cos\theta + 2\cos\varphi\right)$$

$$(N-E) \times \vec{\Psi}' = 4L\, \vec{e}_2 \times (\Psi_x'\, \vec{e}_x + \Psi_z'\, \vec{e}_z) = 4L(-\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y) \times$$

$$\times (\Psi_x'\, \vec{e}_x + \Psi_z'\, \vec{e}_z) = 4L(+\sin\varphi\, \Psi_z'\, \vec{e}_y + \cos\varphi(\Psi_x'\, \vec{e}_z + \Psi_z'\, \vec{e}_x))$$

$$= 4L(\cos\varphi\, \Psi_z'\, \vec{e}_x + \sin\varphi\, \Psi_z'\, \vec{e}_y - \Psi_x'\cos\varphi\, \vec{e}_z)$$

$$(N-E) \times \vec{\Psi}' \cdot \vec{e}_x = 4L\cos\varphi\, \Psi_z'$$

Il lato destro della (15.6) si può calcolare come

$$\frac{d}{dt}\vec{L}_E\cdot\vec{e}_x = \frac{d}{dt}\left(\vec{L}_E\cdot\vec{e}_x\right),$$

dove

$$\vec{L}_E\cdot\vec{e}_x = I_E(\vec{\omega})\cdot\vec{e}_x + \cancel{(G-E)\times m\vec{v}_E\cdot\vec{e}_x} \qquad \vec{v}_E \parallel \vec{e}_x$$

$$= mL^2\left[-\left(\frac{13}{3}\sin\theta\,\dot\varphi+\dot\theta\right)\vec{i}\cdot\vec{e}_x + \left(\sin\theta\,\dot\varphi+\frac{1}{3}\dot\theta\right)\vec{j}\cdot\vec{e}_x + \frac{14}{3}\cos\theta\,\dot\varphi\,\vec{k}\cdot\vec{e}_x\right]$$

$$= mL^2\left[-\left(\frac{13}{3}\sin\theta\,\dot\varphi+\dot\theta\right)(\cos\varphi\cos\theta) + \left(\sin\theta\,\dot\varphi+\frac{1}{3}\dot\theta\right)(-\sin\varphi) + \frac{14}{3}\cos\theta\,\dot\varphi\cos\varphi\sin\theta\right]$$

$$= mL^2\left[\left(-\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\dot\varphi - \left(\frac{1}{3}\sin\varphi+\cos\varphi\cos\theta\right)\dot\theta\right]$$

Quindi,

$$\frac{d}{dt}\left(\vec{L}_E\cdot\vec{e}_x\right) = mL^2\Big[\left(-\cos\varphi\sin\theta\,\dot\varphi - \cancel{\sin\varphi\cos\theta\,\dot\theta} - \frac{1}{6}\sin\varphi\sin 2\theta\,\dot\varphi + \frac{1}{3}\cos\varphi\cos 2\theta\,\dot\theta\right)\dot\varphi +$$

$$+ \left(-\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi +$$

$$- \left(\frac{1}{3}\cos\varphi\,\dot\varphi - \cancel{\sin\varphi\cos\theta\,\dot\varphi} - \cos\varphi\sin\theta\,\dot\theta\right)\dot\theta$$

$$- \left(\frac{1}{3}\sin\varphi+\cos\varphi\cos\theta\right)\ddot\theta\Big]$$

$$= mL^2\Big[\left(-\cos\varphi\sin\theta - \frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2 + \frac{1}{3}\cos\varphi(\cos 2\theta - 1)\,\dot\theta\dot\varphi$$

$$+ \left(-\sin\varphi\sin\theta + \frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi$$

$$- \left(\frac{1}{3}\sin\varphi + \cos\varphi\cos\theta\right)\ddot\theta + \cos\varphi\sin\theta\,\dot\theta^2\Big]$$

Dunque, la (15.6) si scrive

$$+cL^2\left(2\sin\varphi\cos\theta+6\cos\varphi\right)-mgL\left(\frac{1}{2}\sin\varphi\cos\theta+2\cos\varphi\right)+4L\cos\varphi\,\Psi'_z=$$

(18.1)
$$=mL^2\Big[-\left(\cos\varphi\sin\theta+\frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}\cos\varphi\sin^2\theta\,\dot\theta\,\dot\varphi+$$
$$+\left(-\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi-\left(\frac{1}{3}\sin\varphi+\cos\varphi\cos\theta\right)\ddot\theta+$$
$$+\cos\varphi\sin\theta\,\dot\theta^2\Big]$$

se $\cos\varphi\neq0 \iff \varphi\neq\pm\frac{\pi}{2}$ (Se $\cos\varphi=0$, si può usare la II ECD lungo $\vec{e}_4$)

(18.2)
$$\Psi'_z=\frac{1}{4\cancel{L}\cos\varphi}\left[-cL^{\cancel{2}}\left(2\sin\varphi\cos\theta+6\cos\varphi\right)+mg\cancel{L}\left(\frac{1}{2}\sin\varphi\cos\theta+2\cos\varphi\right)\right]+$$
$$+\frac{mL^{\cancel{2}}}{4\cancel{L}\cos\varphi}\Big[-\left(\cos\varphi\sin\theta+\frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}\cos\varphi\sin^2\theta\,\dot\theta\dot\varphi+$$
$$+\left(-\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi-\left(\frac{1}{3}\sin\varphi+\cos\varphi\cos\theta\right)\ddot\theta+$$
$$+\cos\varphi\sin\theta\,\dot\theta^2\Big]$$

Sostituendo la (18.2) nella (15.5) si trova

(18.3)
$$\Phi'_z=-cL\left(2+\sin\theta\right)+mg+m\frac{L}{2}\left(\cos\theta\,\ddot\theta-\sin\theta\,\dot\theta^2\right)+$$
$$-\frac{1}{4}\Big[-cL\left(2\,\mathrm{tg}\varphi\cos\theta+6\right)+mg\left(\frac{1}{2}\mathrm{tg}\varphi\cos\theta+2\right)$$
$$-mL\left(\sin\theta+\frac{1}{6}\mathrm{tg}\varphi\,\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}mL\sin^2\theta\,\dot\theta\dot\varphi$$
$$mL\left(-\mathrm{tg}\varphi\sin\theta+\frac{1}{6}\sin 2\theta\right)\ddot\varphi-mL\left(\frac{1}{3}\mathrm{tg}\varphi+\cos\theta\right)\ddot\theta+$$
$$+mL\sin\theta\,\dot\theta^2\Big]$$

N.B. Si può verificare che $\vec{\Phi}'_{E|\vec{q}_e^{(1)}}=\vec{\Phi}_{E|\vec{q}_e^{(1)}}$, $\vec{\Psi}'_{N|\vec{q}_e^{(1)}}=\vec{\Psi}_{N|\vec{q}_e^{(1)}}$. Analogamente per $\vec{q}_e^{(2)}$.