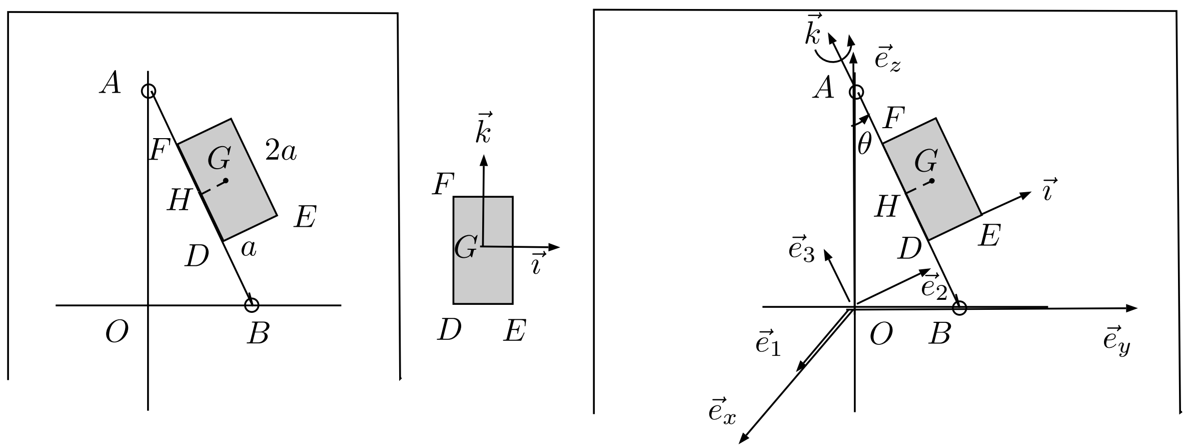
Compito di Meccanica Razionale (9 CFU)

Trieste, 11 gennaio 2019. (G. Tondo)

Un rigido è costituito da una lamina rettangolare omogenea di massa m e lati a e $2a$, saldata, lungo il lato DF, ad un'asta AB di lunghezza $4a$ e di massa trascurabile, in modo che il punto medio H del lato DF coincida con il punto medio di AB. Gli estremi dell'asta sono vincolati, come in figura, a scorrere senza attrito lungo due guide fisse ortogonali, tramite due cerniere sferiche "bucate". Sul rigido agisce solo il peso proprio opposto ad $\vec{e}_z$.



Oltre alla terna fissa $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, si suggerisce di usare anche una terna intermedia formata dai versori $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, con $\vec{e}_1 = \vec{e}_x$, $\vec{e}_3 = vers(A - D)$, $\vec{e}_2 = \vec{e}_3 \times \vec{e}_1$. Inoltre, si consiglia di prendere una terna solidale alla lamina $(\vec{\imath}, \vec{\jmath}, \vec{k})$, con il versore $\vec{\imath} = vers(E - D)$, $\vec{k} = \vec{e}_3$ e $\vec{\jmath} = \vec{k} \times \vec{\imath}$.

Sia $-\pi < \theta \leq \pi$ l'angolo compreso tra $\vec{e}_z$ e $\vec{k}$, e sia $-\pi < \psi \leq \pi$ quello compreso tra $\vec{e}_1$ e $\vec{\imath}$.

STATICA

Determinare:

1) le configurazioni di equilibrio del modello $\vec{q}_e = (\theta_e, \psi_e)$ e la loro stabilità;

2) le reazioni vincolari esterne sul rigido in A, agli equilibri;

3) le reazioni vincolari esterne sul rigido in B, agli equilibri.

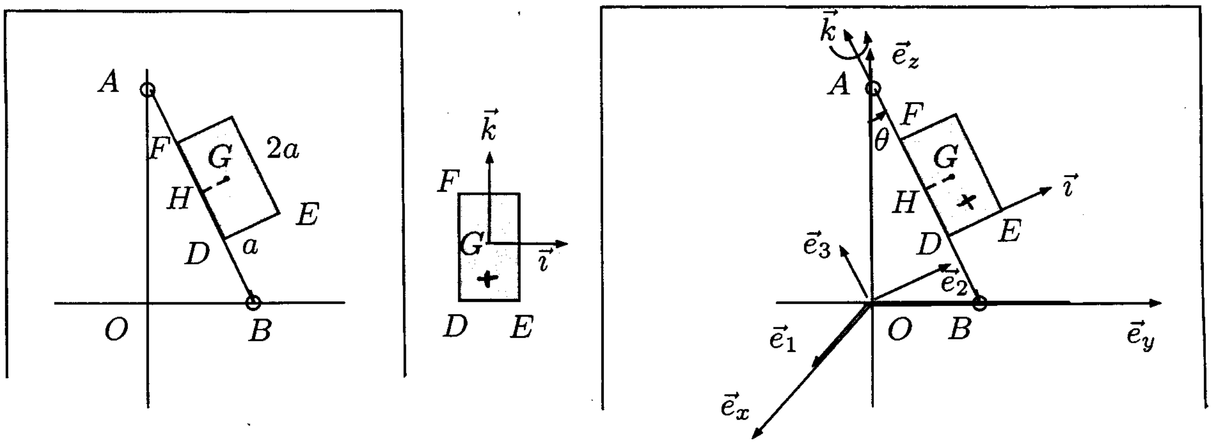
DINAMICA

4) Scrivere le equazione differenziali pure di moto;

5) linearizzare le equazioni di moto intorno alle configurazioni di equilibrio stabile;

6) calcolare le reazioni vincolare esterne nei punti A e B, durante il moto.

Tema del 11/01/2019



Il modello è formato da un solo rigido vincolato con 2 cerniere sferiche lisce. Con il metodo dei congelamenti successivi possiamo osservare che se blocchiamo lo scorrimento del punto B del rigido sull'asse $(O, \vec{e}_y)$, il rigido può ancora ruotare attorno all'asse $(B, \vec{k})$. Dunque, il rigido ha 2 g.l. e, come coordinate libere possiamo prendere gli angoli

$$-\pi < \theta \le \pi \quad , \quad -\pi < \psi \le \pi$$

Quindi, ogni configurazione del rigido è individuata da

$$\vec{q} = (\theta, \psi)$$

Consideriamo le 3 basi

$\mathcal{B} = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$: "fissa"

$\mathcal{B}' = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$: "intermedia"

$\mathcal{B}'' = (\vec{\imath}, \vec{\jmath}, \vec{k})$: solidale al rigido

Le equazioni di trasformazione sono

$$
(2.1)\begin{cases} \vec{e}_1 = \vec{e}_x \\ \vec{e}_2 = \cos\theta\, \vec{e}_y + \sin\theta\, \vec{e}_z \\ \vec{e}_3 = -\sin\theta\, \vec{e}_y + \cos\theta\, \vec{e}_z \end{cases}
\qquad
[\vec{e}_1, \vec{e}_2, \vec{e}_3] = [\vec{e}_x, \vec{e}_y, \vec{e}_z] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}}_{R_\theta}
$$

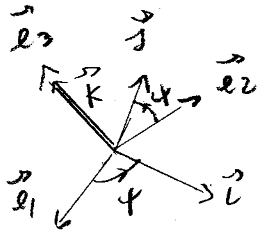
Quindi,

$$R_\theta R_\theta^T = \mathbb{1}$$

$$[\vec{e}_x, \vec{e}_y, \vec{e}_z] = [\vec{e}_1, \vec{e}_2, \vec{e}_3]\, R_\theta^T = [\vec{e}_1, \vec{e}_2, \vec{e}_3] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{R_\theta^T}$$

cioè:

$$
(2.2)\begin{cases} \vec{e}_x = \vec{e}_1 \\ \vec{e}_y = \cos\theta\, \vec{e}_2 - \sin\theta\, \vec{e}_3 \\ \vec{e}_z = \sin\theta\, \vec{e}_2 + \cos\theta\, \vec{e}_3 \end{cases}
$$



Analogamente,

$$
(2.3)\begin{cases} \vec{i} = \cos\psi\, \vec{e}_1 + \sin\psi\, \vec{e}_2 \\ \vec{j} = -\sin\psi\, \vec{e}_1 + \cos\psi\, \vec{e}_2 \\ \vec{k} = \vec{e}_3 \end{cases}
\qquad
[\vec{i}, \vec{j}, \vec{k}] = [\vec{e}_1, \vec{e}_2, \vec{e}_3] \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\psi}
$$

Quindi

$$(2.4)\quad [\vec{e}_1, \vec{e}_2, \vec{e}_3] = [\vec{i}, \vec{j}, \vec{k}]\, R_\psi^T = [\vec{i}, \vec{j}, \vec{k}] \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\psi^T}$$

cioè

$$
(2.5)\begin{cases} \vec{e}_1 = \cos\psi\, \vec{i} - \sin\psi\, \vec{j} \\ \vec{e}_2 = \sin\psi\, \vec{i} + \cos\psi\, \vec{j} \\ \vec{e}_3 = \vec{k} \end{cases}
$$

Componendo le trasformazioni (2.2) e (2.4) o, equivalentemente, moltiplicando le matrici R_θ^T e R_φ^T, si trova

$$(3.1)\qquad [\vec{e}_x, \vec{e}_y, \vec{e}_z] = [\vec{e}_1, \vec{e}_2, \vec{e}_3]\, R_\theta^T \overset{(2.4)}{=} [\vec{i}, \vec{j}, \vec{k}]\, R_\varphi^T R_\theta^T$$

cioè

$$(3.2)\begin{cases} \vec{e}_x = \cos\varphi\,\vec{i} - \sin\varphi\,\vec{j} \\ \vec{e}_y = \cos\theta\left(\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\right) - \sin\theta\,\vec{k} = \cos\theta\sin\varphi\,\vec{i} + \cos\theta\cos\varphi\,\vec{j} - \sin\theta\,\vec{k} \\ \vec{e}_z = \sin\theta\left(\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\right) + \cos\theta\,\vec{k} = \sin\theta\sin\varphi\,\vec{i} + \sin\theta\cos\varphi\,\vec{j} + \cos\theta\,\vec{k} \end{cases}$$

$$(3.3)\qquad \begin{cases} \vec{i} = \cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y + \sin\theta\sin\varphi\,\vec{e}_z \\ \vec{j} = -\sin\varphi\,\vec{e}_x + \cos\theta\cos\varphi\,\vec{e}_y + \sin\theta\cos\varphi\,\vec{e}_z \\ \vec{k} = -\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z \end{cases}$$

$$(3.3)\quad G-H = \frac{a}{2}\,\vec{i} = \frac{a}{2}\left(\cos\varphi\,\vec{e}_1 + \sin\varphi\,\vec{e}_2\right) \overset{(2.1)}{=} \frac{a}{2}\left[\cos\varphi\,\vec{e}_x + \sin\varphi\left(\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z\right)\right]$$

$$(3.4)\quad H-B = 2a\,\vec{k} - 2a\,\vec{e}_3 = 2a\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right)$$

$$(3.5)\quad B-O = 4a\sin\theta\,\vec{e}_y$$

Quindi,

$$(3.6)\quad G-O = (G-H) + (H-B) + (B-O) = \frac{a}{2}\,\vec{i} + 2a\,\vec{e}_3 + 4a\sin\theta\,\vec{e}_y$$

$$= \frac{a}{2}\cos\varphi\,\vec{e}_x + \frac{a}{2}\cos\theta\sin\varphi\,\vec{e}_y + \frac{a}{2}\sin\theta\sin\varphi\,\vec{e}_z +$$

$$+ 2a\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right) + 4a\sin\theta\,\vec{e}_y$$

$$= \frac{a}{2}\cos\varphi\,\vec{e}_x + \left(\frac{a}{2}\cos\theta\sin\varphi + 2a\sin\theta\right)\vec{e}_y + \left(\frac{a}{2}\sin\theta\sin\varphi + 2a\cos\theta\right)\vec{e}_z$$

Statica

Dato che la sollecitazione attiva è solo il peso, quindi conservativa, possiamo calcolare l'energia potenziale del rigido

(4.1) $V(\theta,\varphi) = -m\vec{g}\cdot(G-O) = mg\,\vec{e}_z\cdot(G-O) \overset{(3.6)}{=} mga\left(\frac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\right)$

Troviamo i punti stazionari

(4.2) $\frac{\partial V}{\partial \theta} = mga\left(\frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\right) = -Q_\theta$

(4.3) $\frac{\partial V}{\partial \varphi} = mga\left(\frac{1}{2}\sin\theta\cos\varphi\right) = -Q_\varphi$

Dunque, le equazioni pure di equilibrio sono

(4.4) $\begin{cases} \frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta = 0 \\ \sin\theta\cos\varphi = 0 \end{cases}$

Il sistema precedente equivale all'unione di 2 sistemi

(4.5) $\begin{cases} \frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta = 0 \\ \sin\theta = 0 \end{cases} \cup \begin{cases} \frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta = 0 \\ \cos\varphi = 0 \end{cases}$

Il primo sistema (4.5) equivale a

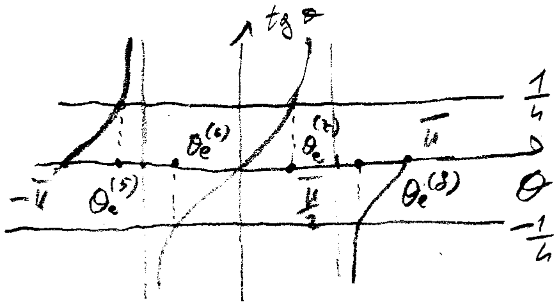
(4.6) $\begin{cases} \sin\varphi = 0 \\ \sin\theta = 0 \end{cases} \Leftrightarrow \begin{cases} \varphi = 0, \pi \\ \theta = 0, \pi \end{cases} \Leftrightarrow \vec{q}_e^{(1)} = (0,0),\ \vec{q}_e^{(2)} = (0,\pi)$

$\vec{q}_e^{(3)} = (\pi,0),\ \vec{q}_e^{(4)} = (\pi,\pi)$

cioè a 4 configurazioni di equilibrio

Il secondo sistema (4.5) equivale, a sua volta, ai 2 sistemi

$$\begin{cases} tg\,\theta = \frac{1}{4} \\ \psi = \frac{\pi}{2} \end{cases} \cup \begin{cases} tg\,\theta = -\frac{1}{4} \\ \psi = -\frac{\pi}{2} \end{cases}$$



Dal grafico, si deducono le soluzioni

$$\vec{q}_e^{\,(5)} = \left(\theta_e^{(5)}, \frac{\pi}{2}\right), \quad \vec{q}_e^{\,(6)} = \left(\theta_e^{(6)}, -\frac{\pi}{2}\right), \quad \vec{q}_e^{\,(7)} = \left(\theta_e^{(7)}, \frac{\pi}{2}\right), \quad \vec{q}_e^{\,(8)} = \left(\theta_e^{(8)}, -\frac{\pi}{2}\right)$$

dove

$$\theta_e^{(5)} = \theta_e^{(7)} - \pi, \quad \theta_e^{(6)} = -\operatorname{arctg}\frac{1}{4}, \quad \theta_e^{(7)} = \operatorname{arctg}\frac{1}{4}, \quad \theta_e^{(8)} = \theta_e^{(6)} + \pi$$

Stabilità

Studiamo la qualità dei punti stazionari di V mediante la matrice Hessiana

$$\mathcal{H}_V(\theta,\psi) = \begin{bmatrix} \frac{\partial^2 V}{\partial \theta^2} & \frac{\partial^2 V}{\partial \psi \partial \theta} \\ \frac{\partial^2 V}{\partial \theta \partial \psi} & \frac{\partial^2 \psi}{\partial \theta^2} \end{bmatrix} = mga \begin{bmatrix} -\left(\frac{1}{2}\sin\theta\sin\psi + 2\cos\theta\right) & \frac{1}{2}\cos\theta\cos\psi \\ \frac{1}{2}\cos\theta\cos\psi & -\frac{1}{2}\sin\theta\sin\psi \end{bmatrix}$$

$$\mathcal{H}_{V|q_e^{(1)}} = mga \begin{bmatrix} -2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \Rightarrow \det \mathcal{H} < 0 \Rightarrow \text{sella} \Rightarrow \text{instabile}$$

$$\mathcal{H}_{V|q_e^{(2)}} = mga \begin{bmatrix} -2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \Rightarrow \det \mathcal{H} < 0 \Rightarrow \text{sella} \Rightarrow \text{instabile}$$

$$\mathcal{H}_{V|q_e^{(3)}} = mga \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \Rightarrow \det \mathcal{H} < 0 \Rightarrow \text{sella} \Rightarrow \text{instabile}$$

$$\mathcal{H}_{V|\vec{q}_e^{(4)}} = mga \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \Rightarrow \det \mathcal{H} < 0 \Rightarrow \text{rella} \Rightarrow \text{instabile}$$

$$\mathcal{H}_{V|\vec{q}_e^{(5)}} = mga \left[\begin{array}{c|c} -\left(\frac{1}{2}\sin\theta_e^{(5)} + 2\cos\theta_e^{(5)}\right) & 0 \\ 0 & -\frac{1}{2}\sin\theta_e^{(5)} \end{array}\right] \Rightarrow \frac{\mathcal{H}_{11}}{mga} = -\cos\theta_e^{(5)}\left(\frac{1}{2}\operatorname{tg}\theta_e^{(5)} + 2\right) = -\cos\theta_e^{(5)}\,\frac{17}{8} > 0$$

$$\frac{\det \mathcal{H}}{mga} = \mathcal{H}_{11}\left(-\frac{1}{2}\sin\theta_e^{(5)}\right) > 0$$

Dunque, $\vec{q}_e^{(5)}$ è un punto di minimo per V, quindi è stabile.

$$\mathcal{H}_{V|\vec{q}_e^{(6)}} = mga \left[\begin{array}{c|c} -\left(-\frac{1}{2}\sin\theta_e^{(6)} + 2\cos\theta_e^{(6)}\right) & 0 \\ \hline 0 & \frac{1}{2}\sin\theta_e^{(6)} \end{array}\right] \Rightarrow \frac{\mathcal{H}_{11}}{mga} = -\cos\theta_e^{(6)}\left(-\frac{1}{2}\operatorname{tg}\theta_e^{(6)} + 2\right) = -\cos\theta_e^{(6)}\,\frac{17}{8} < 0 \Rightarrow$$

$\Rightarrow$ instabile

$$\mathcal{H}_{V|\vec{q}_e^{(7)}} = mga \left[\begin{array}{c|c} -\left(\frac{1}{2}\sin\theta_e^{(7)} + 2\cos\theta_e^{(7)}\right) & 0 \\ \hline 0 & -\frac{1}{2}\sin\theta_e^{(7)} \end{array}\right] \Rightarrow \frac{\mathcal{H}_{11}}{mga} = -\cos\theta_e^{(7)}\left(\frac{1}{2}\operatorname{tg}\theta_e^{(7)} + 2\right) = -\cos\theta_e^{(7)}\,\frac{17}{8} < 0 \Rightarrow$$

$\Rightarrow$ instabile

$$\mathcal{H}_{V|\vec{q}_e^{(8)}} = mga \left[\begin{array}{c|c} -\left(-\frac{1}{2}\sin\theta_e^{(8)} + 2\cos\theta_e^{(8)}\right) & 0 \\ 0 & \frac{1}{2}\sin\theta_e^{(8)} \end{array}\right] \Rightarrow \frac{\mathcal{H}_{11}}{mga} = -\cos\theta_e^{(8)}\left(-\frac{1}{2}\operatorname{tg}\theta_e^{(8)} + 2\right) = -\cos\theta_e^{(8)}\,\frac{17}{8} > 0$$

$$\frac{\det \mathcal{H}}{mga} = \mathcal{H}_{11}\,\frac{1}{2}\sin\theta_e^{(8)} > 0$$

Quindi $\vec{q}_e^{(8)}$ è un punto di minimo per V, dunque è stabile.

2) + 3) Reazioni all'equilibrio in A e B

Le cerniere sferiche sono, per ipotesi, non dissipative e bilatere. Quindi, eserciteranno una sollecitazione reattiva data da

$$(7.1)\qquad \mathcal{L}^{(reatt)} = \{ (A,\vec{\Psi}), (B,\vec{\Phi}) \} \text{ con } \vec{\Psi}_A \cdot \vec{e}_2 = 0,\ \vec{\Phi}_B \cdot \vec{e}_1 = 0$$

Per calcolare $\vec{\Psi}_A$ e $\vec{\Phi}_B$, scriviamo le ECS su tutto il rigido

$$(7.2)\qquad \begin{cases} \vec{R}^{(est,att)} + \vec{\Psi}_A + \vec{\Phi}_B = 0 \\ \vec{M}_B^{(est,att)} + (A-B)\times\vec{\Psi}_A = 0 \end{cases}$$

Dalla II ECS ricaviamo $\vec{\Psi}_A$:

$$(7.3)\qquad \vec{\Psi}_A \times (A-B) = \vec{M}_B^{(est,att)} \iff \vec{\Psi}_A = \frac{A-B}{|A-B|^2} \times \vec{M}_B^{(est,att)} + \lambda\vec{K}$$

$$\vec{M}_B^{(est,att)} = (G-B)\times m\vec{g} = \left(\frac{a}{2}\vec{\imath} + 2a\vec{K}\right)\times(-mg\,\vec{e}_z) =$$

$$(7.4)\qquad = -mg\left(\frac{a}{2}\vec{\imath} + 2a\vec{K}\right)\times\left(\sin\theta\sin\psi\,\vec{\imath} + \sin\theta\cos\psi\,\vec{\jmath} + \cos\theta\,\vec{K}\right)$$

$$= -mg\left[\frac{a}{2}\left(\sin\theta\cos\psi\,\vec{K} - \cos\theta\,\vec{\jmath}\right) + 2a\left(\sin\theta\sin\psi\,\vec{\jmath} - \sin\theta\cos\psi\,\vec{\imath}\right)\right]$$

$$= -mga\left[-2\sin\theta\cos\psi\,\vec{\imath} + \left(2\sin\theta\sin\psi - \frac{1}{2}\cos\theta\right)\vec{\jmath} + \frac{1}{2}\sin\theta\cos\psi\,\vec{K}\right]$$

Dunque, nelle configurazioni di equilibrio

$$\vec{\Psi}_A\Big|_{q_e} = \frac{-\vec{K}}{4a}\times mga\left[-2\sin\theta\cos\psi\,\vec{\imath} + \left(2\sin\theta\sin\psi - \frac{1}{2}\cos\theta\right)\vec{\jmath} + \frac{1}{2}\sin\theta\cos\psi\,\vec{K}\right] + \lambda\vec{K}$$

$$(7.5)\qquad = -\frac{mg}{4}\left[-2\sin\theta\cos\psi\,\vec{\jmath} - \left(2\sin\theta\sin\psi - \frac{1}{2}\cos\theta\right)\vec{\imath}\right] + \lambda\vec{K}\Big|_{q_e}$$

$$= \frac{mg}{4}\left[\left(2\sin\theta_e\sin\psi_e - \frac{1}{2}\cos\theta_e\right)\vec{\imath} + 2\sin\theta_e\cos\psi_e\,\vec{\jmath}\right] + \lambda\vec{K}\Big|_{q_e}$$

Determiniamo la funzione $\lambda(\theta,\varphi)$ nella (7.5) richiedendo che

$$\vec{\Psi}_A \cdot \vec{e}_z = 0 \tag{7.6}$$

Poiché

$$\vec{\Psi}_A\Big|_{q_e} \cdot \vec{e}_z = \frac{mg}{4}\left[\left(2\sin\theta_e \sin\varphi_e - \frac{1}{2}\cos\theta_e\right)\vec{i}\cdot\vec{e}_z + 2\sin\theta_e\cos\varphi_e\,\vec{j}\cdot\vec{e}_z\right] + \lambda\,\vec{K}\cdot\vec{e}_z =$$

$$\overset{(3.2)}{=} \frac{mg}{4}\left[\left(2\sin\theta_e\sin\varphi_e - \frac{1}{2}\cos\theta_e\right)\sin\theta_e\sin\varphi_e + 2\sin\theta_e^2\cos\varphi_e^2\right] + \lambda\cos\theta_e \tag{7.7}$$

Allora, la condizione (7.6) impone

$$\lambda\cos\theta_e + \frac{mg}{4}\left[2\sin^2\theta_e\sin^2\varphi_e - \frac{1}{2}\sin\theta_e\cos\theta_e\sin\varphi_e + 2\sin^2\theta_e\cos\varphi_e^2\right] = 0 \tag{7.8}$$

cioè

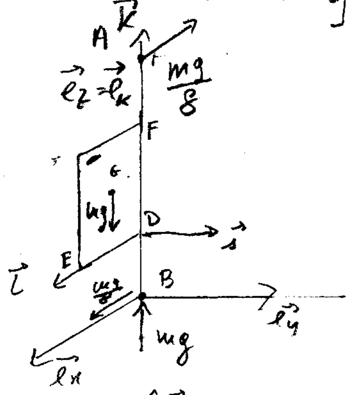
$$\lambda(\theta_e,\varphi_e) = \frac{mg}{4}\sin\theta_e\left(\frac{1}{2}\sin\varphi_e - 2\,\mathrm{tg}\,\theta_e\right) \tag{7.9}$$

Sostituendo la $\vec{F}_\pi$ nella I eq. delle (7.2) si trova

$$\vec{\phi}_{B|_{\vec{q}_e}} = mg\,\vec{e}_z - \frac{mg}{4}\left[\left(2\sin\theta_e\sin\psi_e - \frac{1}{2}\cos\theta_e\right)\vec{i} + 2\sin\theta_e\cos\psi_e\,\vec{j}\right] - \lambda\vec{k}$$

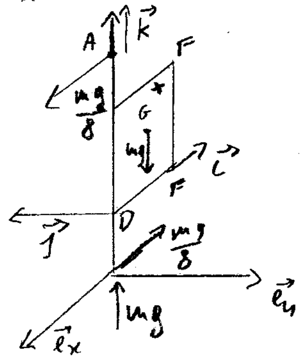
Ricapitolando,

$$\vec{q}_e^{\,(1)} = (0,0) \;:\; \vec{\psi}_A = -\frac{mg}{8}\vec{i} + \cancel{\lambda}\,\vec{e}_z = -\frac{mg}{8}\vec{e}_x$$



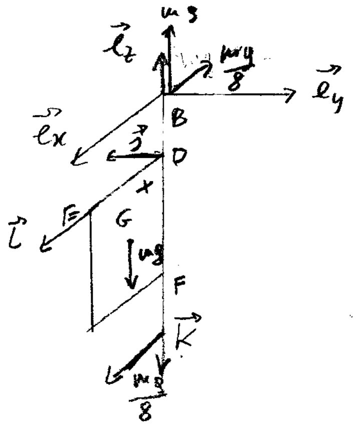
instabile $\quad \vec{\phi}_B = mg\,\vec{e}_z + \frac{mg}{8}\vec{i} - \cancel{\lambda\vec{k}} = mg\left(\vec{e}_z + \frac{1}{8}\vec{e}_x\right)$

$$q_e^{(2)} = (0,\pi) \;:\; \vec{\psi}_A = -\frac{mg}{8}\vec{i} + \cancel{\lambda}\,\vec{e}_z = \frac{mg}{8}\vec{e}_x$$



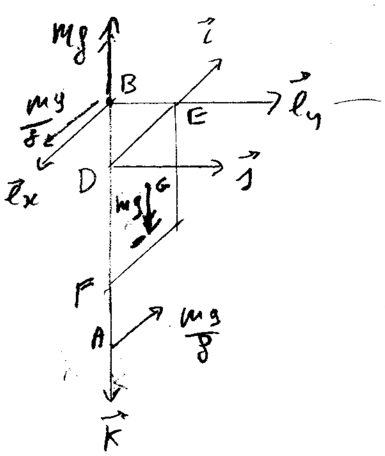
instabile $\quad \vec{\phi}_B = mg\,\vec{e}_z + \frac{mg}{8}\vec{i} - \cancel{\lambda\vec{k}} = mg\left(\vec{e}_z - \frac{1}{8}\vec{e}_x\right)$

$$\vec{q}_e^{\,(3)} = (\pi,0) \;:\; \vec{\psi}_A = \frac{mg}{8}\vec{i} + \cancel{\lambda}\,\vec{e}_z = \frac{mg}{8}\vec{e}_x$$



instabile $\quad \vec{\phi}_B = mg\,\vec{e}_z - \frac{mg}{8}\vec{i} - \cancel{\lambda\vec{k}} = mg\,\vec{e}_z - \frac{mg}{8}\vec{e}_x$

$$\vec{q}_e^{\,(4)} = (\pi,\pi) \;:\; \vec{\psi}_A = \frac{mg}{8}\vec{i} + \cancel{\lambda}\,\vec{e}_z = -\frac{mg}{8}\vec{e}_x$$



instabile $\quad \vec{\phi}_B = mg\,\vec{e}_z - \frac{mg}{8}\vec{i} - \cancel{\lambda\vec{k}} = mg\,\vec{e}_z + \frac{mg}{8}\vec{e}_x$

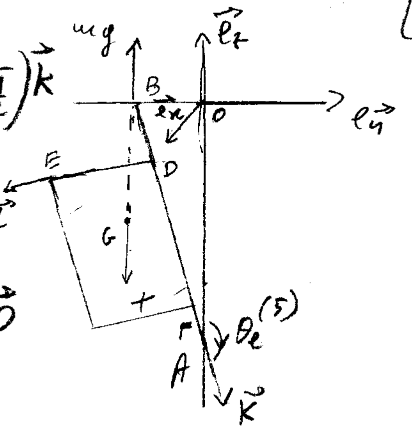
$q_e^{(5)} = \left(\theta_e^{(5)}, \frac{\pi}{2}\right)$: $\vec{\Psi}_A = \frac{mg}{4}\left(2\sin\theta_e^{(5)} - \frac{1}{2}\cos\theta_e^{(5)}\right)\vec{i} + \lambda\left(\theta_e^{(5)}, \frac{\pi}{2}\right)\vec{k}$

stabile

$$= \frac{mg}{\cancel{4}\,2}\,\cancel{2}\cos\theta_e^{(5)}\cancel{\left(tg\,\theta_e^{(5)} - \frac{1}{4}\right)}\vec{i} +$$

$$+ \frac{mg}{4}\sin\theta_e\cancel{\left(\frac{1}{2} - 2\,tg\,\theta_e^{(5)}\right)}\vec{k} = \vec{0}$$

$$\vec{\Phi}_B = mg\,\vec{e}_z$$



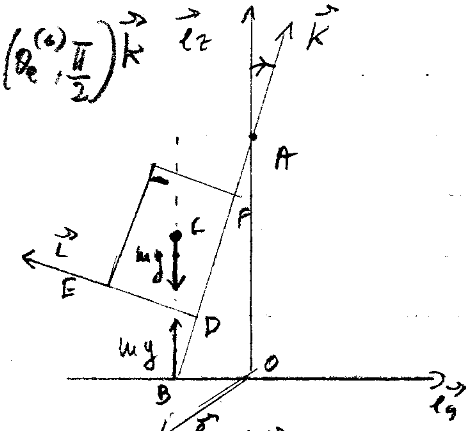
$\vec{q}_e^{(6)} = \left(\theta_e^{(6)}, -\frac{\pi}{2}\right)$: $\vec{\Psi}_A = \frac{mg}{4}\left(-2\sin\theta_e^{(6)} - \frac{1}{2}\cos\theta_e^{(6)}\right)\vec{i} + \lambda\left(\theta_e^{(6)}, \frac{\pi}{2}\right)\vec{k}$

instabile

$$= -\frac{mg}{\cancel{4}\,2}\,\cancel{2}\cos\theta_e^{(6)}\cancel{\left(tg\,\theta_e^{(6)} + \frac{1}{4}\right)}\vec{i} +$$

$$+ \frac{mg}{4}\sin\theta_e^{(6)}\cancel{\left(-\frac{1}{2} - 2\,tg\,\theta_e^{(6)}\right)}\vec{k} = \vec{0}$$

$$\vec{\Phi}_B = mg\,\vec{e}_z$$



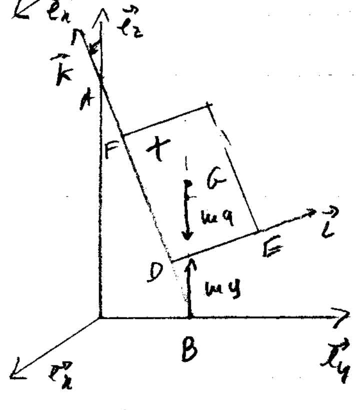
$\vec{q}_e^{(7)} = \left(\theta_e^{(7)}, \frac{\pi}{2}\right)$: $\vec{\Psi}_A = \frac{mg}{4}\left(2\sin\theta_e^{(7)} - \frac{1}{2}\cos\theta_e^{(7)}\right)\vec{i} + \lambda\left(\theta_e^{(7)}, \frac{\pi}{2}\right)$

instabile

$$= \frac{mg}{2}\cos\theta_e^{(7)}\cancel{\left(tg\,\theta_e^{(7)} - \frac{1}{4}\right)}\vec{i} +$$

$$+ \frac{mg}{4}\sin\theta_e^{(7)}\cancel{\left(\frac{1}{2} - 2\,tg\,\theta_e^{(7)}\right)}\vec{K} = \vec{0}$$

$$\vec{\Phi}_B = mg\,\vec{e}_z$$



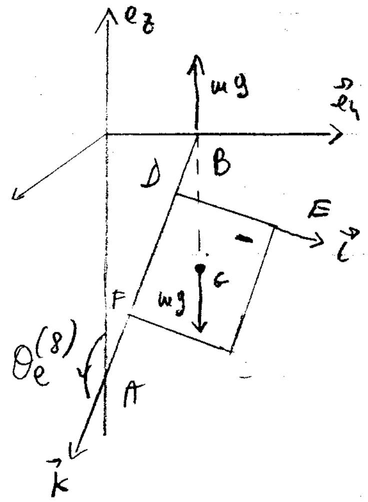
$\vec{q}_e^{(8)} = \left(\theta_e^{(8)}, -\frac{\pi}{2}\right)$: $\vec{\Psi}_A = \frac{mg}{4}\left(-2\sin\theta_e^{(8)} - \frac{1}{2}\cos\theta_e^{(8)}\right)\vec{i} + \lambda\left(\theta_e^{(8)}, -\frac{\pi}{2}\right)\vec{k}$

stabile

$$= -\frac{mg}{2}\cos\theta_e^{(8)}\cancel{\left(+tg\,\theta_e^{(8)} + \frac{1}{4}\right)}\vec{i} +$$

$$+ \frac{mg}{4}\sin\theta_e^{(8)}\cancel{\left(-\frac{1}{2} - 2\,tg\,\theta_e^{(8)}\right)}\vec{k} = \vec{0}$$

$$\vec{\Phi}_B = mg\,\vec{e}_z$$



4) Scriviamo le 2 EL relative a θ e φ. A tale scopo calcoliamo l'energia cinetica del rigido

$$K = \frac{1}{2} m |\vec{v}_H|^2 + \frac{1}{2}\vec{\omega} \cdot I_H(\vec{\omega}) + m \vec{v}_H \cdot \vec{\omega} \times (G-H)$$

Dalle (3.4) e (3.5) segue che

$$H-O = (H-B)+(B-O) = 2a\vec{k} + 4a\sin\theta\,\vec{e}_y = 2a(\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z)$$

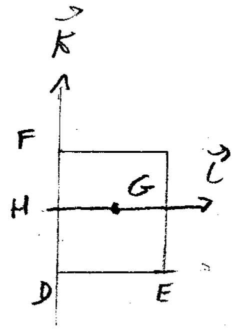
Quindi,

$$\vec{v}_H = \frac{d}{dt}(H-O) = 2a(\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z)\dot{\theta}$$

$$|\vec{v}_H|^2 = 4a^2(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 = 4a^2\dot{\theta}^2$$

Dal Teo. di Frini segue che

$$\vec{\omega} = \dot{\theta}\vec{e}_1 + \dot{\varphi}\vec{K} \overset{(3.5)}{=} \dot{\theta}(\cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}) + \dot{\varphi}\vec{K}$$



La terna $(H, \vec{i}, \vec{j}, \vec{k})$ è una TPI(H) (perchè?) quindi

$$I_H^{(B'')} = ma^2 \begin{bmatrix} \frac{4}{12} & & \\ & \frac{2}{3} & \\ & & \frac{1}{3} \end{bmatrix} = \frac{ma^2}{3}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Dunque

$$\frac{1}{2}\vec{\omega}\cdot I_H(\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot\frac{ma^2}{3}\begin{bmatrix} 1 & & \\ & 2 & \\ & & 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}\cos\varphi \\ -\dot{\theta}\sin\varphi \\ \dot{\varphi} \end{bmatrix} = \frac{ma^2}{6}\left[\dot{\theta}\cos\varphi, -\dot{\theta}\sin\varphi, \dot{\varphi}\right]\begin{bmatrix} \dot{\theta}\cos\varphi \\ -2\dot{\theta}\sin\varphi \\ \dot{\varphi} \end{bmatrix} =$$

$$= \frac{ma^2}{6}\left(\dot{\theta}^2\cos^2\varphi + 2\dot{\theta}^2\sin^2\varphi + \dot{\varphi}^2\right) =$$

$$= \frac{ma^2}{6}\left[(1+\sin^2\varphi)\dot{\theta}^2 + \dot{\varphi}^2\right]$$

$$\vec{\omega} \times (G-H) = \left(\dot\theta \cos\psi\, \vec{i} - \dot\theta \sin\psi\, \vec{j} + \dot\psi\, \vec{k}\right) \times \frac{a}{2}\vec{i} =$$

$$= \frac{a}{2}\left(\dot\theta \sin\psi\, \vec{k} + \dot\psi\, \vec{j}\right)$$

Inoltre, riscrivendo $\vec{v}_H$ nella base solidale $\mathcal{B}''$, si ottiene

$$\vec{v}_H = 2a\dot\theta\left(\cos\theta\, \vec{e}_y - \sin\theta\, \vec{e}_x\right) = 2a\dot\theta\left[\cos\theta\left(\sin\psi\, \vec{i} + \cos\psi\, \vec{j}\right) - \sin\theta\, \vec{k}\right]\cos\theta +$$

$$-2a\dot\theta\left[\sin\theta\left(\sin\psi\, \vec{i} + \cos\psi\, \vec{j}\right) + \cos\theta\, \vec{k}\right]\sin\theta$$

$$= 2a\dot\theta\left[\left(\cos^2\theta - \sin^2\theta\right)\left(\sin\psi\, \vec{i} + \cos\psi\, \vec{j}\right) - 2\sin\theta\cos\theta\, \vec{k}\right] =$$

$$= 2a\dot\theta\left[\left(1 - 2\sin^2\theta\right)\left(\sin\psi\, \vec{i} + \cos\psi\, \vec{j}\right) - \sin 2\theta\, \vec{k}\right]$$

Quindi, il termine misto delle (10.1) vale

$$\vec{v}_H \cdot \vec{\omega} \times (G-H) = a^2\dot\theta\left[-\dot\theta \sin 2\theta \sin\psi + \left(1 - 2\sin^2\theta\right)\cos\psi\, \dot\psi\right]$$

Dunque,

$$K = \frac{1}{2} m 4 a^2 \dot\theta^2 + \frac{1}{6} m a^2\left[\left(1+\sin^2\psi\right)\dot\theta^2 + \dot\psi^2\right] + a^2\dot\theta\left[-\sin 2\theta \sin\psi\, \dot\theta + \left(1-2\sin^2\theta\right)\cos\psi\, \dot\psi\right]$$

$$= m a^2\left[\left(\frac{13}{6} + \frac{1}{6}\sin^2\psi - \sin 2\theta \sin\psi\right)\dot\theta^2 + \frac{1}{6}\dot\psi^2 + \left(1-2\sin^2\theta\right)\cos\psi\, \dot\theta\dot\psi\right]$$

$$= \frac{1}{2} m a^2 \left[\dot\theta, \dot\psi\right] \begin{bmatrix} \frac{13}{3} + \frac{1}{3}\sin^2\psi - 2\sin 2\theta \sin\psi & \left(1-2\sin^2\theta\right)\cos\psi \\ \left(1-2\sin^2\theta\right)\cos\psi & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \dot\theta \\ \dot\psi \end{bmatrix}$$

Scriviamo le EL.

$$\frac{\partial K}{\partial \dot\theta} = 2ma^2\left(\frac{13}{6}+\frac{1}{6}\sin^2\varphi - \sin 2\theta \sin\varphi\right)\dot\theta + (1-2\sin^2\theta)\cos\varphi\,\dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\theta}\right) = 2ma^2\left(\frac{1}{3}\sin\varphi\cos\varphi\,\dot\varphi - 2\cos 2\theta\sin\varphi\,\dot\theta - \sin 2\theta\cos\varphi\,\dot\varphi\right)\dot\theta +$$

$$+\,2ma^2\left(\frac{13}{6}+\frac{1}{6}\sin^2\varphi - \sin 2\theta\sin\varphi\right)\ddot\theta +$$

$$+\left(-4\sin\theta\cos\theta\,\dot\theta\right)\cos\varphi\,\dot\varphi - (1-2\sin^2\theta)\sin\varphi\,\dot\varphi^2 + (1-2\sin^2\theta)\cos\varphi\,\ddot\varphi$$

$$\frac{\partial K}{\partial \theta} = ma^2\left(-2\cos 2\theta\sin\varphi\,\dot\theta^2 - 4\sin\theta\cos\theta\cos\varphi\,\dot\theta\dot\varphi\right)$$

$$EL_\theta:\ 2ma^2\Big[\left(\frac{1}{3}\sin\varphi - \sin 2\theta\right)\cos\varphi\,\dot\theta\dot\varphi - \cos 2\theta\sin\varphi\,\dot\theta^2 +$$

$$+\left(\frac{13}{6}+\frac{1}{6}\sin^2\varphi - \sin 2\theta\sin\varphi\right)\ddot\theta - (1-2\sin^2\theta)\sin\varphi\,\dot\varphi^2 + (1-2\sin^2\theta)\cos\theta\,\ddot\varphi\Big] =$$

$$= -mga\left(\frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\right)$$

$$\frac{\partial K}{\partial \dot\varphi} = ma^2\left[\frac{1}{3}\dot\varphi + (1-2\sin^2\theta)\cos\varphi\,\dot\theta\right],\quad \frac{d}{dt}\left(\frac{\partial K}{\partial\dot\varphi}\right) = ma^2\left[\frac{1}{3}\ddot\varphi + (1-2\sin^2\theta)\right.$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\varphi}\right) = ma^2\left[\frac{1}{3}\ddot\varphi + \left(-4\sin\theta\cos\theta\cos\varphi\,\dot\theta^2\right) + (1-2\sin^2\theta)\left(-\sin\varphi\,\dot\varphi\dot\theta + \cos\varphi\,\ddot\theta\right)\right]$$

$$\frac{\partial K}{\partial \varphi} = ma^2\left[\left(\frac{1}{3}\sin\varphi\cos\varphi - \sin 2\theta\cos\varphi\right)\dot\theta^2 - (1-2\sin^2\theta)\sin\varphi\,\dot\theta\dot\varphi\right]$$

$$EL_\varphi:\ ma^2\left[\frac{1}{3}\ddot\varphi - \left(\frac{1}{3}\sin\varphi + \sin 2\theta\right)\cos\varphi\,\dot\theta^2 + (1-2\sin^2\theta)\cos\varphi\,\ddot\theta\right] = -mga\left(\frac{1}{2}\sin\theta\cos\varphi\right)$$

5) Linearizzazione

Poiché la sollecitazione è conservativa, possiamo usare la formula

$$(13.1)\quad A(\vec{q}_e)\ddot{\vec{x}} + \mathcal{H}_V(\vec{q}_e)\vec{x} = 0 \qquad \vec{x} = \frac{\vec{q}(t)-\vec{q}_e}{\varepsilon}$$

Quindi

$$(13.2)\quad ma^2\begin{bmatrix} \frac{13}{3}+\frac{1}{3}\sin^2\varphi_e - 2\sin 2\theta_e \sin\varphi_e & (1-2\sin^2\theta_e)\cos\varphi_e \\ (1-2\sin^2\theta_e)\cos\varphi_e & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \mathcal{H}_V(\vec{q}_e)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Nella configurazione $\vec{q}_e^{(5)}$ si ottiene

$$(13.3)\quad a\begin{bmatrix} \frac{13}{3}+\frac{1}{3}-2\sin 2\theta_e^{(5)} & 0 \\ 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + g\begin{bmatrix} -\cos\theta_e^{(5)}\left(\frac{1}{2}\operatorname{tg}\theta_e^{(5)}+2\right) & 0 \\ 0 & -\frac{1}{2}\sin\theta_e^{(5)}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

cioè

$$(13.4)\quad \begin{cases} a\left(\frac{14}{3}-2\sin 2\theta_e^{(5)}\right)\ddot{x}_1 - g\cos\theta_e^{(5)}\,\frac{17}{8}\,x_1 = 0 \\ \frac{1}{3}a\ddot{x}_2 - \frac{1}{2}g\sin\theta_e^{(5)}\,x_2 = 0 \end{cases}$$

Tenendo conto che

$$(13.5)\quad \sin\theta = \pm\sqrt{\frac{\operatorname{tg}^2\theta}{1+\operatorname{tg}^2\theta}}\,,\quad \cos\theta = \pm\frac{1}{\sqrt{1+\operatorname{tg}^2\theta}}\,,\quad \sin 2\theta = 2\sin\theta\cos\theta,$$

si ha:

$$(13.6)\quad \sin\theta_e^{(5)} = -\frac{1}{\sqrt{17}}\,,\quad \cos\theta_e^{(5)} = -\frac{4}{\sqrt{17}}\,,\quad \sin 2\theta_e^{(5)} = 2\cdot\frac{4}{17} = \frac{8}{17}$$

$$(14.1)\quad \begin{cases} a\left(\frac{14}{3} - 2\cdot\frac{8}{17}\right)\ddot{x}_1 + g\frac{4}{\sqrt{17}}\cdot\frac{17}{8_2}\; x_1 = 0 \\ \frac{1}{3}a\,\ddot{x}_2 + g\frac{1}{2\sqrt{17}}\; x_2 = 0 \end{cases}$$

Eq. disaccoppiate!

Analogamente, nella configurazione $\vec{q}_e^{\,(8)}$ si ottiene

$$(14.2)\quad a\begin{bmatrix} \frac{13}{3} + \frac{1}{3} + 2\sin 2\theta_e^{(8)} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + g\begin{bmatrix} -\cos\theta_e^{(8)}\left(-\frac{1}{2}\operatorname{tg}\theta_e^{(8)} + 2\right) & 0 \\ 0 & \frac{1}{2}\sin\theta_e^{(8)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Tenendo conto della (13.5) si trova:

$$(14.3)\quad \sin\theta_e^{(8)} = \frac{1}{\sqrt{17}}, \quad \cos\theta_e^{(8)} = -\frac{4}{\sqrt{17}}, \quad \sin 2\theta^{(8)} = 2\left(-\frac{4}{17}\right) = -\frac{8}{17}$$

Quindi, il sistema delle eq. linearizzate in $\vec{q}_e^{\,(8)}$ è

$$(14.4)\quad \begin{cases} a\left(\frac{14}{3} - \frac{16}{17}\right)\ddot{x}_1 + g\,\frac{4}{\sqrt{17}}\,\frac{17}{8_2}\,x_1 = 0 \\ \frac{1}{3}a\,\ddot{x}_2 + g\,\frac{1}{2}\,\frac{1}{\sqrt{17}}\,x_2 = 0 \end{cases}$$

Eq. disaccoppiate!

e coincide con quello in $\vec{q}_e^{\,(5)}$.

e) Reazioni dinamiche in A e B.

Scriviamo le 2 ECD, scegliendo come polo per la II il punto $B \in R$

$$(15.1)\quad \begin{cases} \vec{R}^{(est,att)} + \vec{\Psi}_A + \vec{\phi}_B = m\,\vec{a}_G \\ \vec{M}_B^{(est,att)} + (A-B)\times\vec{\Psi}_A = \dfrac{d\vec{L}_B}{dt} + \vec{v}_B \times \vec{p} \end{cases}$$

Proiettando la I ECD sulla base fissa $\mathcal{B}_1$, possiamo ricavare subito $\vec{\Psi}_A\cdot\vec{e}_y$ e $\vec{\phi}_B\cdot\vec{e}_z$.

$$(15.2)\quad \vec{\Psi}_A\cdot\vec{e}_x + \vec{\phi}_B\cdot\vec{e}_x = m g\,\vec{e}_z\cdot\vec{e}_x + m\,\vec{a}_G\cdot\vec{e}_x$$

$$(15.3)\quad \vec{\Psi}_A\cdot\vec{e}_y = m g\,\cancel{\vec{e}_z\cdot\vec{e}_y} - \cancel{\vec{\phi}_B\cdot\vec{e}_y} + m\,\vec{a}_G\cdot\vec{e}_y$$

$$(15.4)\quad \vec{\phi}_B\cdot\vec{e}_z = m g\,\vec{e}_z\cdot\vec{e}_z - \cancel{\vec{\Psi}_A\cdot\vec{e}_z} + m\,\vec{a}_G\cdot\vec{e}_z$$

Quindi, dobbiamo calcolare le componenti di $\vec{a}_G$ su $\mathcal{B}$.
Facciamolo derivando 2 volte risp. al tempo le (3.6)

$$\vec{v}_G = \frac{d}{dt}(G-O) = a\Big[-\frac{1}{2}\sin\psi\,\dot\psi\,\vec{e}_x + \Big(-\frac{1}{2}\sin\theta\sin\psi\,\dot\theta + \frac{1}{2}\cos\theta\cos\psi\,\dot\psi + 2\cos\theta\,\dot\theta\Big)\vec{e}_y$$

$$(15.5)\qquad + \Big(\frac{1}{2}\cos\theta\sin\psi\,\dot\theta + \frac{1}{2}\sin\theta\cos\psi\,\dot\psi - 2\sin\theta\,\dot\theta\Big)\vec{e}_z\Big]$$

$$= a\Big\{-\frac{1}{2}\sin\psi\,\dot\psi\,\vec{e}_x + \Big[\Big(-\frac{1}{2}\sin\theta\sin\psi + 2\cos\theta\Big)\dot\theta + \frac{1}{2}\cos\theta\cos\psi\,\dot\psi\Big]\vec{e}_y +$$

$$+ \Big[\Big(\frac{1}{2}\cos\theta\sin\psi - 2\sin\theta\Big)\dot\theta + \frac{1}{2}\sin\theta\cos\psi\,\dot\psi\Big]\vec{e}_z\Big\}$$

$$\vec{a}_G = a\Big\{ -\frac{1}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right)\vec{e}_x \,+$$

$$+\Big[\left(-\frac{1}{2}\left(\cos\theta\sin\varphi\,\dot{\theta} + \sin\theta\cos\varphi\,\dot{\varphi}\right) - 2\sin\theta\,\dot{\theta}\right)\dot{\theta} + \left(-\frac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\right)\ddot{\theta}$$

$$+\frac{1}{2}\left(-\sin\theta\cos\varphi\,\dot{\theta} + \cos\theta\sin\varphi\,\dot{\varphi}\right)\dot{\varphi} + \frac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_y$$

$$+\Big[\left(\frac{1}{2}\left(-\sin\theta\sin\varphi\,\dot{\theta} + \cos\theta\cos\varphi\,\dot{\varphi}\right) - 2\cos\theta\,\dot{\theta}\right)\dot{\theta} + \left(\frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\right)\ddot{\theta} \,+ \tag{16.1}$$

$$+\frac{1}{2}\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right)\dot{\varphi} + \frac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_z\Big\}$$

$$= a\Big\{ -\frac{1}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right)\vec{e}_x \,+$$

$$+\Big[-\left(\frac{1}{2}\cos\theta\sin\varphi + 2\sin\theta\right)\dot{\theta}^2 - \sin\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \left(-\frac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\right)\ddot{\theta}$$

$$-\frac{1}{2}\cos\theta\sin\varphi\,\dot{\varphi}^2 + \frac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_y \,+$$

$$+\Big[-\left(\frac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\right)\dot{\theta}^2 + \cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \left(\frac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\right)\ddot{\theta}$$

$$-\frac{1}{2}\sin\theta\sin\varphi\,\dot{\varphi}^2 + \frac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_z\Big\}$$

Dunque, la (15.3) fornisce

$$(17.1)\quad \vec{\Psi}_A \cdot \vec{e}_y = ma\Big[-\Big(\tfrac{1}{2}\cos\theta\sin\varphi + 2\sin\theta\Big)\dot{\theta}^2 - \sin\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \\ + \Big(-\tfrac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\Big)\ddot{\theta} - \tfrac{1}{2}\cos\theta\sin\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big]$$

e la (15.4)

$$(17.2)\quad \vec{\Phi}_B \cdot \vec{e}_z = mg + ma\Big[-\Big(\tfrac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\Big)\dot{\theta}^2 + \cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \\ + \Big(\tfrac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\Big)\ddot{\theta} - \tfrac{1}{2}\sin\theta\sin\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big]$$

Inoltre, la (15.2) fornisce la somma delle reazioni lungo $\vec{e}_x$

$$(17.3)\quad (\vec{\Psi}_A + \vec{\Phi}_B)\cdot\vec{e}_x = -\frac{ma}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right)$$

Per determinare ognuna delle 2 incognite $\vec{\Psi}_A\cdot\vec{e}_x$, $\vec{\Phi}_B\cdot\vec{e}_x$ dobbiamo usare la II ECD (15.1). Osserviamo che

$$(A-B)\times\vec{\Psi}_A = 4a\,\vec{k}\times(\Psi_x\vec{e}_x + \Psi_y\vec{e}_y) =$$

$$(17.4)\qquad = 4a(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z)\times(\Psi_x\vec{e}_x + \Psi_y\vec{e}_y)$$

$$= 4a(\Psi_x\sin\theta\,\vec{e}_z + \Psi_x\cos\theta\,\vec{e}_y - \Psi_y\cos\theta\,\vec{e}_x)$$

Allora, se proiettiamo la II ECD lungo $\vec{e}_y$, otteniamo

$$(17.5)\quad 4a\cos\theta\,\Psi_x = -(G-B)\times m\vec{g}\cdot\vec{e}_y + \frac{d\vec{L}_B}{dt}\cdot\vec{e}_y + \cancel{\vec{v}_0\times\vec{p}\cdot\vec{e}_y} \qquad \vec{v}_0 \parallel \vec{e}_y$$

Il momento della forza peso vale

$$(G-B)\times m\vec g = \left(\frac{a}{2}\vec i + 2a\vec k\right)\times(-mg\vec e_z) = -mg\left(\frac{a}{2}\vec i\times\vec e_z + 2a\vec k\times\vec e_z\right)$$

$$= -mg\left[\frac{a}{2}(\cos\varphi\,\vec e_x + \cos\theta\sin\varphi\,\vec e_y)\times\vec e_z + 2a(-\sin\theta)\,\vec e_y\times\vec e_z\right]$$

$$= -mga\left[-\frac{1}{2}\cos\varphi\,\vec e_y + \cos\theta\sin\varphi\,\vec e_x - 2\sin\theta\,\vec e_x\right]$$

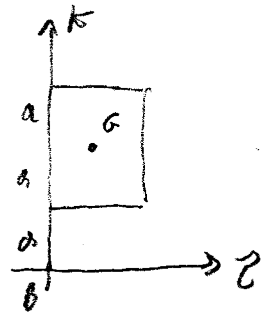
$$= -mga\left[(\cos\theta\sin\varphi - 2\sin\theta)\,\vec e_x - \frac{1}{2}\cos\varphi\,\vec e_y\right]$$

Calcoliamo

$$\frac{d\vec L_B}{dt}\cdot\vec e_y = \frac{d}{dt}\left(\vec L_B\cdot\vec e_y\right)$$

$$\vec L_B\cdot\vec e_y = I_B(\vec\omega)\cdot\vec e_y + \cancel{(G-B)\times m\vec v_B\cdot\vec e_y} \qquad \vec v_B \parallel \vec e_y$$

Determiniamo I_B con il Teo di Huygens-Steiner



$$[I_B]^{B''} = [I_G]^{B''} + m\begin{bmatrix} y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2\end{bmatrix}$$

dove con (x,y,z) abbiamo indicato le coordinate di G rs. a (B, B'').

$$[I_B]^{B''} = ma^2\begin{bmatrix}\frac{4}{12}&&\\&\frac{5}{12}&\\&&\frac{1}{12}\end{bmatrix} + m\begin{bmatrix}4a^2&0&-\frac{a}{2}2a\\0&\frac{17}{4}a^2&0\\-a^2&0&\frac{a^2}{4}\end{bmatrix} = ma^2\begin{bmatrix}\frac{13}{3}&0&-1\\0&\frac{14}{3}&0\\-1&0&\frac{1}{3}\end{bmatrix}$$

Allora

$$I_B(\vec{\omega}) = ma^2 [\vec{i}, \vec{j}, \vec{k}] \begin{bmatrix} \frac{13}{3} & 0 & -1 \\ 0 & \frac{14}{3} & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \dot\theta \cos\psi \\ -\dot\theta \sin\psi \\ \dot\psi \end{bmatrix} =$$

$$= ma^2 [\vec{i}, \vec{j}, \vec{k}] \begin{bmatrix} \frac{13}{3}\cos\psi\,\dot\theta - \dot\psi \\ -\frac{14}{3}\sin\psi\,\dot\theta \\ -\cos\psi\,\dot\theta + \frac{1}{3}\dot\psi \end{bmatrix}$$

Quindi,

$$I_B(\vec{\omega}) = ma^2 \left[\left(\frac{13}{3}\cos\psi\,\dot\theta - \dot\psi\right)\vec{i} - \frac{14}{3}\sin\psi\,\dot\theta\,\vec{j} + \left(\frac{1}{3}\dot\psi - \cos\psi\,\dot\theta\right)\vec{k} \right]$$

$$\overset{(3.3)}{=} ma^2 \Big[\left(\frac{13}{3}\cos\psi\,\dot\theta - \dot\psi\right)\left(\cos\psi\,\vec{e}_x + \cos\theta\sin\psi\,\vec{e}_y + \sin\theta\sin\psi\,\vec{e}_z\right) +$$

$$-\frac{14}{3}\sin\psi\,\dot\theta\left(-\sin\psi\,\vec{e}_x + \cos\theta\cos\psi\,\vec{e}_y + \sin\theta\cos\psi\,\vec{e}_z\right) +$$

$$+\left(\frac{1}{3}\dot\psi - \cos\psi\,\dot\theta\right)\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right)\Big]$$

$$= ma^2 \Big\{ \left[\left(\frac{13}{3}\cos\psi\,\dot\theta - \dot\psi\right)\cos\psi + \frac{14}{3}\sin^2\psi\,\dot\theta \right]\vec{e}_x +$$

$$+ \left[\left(\frac{13}{3}\cos\psi\,\dot\theta - \dot\psi\right)\cos\theta\sin\psi - \frac{14}{3}\dot\theta\cos\theta\sin\psi\cos\psi - \sin\theta\left(\frac{1}{3}\dot\psi - \cos\psi\,\dot\theta\right) \right]\vec{e}_y$$

$$+ \left[\left(\frac{13}{3}\cos\psi\,\dot\theta - \dot\psi\right)\sin\theta\sin\psi - \frac{14}{3}\dot\theta\sin\theta\sin\psi\cos\psi + \left(\frac{1}{3}\dot\psi - \cos\psi\,\dot\theta\right)\cos\theta \right]\vec{e}_z$$

$$I_B(\vec{\omega}) \cdot \vec{e}_y = ma^2 \left[\cos\psi\,\dot\theta\left(-\frac{1}{3}\cos\theta\sin\psi + \sin\theta\right) - \left(\frac{1}{3}\sin\theta + \cos\theta\sin\psi\right)\dot\psi \right]$$

Dunque,

$$\frac{d}{dt}\left(\vec{L}_B\cdot\vec{e}_y\right) = \frac{d}{dt}\left(I_B(\vec{\omega})\cdot\vec{e}_y\right) =$$

$$= ma^2\Big[-\sin\psi\,\dot\psi\,\dot\theta\left(-\frac13\cos\theta\sin\psi+\cancel{\sin\theta}\right)+\cos\psi\,\ddot\theta\left(-\frac13\cos\theta\sin\psi+\sin\theta\right)+$$

(20.1) $$+\cos\psi\,\dot\theta\left(+\frac13\sin\theta\,\dot\theta\sin\psi-\frac13\cos\theta\cos\psi\,\dot\psi+\cos\theta\,\dot\theta\right)+$$

$$-\left(\frac13\cos\theta\,\dot\theta-\sin\theta\cancel{\sin\psi}\,\dot\theta+\cos\theta\cos\psi\,\dot\psi\right)\dot\psi-\left(\frac13\sin\theta+\cos\theta\sin\psi\right)\ddot\psi\Big]=$$

$$=ma^2\Big[\left(\frac13\cos\theta\left(\sin^2\psi-\cos^2\psi\right)-\frac13\cos\theta\right)\dot\psi\dot\theta+\cos\psi\left(\sin\theta-\frac13\cos\theta\sin\psi\right)\ddot\theta+\cos\psi\left(\frac13\sin\theta\sin\psi+\cos\theta\right)\dot\theta^2$$
$$-\cos\theta\cos\psi\,\dot\psi^2-\left(\frac13\sin\theta+\cos\theta\sin\psi\right)\ddot\psi\Big]$$

Allora, la (17.5) si scrive:

$$4a\cos\theta\,\Psi_x=-\frac{mga}{2}\cos\psi+ma^2\Big[-\frac13\cos\theta\left(\cos2\psi+1\right)\dot\psi\dot\theta+\cos\psi\left(\sin\theta-\frac13\cos\theta\sin\psi\right)\ddot\theta$$

(20.2) $$+\cos\psi\left(\frac13\sin\theta\sin\psi+\cos\theta\right)\dot\theta^2-\cos\theta\cos\psi\,\dot\psi^2-\left(\frac13\sin\theta+\cos\theta\sin\psi\right)\ddot\psi\Big]$$

Quindi, se $\cos\theta\neq0 \Leftrightarrow \theta\neq\pm\pi/2$, si trova

(20.3) $$\Psi_x=-\frac{mg}{8}\frac{\cos\psi}{\cos\theta}+\frac{ma}{4\cos\theta}\Big[-\frac13\cos\theta\left(1+\cos2\psi\right)\dot\theta\dot\psi+\cos\psi\left(-\frac13\cos\theta\sin\psi+\sin\theta\right)\ddot\theta$$
$$+\left(\frac16\sin\theta\sin2\psi+\cos\theta\right)\dot\theta^2-\cos\theta\cos\psi\,\dot\psi^2-\left(\frac13\sin\theta+\cos\theta\sin\psi\right)\ddot\psi\Big]+$$

Sostituendo nella (17.3), otteniamo

(20.4) $$\Phi_x=-\frac{ma}{2}\left(\cos\psi\,\dot\psi^2+\sin\psi\,\ddot\psi\right)+\frac{mg}{8}\frac{\cos\psi}{\cos\theta}+$$

$$-\frac{ma}{4}\Big[-\frac23\cos^2\psi\,\dot\theta\dot\psi+\cos\psi\left(-\frac13\sin\psi+\mathrm{tg}\,\theta\right)\ddot\theta+\frac16\left(\mathrm{tg}\,\theta\sin2\psi+1\right)\dot\theta^2-\cos\psi\,\dot\psi^2+$$

$$-\left(\frac13\,\mathrm{tg}\,\theta+\sin\psi\right)\ddot\psi\Big]$$