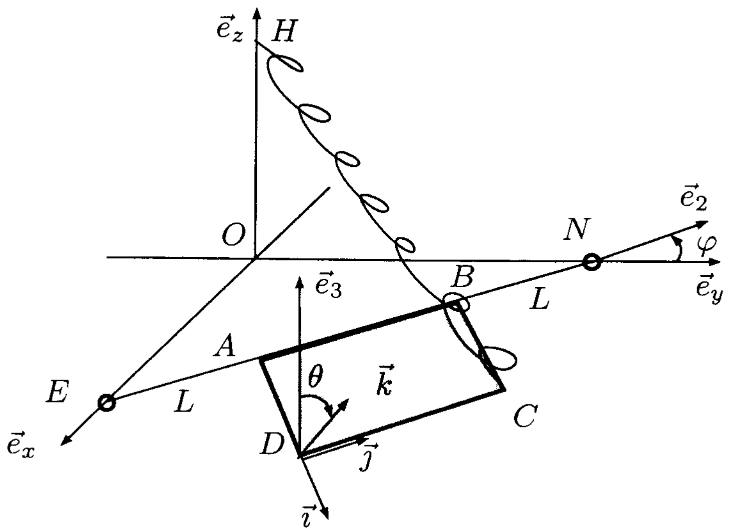
Compito di Meccanica Razionale

Trieste, 16 luglio 2018. (G. Tondo)



Una lamina omogenea rettangolare di lati L e $3L$ e massa m è saldata, lungo il lato più lungo, ad un'asta EN di lunghezza $5L$ e di massa trascurabile. Gli estremi dell'asta sono vincolati, come in figura, a scorrere senza attrito lungo due guide fisse ortogonali, tramite due cerniere sferiche "bucate". Sul rigido agisce il peso proprio opposto ad $\vec{e}_z$, la forza di richiamo di una molla lineare, di costante elastica c, fissata al vertice C e nel punto fisso H, di quota $3L$.

STATICA

Siano φ l'angolo tra i versori $\vec{e}_y$ ed $\vec{e}_2$, e θ l'angolo di rotazione della lamina intorno all'asse EN orientato come $\vec{e}_2$, misurato tra i versori $\vec{e}_3$ e $\vec{k}$: si considerino entrambi gli angoli appartenenti all'intervallo $]-\pi, \pi]$.

1) Verificare che le configurazioni $\vec{q} = (\varphi = 0, \theta = \pm\frac{\pi}{2})$ sono di equilibrio e determinare la loro stabilità in funzione dei parametri del modello;

2) le reazioni vincolari esterne sul rigido in E, nelle configurazioni di equilibrio suddette;

3) le reazioni vincolari esterne sul rigido in N, nelle configurazioni di equilibrio suddette.

DINAMICA

4) Scrivere le equazione differenziali pure di moto;

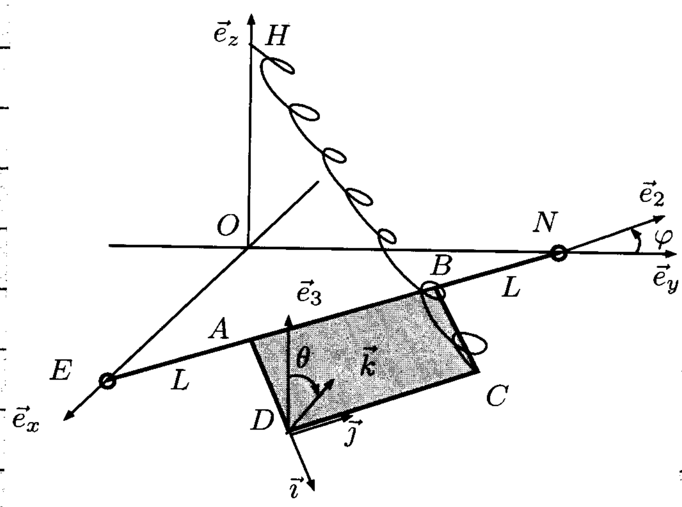
5) linearizzare le equazioni di moto intorno alle configurazioni di equilibrio del punto **1**;

6) calcolare le reazioni vincolare esterne nei punti E e N, durante il moto.

Nota Bene. Si suggerisce di usare, oltre alla base fissa $\mathcal{B} = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$ anche una base intermedia formata dai versori $\mathcal{B}' = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$, con $\vec{e}_2$ diretto come l'asta, $\vec{e}_3 = \vec{e}_z$ ed $\vec{e}_1 = \vec{e}_2 \times \vec{e}_3$. Inoltre, si consiglia di prendere una base $(\vec{\imath}, \vec{\jmath}, \vec{k})$, solidale alla lamina, con $\vec{\jmath} = \vec{e}_2$, il versore $\vec{\imath}$ lungo il lato AD e $\vec{k} = \vec{\imath} \times \vec{\jmath}$.

Tema del 16/07/2018 $\quad$ 1

Il modello è un rigido $\mathcal{R}$ vincolato con 2 cerniere sferiche "bucate" alle guide $(O,\vec{e}_x)$ e $(O,\vec{e}_y)$. Con il metodo dei congelamenti successivi si può osservare che, una volta bloccato lo spostamento virtuale di E, non rimane che lo spostamento virtuale rotatorio intorno all'asse EN. Dunque il rigido ha 2 g.l. Coordinate libere: $-\pi<\varphi\le\pi$, $-\pi<\theta\le\pi$.



Conviene usare le seguenti basi ON:

$$\mathcal{B}=(\vec{e}_x,\vec{e}_y,\vec{e}_z),\quad \mathcal{B}'=(\vec{e}_1,\vec{e}_2,\vec{e}_3),\quad \mathcal{B}''=(\vec{i},\vec{j},\vec{k})$$

$$(1.1)\begin{cases}\vec{e}_1=\vec{e}_2\times\vec{e}_3=\cos\varphi\,\vec{e}_x+\sin\varphi\,\vec{e}_y\\ \vec{e}_2=\cos\varphi\,\vec{e}_y-\sin\varphi\,\vec{e}_x\\ \vec{e}_3=\vec{e}_z\end{cases}\qquad [\vec{e}_1,\vec{e}_2,\vec{e}_3]=[\vec{e}_x,\vec{e}_y,\vec{e}_z]\underbrace{\begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}}_{R_\varphi}$$

$$(1.2)\begin{cases}\vec{i}=\vec{j}\times\vec{k}=\cos\theta\,\vec{e}_1-\sin\theta\,\vec{e}_3\\ \vec{j}=\vec{e}_2\\ \vec{k}=\cos\theta\,\vec{e}_3+\sin\theta\,\vec{e}_1\end{cases}\qquad [\vec{i},\vec{j},\vec{k}]=[\vec{e}_1,\vec{e}_2,\vec{e}_3]\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$(1.3)\quad [\vec{i},\vec{j},\vec{k}]=[\vec{e}_x,\vec{e}_y,\vec{e}_z]\,R_\varphi R_\theta=[\vec{e}_x,\vec{e}_y,\vec{e}_z]\begin{bmatrix}\cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta\\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$E-O = 5L\sin\varphi\,\vec{e}_x$$

$$G-E = L\left(\frac{1}{2}\vec{i}+\frac{5}{2}\vec{j}\right) = \frac{L}{2}\left(\cos\theta\,\vec{e}_1 + 5\,\vec{e}_2 - \sin\theta\,\vec{e}_3\right)$$

$$G-O = (G-E)+(E-O) = 5L\sin\varphi\,\vec{e}_x + \frac{L}{2}\left(\vec{i}+5\vec{j}\right) =$$

$$= 5L\sin\varphi\,\vec{e}_x + \frac{L}{2}\cos\varphi\cos\theta\,\vec{e}_x + \frac{L}{2}\sin\varphi\cos\theta\,\vec{e}_y - \frac{L}{2}\sin\theta\,\vec{e}_z + \frac{5L}{2}\left(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\right)$$

$$= \left(\frac{5}{2}L\sin\varphi + \frac{L}{2}\cos\varphi\cos\theta\right)\vec{e}_x + \left(\frac{L}{2}\sin\varphi\cos\theta + \frac{5}{2}L\cos\varphi\right)\vec{e}_y - \frac{L}{2}\sin\theta\,\vec{e}_z$$

$$C-B = L\vec{i}, \quad B-E = 4L\vec{j}$$

$$C-O = (C-B)+(B-E)+(E-O) = L\left(\vec{i}+4\vec{j}\right) + 5L\sin\varphi\,\vec{e}_x$$

$$H-O = 3L\,\vec{e}_z$$

$$C-H = (C-O)+(O-H) = L\left(\vec{i}+4\vec{j}\right) + 5L\sin\varphi\,\vec{e}_x - 3L\,\vec{e}_z$$

$$\overline{|CH|}^2 = (C-H)\cdot(C-H) = L^2\left(\vec{i}+4\vec{j}+5\sin\varphi\,\vec{e}_x - 3\,\vec{e}_z\right)\cdot\left(\vec{i}+4\vec{j}+5\sin\varphi\,\vec{e}_x - 3\vec{e}_z\right)$$

$$= L^2\Big[1+16+25\sin^2\varphi + 9 + \cancel{8\,\vec{i}\cdot\vec{j}} + 10\sin\varphi\,\vec{i}\cdot\vec{e}_x - 6\,\vec{i}\cdot\vec{e}_z +$$

$$+ 40\sin\varphi\,\vec{j}\cdot\vec{e}_x - \cancel{24\,\vec{j}\cdot\vec{e}_z} - \cancel{30\sin\varphi\,\vec{e}_x\cdot\vec{e}_z}\Big]$$

$$= L^2\left(26 + 25\sin^2\varphi + 10\sin\varphi\cos\varphi\cos\theta + 6\sin\theta + 40\sin\varphi(-\sin\varphi)\right)$$

$$= L^2\left(26 - 15\sin^2\varphi + 5\sin 2\varphi\cos\theta + 6\sin\theta\right)$$

$$C-H = L\left(\vec{i}+4\vec{j}\right) + 5L\sin\varphi\,\vec{e}_x - 3L\,\vec{e}_z =$$

$$= L\left(\cos\varphi\cos\theta\,\vec{e}_x + \sin\varphi\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z\right) +$$

$$+ 4L\left(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\right) + 5L\sin\varphi\,\vec{e}_x - 3L\,\vec{e}_z$$

$$= L\left(\cos\varphi\cos\theta + \sin\varphi\right)\vec{e}_x + L\left(\sin\varphi\cos\theta + 4\cos\varphi\right)\vec{e}_y - L\left(\sin\theta + 3\right)\vec{e}_z$$

Statica

Dato che la sollecitazione è conservativa, cerchiamo i punti stazionari dell'energia potenziale

$$V(\varphi,\theta) = V^{(peso)} + V^{(molla)}$$

$$V^{(peso)} = -m\vec{g}\cdot(G-O) = mg\,\vec{e}_z\cdot\left(5L\sin\varphi\,\vec{e}_x + \frac{L}{2}\left(\vec{i}+5\vec{j}\right)\right)$$

$$= mgL\,\frac{1}{2}\left(\vec{e}_z\cdot\vec{i} + \cancel{5\,\vec{e}_z\cdot\vec{j}}\right) =$$

$$= mg\frac{L}{2}(-\sin\theta)$$

$$V^{(molla)} = \frac{1}{2}c\overline{CH}^2 = \frac{1}{2}cL^2\left(-15\sin^2\varphi + 5\sin 2\varphi\cos\theta + 6\sin\theta\right)$$

$$V = -mg\frac{L}{2}\sin\theta + \frac{1}{2}cL^2\left(-15\sin^2\varphi + 5\sin 2\varphi\cos\theta + 6\sin\theta\right)$$

$$= \left(3cL^2 - mg\frac{L}{2}\right)\sin\theta - \frac{15}{2}cL^2\sin^2\varphi + \frac{5}{2}cL^2\sin 2\varphi\cos\theta$$

$$-Q_\varphi = \frac{\partial V}{\partial\varphi} = -15cL^2\sin\varphi\cos\varphi + 5cL^2\cos 2\varphi\cos\theta$$

$$= cL^2\left(-\frac{15}{2}\sin 2\varphi + 5\cos 2\varphi\cdot\cos\theta\right)$$

$$-Q_\theta = \frac{\partial V}{\partial\theta} = \left(3cL - \frac{mg}{2}\right)L\cos\theta - \frac{5}{2}cL^2\sin 2\varphi\sin\theta$$

$$\left.\frac{\partial V}{\partial\varphi}\right|_{\substack{\varphi=0\\ \theta=\pm\pi/2}} = 0 \quad , \quad \left.\frac{\partial V}{\partial\theta}\right|_{\substack{\varphi=0\\ \theta=\pm\pi/2}} = 0 \Rightarrow \vec{q} = \left(0, \pm\frac{\pi}{2}\right) \text{ sono}$$

configurazioni di equilibrio.

<u>Stabilità</u>

$$\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right) \quad , \quad \vec{q}_e^{(2)} = \left(0, -\frac{\pi}{2}\right)$$

Determiniamo la matrice Hessiana di $V(\varphi, \theta)$ e calcoliamola in $\vec{q}_e^{(1)}$ e $\vec{q}_e^{(2)}$.

$$\frac{\partial^2 V}{\partial \varphi^2} = c l^2 \left(-15 \cos 2\varphi - 10 \sin 2\varphi \cos\theta\right)$$

$$\frac{\partial^2 V}{\partial \varphi \partial \theta} = -5 c l^2 \cos 2\varphi \sin\theta$$

$$\frac{\partial^2 V}{\partial \theta^2} = -\left(3cl - \frac{mg}{2}\right) l \sin\theta - \frac{5}{2} c l^2 \sin 2\varphi \cos\theta$$

$$\mathcal{H} = \begin{bmatrix} -5cl^2\left(3\cos 2\varphi + 2\sin 2\varphi \cos\theta\right) & -5cl^2 \cos 2\varphi \sin\theta \\ -5cl^2 \cos 2\varphi \sin\theta & -\left(3cl - \frac{mg}{2}\right) l \sin\theta - \frac{5}{2} c l^2 \sin 2\varphi \cos\theta \end{bmatrix}$$

$$\mathcal{H}_{11}\Big|_{\substack{\varphi = 0 \\ \theta = \pm\frac{\pi}{2}}} = -15 c l^2 < 0 \implies \text{instabilità}$$

2) e 3) Reazioni in E e N agli equilibri

Poiché i vincoli nei punti in E e N sono cerniere sferiche "bucate" lisce, si ha che

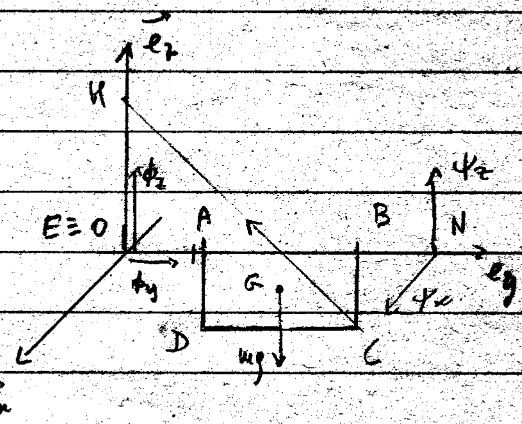
$$(5.1) \quad \mathcal{L}^{vinc} = \left\{ \left(E, \vec{\phi} = \phi_y \vec{e}_y + \phi_z \vec{e}_z\right), \left(N, \vec{\Psi} = \Psi_x \vec{e}_x + \Psi_z \vec{e}_z\right) \right\}$$

Dunque, abbiamo 4 incognite

$$(5.2) \qquad (\phi_y, \phi_z, \Psi_x, \Psi_z)$$

Per determinarle, scriviamo le ECS che devono essere soddisfatte nelle configurazioni di equilibrio $\vec{q}_e^{(1)}, \vec{q}_e^{(2)}$

$$(5.3) \quad \begin{cases} \vec{R}^{(ext,att)} + \vec{\phi}_E + \vec{\Psi}_N = \vec{0} \\ \vec{M}_E^{(ext,att)} + (N-E) \times \vec{\Psi}_N = \vec{0} \end{cases}$$



In $\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right)$

$$(5.4) \quad \vec{F}_c = -c(C-H) = -c(4L\vec{e}_y - 4L\vec{e}_z)$$

$$(5.5) \quad \vec{M}_E^{(peso)} = (G-E) \times m\vec{g} = -\frac{5}{2} mgL\,\vec{e}_x$$

$$(5.6) \quad \vec{M}_E^{(molla)} = (C-E) \times \vec{F}_c = (4L\vec{e}_y - L\vec{e}_z) \times \left[-c(4L\vec{e}_y - 4L\vec{e}_z)\right]$$

$$= -c(-16L^2\vec{e}_x + 4L^2\vec{e}_x) = 12cL^2\vec{e}_x$$

$$(5.7) \quad \vec{M}_E^{(ext,reatt)} = (N-E) \times \vec{\Psi}_N = 5L\vec{e}_y \times (\Psi_x\vec{e}_x + \Psi_z\vec{e}_z) = 5L(-\Psi_x\vec{e}_z + \Psi_z\vec{e}_x)$$

Proiettando la I ECS lungo $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ si trova

$$(5.8) \quad \vec{e}_x: \quad \Psi_x = 0$$

$$(5.9) \quad \vec{e}_y: \quad -4cL + \phi_y = 0 \iff \phi_y = 4cL$$

$$(5.10) \quad \vec{e}_z: \quad 4cL - mg + \phi_z + \Psi_z = 0 \iff \phi_z = mg - 4cL - \Psi_z$$

Resta da determinare l'incognita ψ_z. A tale scopo, utilizziamo la II ECS (5.3) proiettata lungo $\vec{e}_x$:

(6.1) $\vec{e}_x$: $-\frac{5}{2} mgL + 12cL^2 + 5L\psi_z = 0$

Quindi,

(6.2) $$\psi_z = \frac{\frac{5}{2}mgL - 12cL^2}{5L} = \frac{mg}{2} - \frac{12}{5}cL$$

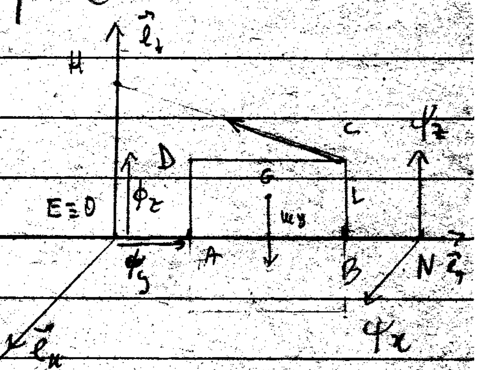
che, sostituito nella (5.10) fornisce

(6.3) $$\phi_z = mg - 4cL - \left(\frac{mg}{2} - \frac{12}{5}cL\right) = \frac{mg}{2} - \frac{8}{5}cL$$

Dunque, in $\vec{q}_e^{(1)}$

(6.4) $$\vec{\Phi}_E\Big|_{\vec{q}_e^{(1)}} = 4cL\vec{e}_y + \left(\frac{mg}{2} - \frac{8}{5}cL\right)\vec{e}_z, \quad \vec{\psi}_N\Big|_{\vec{q}_e^{(1)}} = \left(\frac{mg}{2} - \frac{12}{5}cL\right)\vec{e}_z$$

Analogamente, in $\vec{q}_e^{(2)} = \left(0, -\frac{\pi}{2}\right)$



(6.5) $\vec{F}_c = -c(C-H) = -c(4L\vec{e}_y - 2L\vec{e}_z)$

(6.6) $\vec{M}_C^{(peso)} = (G-E)\times m\vec{g} = -\frac{5}{2}mgL\,\vec{e}_x$

(6.7) $\vec{M}_E^{(molla)} = (C-E)\times\vec{F}_c = (4L\vec{e}_y + L\vec{e}_z)\times\left[-c(4L\vec{e}_y - 2L\vec{e}_z)\right] =$

$$= -c\left(-8L^2\vec{e}_x + 4L^2(-\vec{e}_x)\right) = 12cL^2\vec{e}_x$$

(6.8) $\vec{M}_E^{(est,vinc)} = (N-E)\times\vec{\psi}_N = 5L(-\psi_x\vec{e}_z + \psi_z\vec{e}_x)$

(6.9) $\vec{e}_x$: $\psi_x = 0$

(6.10) $\vec{e}_y$: $-4cL + \phi_y = 0 \iff \phi_y = 4cL$

(6.11) $\vec{e}_z$: $2cL - mg + \phi_z + \psi_z = 0 \iff \phi_z = mg - 2cL - \psi_z$

Proiettando la II ECS (5.3) lungo $\vec{e}_x$, ritrovo

(7.1) $\vec{e}_x$: $-\frac{5}{2}mgL + 12cL^2 + 5L\psi_z = 0 \iff \psi_z = \frac{\frac{5}{2}mgL - 12cL^2}{5L} = \left(\frac{mg}{2} - \frac{12}{5}cL\right)$

che, sostituita nella (6.11) fornisce

(7.2) $\phi_z = mg - 2cL - \left(\frac{mg}{2} - \frac{12}{5}cL\right) = \frac{mg}{2} + \frac{2}{5}cL > 0$

Dunque, in $\vec{q}_e^{\,(2)}$

(7.3) $\vec{\phi}_E\Big|_{\vec{q}_e^{\,(2)}} = 4cL\,\vec{e}_y + \left(\frac{mg}{2} + \frac{2}{5}cL\right)\vec{e}_z$, $\quad \vec{\psi}_N\Big|_{\vec{q}_e^{\,(2)}} = \left(\frac{mg}{2} - \frac{12}{5}cL\right)\vec{e}_z$

N.B. Si osservi che, sia in $\vec{q}_e^{\,(1)}$, sia in $\vec{q}_e^{\,(2)}$, le componenti della II ECS lungo $\vec{e}_y$ ed $\vec{e}_z$ sono identicamente soddisfatte. Infatti,

$\vec{e}_y$: $0 = 0$

$\vec{e}_z$: $-5L\psi_x = 0$

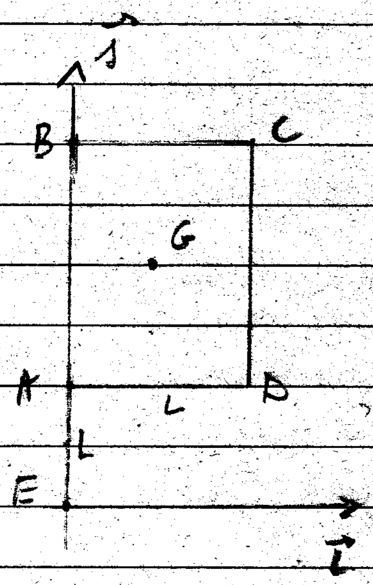
4) Scriviamo le EL. A tale scopo, calcoliamo l'energia cinetica del rigido

$$K = \frac{1}{2} m |\vec{v}_E|^2 + \frac{1}{2} \vec{\omega} \cdot I_E(\vec{\omega}) + m \vec{v}_E \cdot \vec{\omega} \times (G-E) \tag{8.1}$$

$$\vec{v}_E = \frac{d}{dt}(E-O) = 5L\cos\varphi\,\dot{\varphi}\,\vec{e}_x \tag{8.2}$$

$$|\vec{v}_E|^2 = 25L^2\cos^2\varphi\,\dot{\varphi}^2$$

$$\vec{\omega} = \dot{\varphi}\,\vec{e}_z + \dot{\theta}\,\vec{e}_2 = \dot{\varphi}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{k}) + \dot{\theta}\,\vec{j}$$



Calcolo di $[I_E]^{B''}$

$$[I_E]^{B''} = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

$$I_{11} = I_E(\vec{i})\cdot\vec{i} = I_G(\vec{i})\cdot\vec{i} + m\left(\frac{5L}{2}\right)^2 = \frac{1}{12}m(3L)^2 + \frac{25}{4}mL^2 = \frac{84}{12}mL^2 = 7mL^2$$

$$I_{22} = I_E(\vec{j})\cdot\vec{j} = \frac{1}{3}mL^2, \quad I_{33} = I_{11} + I_{22} = \frac{22}{3}mL^2$$

$$I_{12} = I_E(\vec{i})\cdot\vec{j} = -\frac{m}{3L^2}\int_0^L x\,dx\int_L^{4L} y\,dy = -\frac{m}{3L^2}\left[\frac{x^2}{2}\right]_0^L\left[\frac{y^2}{2}\right]_L^{4L} =$$

$$= -\frac{m}{3L^2}\frac{L^2}{2}\frac{(4L)^2-L^2}{2} = -\frac{m\,\cancel{15}^{5}L^2}{\cancel{3}\cdot 4} = -\frac{5mL^2}{4}$$

$$[I_E]^{B''} = mL^2\begin{bmatrix} 7 & -\frac{5}{4} & 0 \\ -\frac{5}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{22}{3} \end{bmatrix}$$

Termine polare di **K**:

$$\frac{1}{2}\vec{\omega}\cdot I_E(\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot mL^2\begin{bmatrix} 7 & -\frac{5}{4} & 0 \\ -\frac{5}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{22}{3}\end{bmatrix}\begin{bmatrix} -\dot{\varphi}\sin\theta \\ \dot{\theta} \\ \dot{\varphi}\cos\theta\end{bmatrix} =$$

$$= \frac{1}{2}mL^2\left[-\dot{\varphi}\sin\theta,\ \dot{\theta},\ \dot{\varphi}\cos\theta\right]\cdot\begin{bmatrix} -7\dot{\varphi}\sin\theta - \frac{5}{4}\dot{\theta} \\ \frac{5}{4}\dot{\varphi}\sin\theta + \frac{1}{3}\dot{\theta} \\ \frac{22}{3}\dot{\varphi}\cos\theta\end{bmatrix} =$$

$$= \frac{1}{2}\left(7\dot{\varphi}^2\sin^2\theta + \frac{5}{4}\dot{\varphi}\dot{\theta}\sin\theta + \frac{5}{4}\dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2 + \frac{22}{3}\dot{\varphi}^2\cos^2\theta\right)$$

$$= \frac{1}{2}mL^2\left[\left(7\sin^2\theta + \frac{22}{3}\cos^2\theta\right)\dot{\varphi}^2 + \frac{5}{2}\dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2\right]$$

$$= \frac{1}{2}\cdot mL^2\left[\left(7 + \frac{1}{3}\cos^2\theta\right)\dot{\varphi}^2 + \frac{5}{2}\dot{\varphi}\dot{\theta}\sin\theta + \frac{1}{3}\dot{\theta}^2\right]$$

Termine misto di K:

$$m\vec{v}_F\cdot\vec{\omega}\times(G-E)=m5L\cos\varphi\,\dot\varphi\,\vec{e}_x\cdot\left(\dot\varphi\,\vec{e}_3+\dot\theta\,\vec{e}_2\right)\times L\left(\frac{1}{2}\vec{i}+\frac{5}{2}\vec{j}\right)$$

$$=m5L^2\cos\varphi\,\dot\varphi\,\vec{e}_x\cdot\left(\frac{\dot\varphi}{2}\,\vec{e}_3\times\vec{i}+\frac{5}{2}\dot\varphi\,\vec{e}_3\times\vec{e}_2+\frac{\dot\theta}{2}\,\vec{j}\times\vec{i}\right)$$

$$=m5L^2\cos\varphi\,\dot\varphi\,\vec{e}_x\cdot\left(\frac{\dot\varphi}{2}\cos\theta\,\vec{e}_2-\frac{5}{2}\dot\varphi\,\vec{e}_1-\frac{\dot\theta}{2}\vec{k}\right)$$

$$=m5L^2\cos\varphi\,\dot\varphi\left(\frac{\dot\varphi}{2}\cos\theta\,\vec{e}_x\cdot\vec{e}_2-\frac{5}{2}\dot\varphi\,\vec{e}_x\cdot\vec{e}_1-\frac{\dot\theta}{2}\vec{e}_x\cdot\vec{k}\right)$$

$$=m5L^2\cos\varphi\,\dot\varphi\left(\frac{\dot\varphi}{2}\cos\theta(-\sin\varphi)-\frac{5}{2}\dot\varphi\cos\varphi-\frac{\dot\theta}{2}\cos\varphi\sin\theta\right)$$

$$=-m5L^2\cos\varphi\,\dot\varphi\left(+\frac{\dot\varphi}{2}\sin\varphi\cos\theta+\frac{5}{2}\dot\varphi\cos\varphi+\frac{\dot\theta}{2}\cos\varphi\sin\theta\right)=$$

$$=-\frac{5}{2}mL^2\left[\cos\varphi\left(\sin\varphi\cos\theta+5\cos\varphi\right)\dot\varphi^2+\cos^2\varphi\sin\theta\,\dot\theta\,\dot\varphi\right]$$

Dunque,

$$K=\frac{1}{2}m\,\cancel{25L^2\cos^2\varphi\,\dot\varphi^2}+\frac{1}{2}mL^2\left[\left(7+\frac{1}{3}\cos^2\theta\right)\dot\varphi^2+\frac{5}{2}\dot\varphi\dot\theta\sin\theta+\frac{1}{3}\dot\theta^2\right]+$$

$$-\frac{5}{2}mL^2\left[\cos\varphi\left(\sin\varphi\cos\theta+\cancel{5\cos\varphi}\right)\dot\varphi^2+\cos^2\varphi\sin\theta\,\dot\varphi\dot\theta\right]$$

$$=mL^2\left[\frac{1}{2}\left(7+\frac{1}{3}\cos^2\theta-\frac{5}{2}\sin2\varphi\cos\theta\right)\dot\varphi^2+\frac{1}{6}\dot\theta^2+\frac{5}{2}\left(\frac{1}{2}-\cos^2\varphi\right)\sin\theta\,\dot\varphi\dot\theta\right]$$

$$=\frac{1}{2}mL^2\left[\dot\varphi,\dot\theta\right]\begin{bmatrix}\left(7+\frac{1}{3}\cos^2\theta-\frac{5}{2}\sin2\varphi\cos\theta\right) & 5\left(\frac{1}{2}-\cos^2\varphi\right)\sin\theta\\ 5\left(\frac{1}{2}-\cos^2\varphi\right)\sin\theta & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\dot\varphi\\ \dot\theta\end{bmatrix}$$

$$\frac{\partial K}{\partial \dot\varphi} = mL^2\left(7 + \frac{1}{3}\cos^2\theta - \frac{5}{2}\sin 2\varphi\cos\theta\right)\dot\varphi + mL^2\frac{5}{2}\left(\frac{1}{2} - \cos^2\varphi\right)\sin\theta\,\dot\theta$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\varphi}\right) = mL^2\left(7 + \frac{1}{3}\cos^2\theta - \frac{5}{2}\sin 2\varphi\cos\theta\right)\ddot\varphi +$$

$$+ mL^2\left(-\frac{2}{3}\cos\theta\sin\theta\,\dot\theta - 5\cos 2\varphi\cos\theta\,\dot\varphi + \frac{5}{2}\sin 2\varphi\sin\theta\,\dot\theta\right)\dot\varphi$$

$$+ mL^2\frac{5}{2}\left(\frac{1}{2} - \cos^2\varphi\right)\left(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta\right) + mL^2\, 5\cos\varphi\sin\varphi\sin\theta\,\dot\varphi\,\dot\theta,$$

$$\frac{\partial K}{\partial \varphi} = mL^2\left(-\frac{5}{2}\cos 2\varphi\cos\theta\,\dot\varphi^2 + 5\cos\varphi\sin\varphi\sin\theta\,\dot\varphi\,\dot\theta\right)$$

Quindi

$$EL_\varphi:\quad mL^2\left(7 + \frac{1}{3}\cos^2\theta - \frac{5}{2}\sin 2\varphi\cos\theta\right)\ddot\varphi - \frac{5}{2}mL^2\cos 2\varphi\cos\theta\,\dot\varphi^2$$

$$+ mL^2\left(-\frac{2}{3}\cos\theta + \frac{5}{2}\sin 2\varphi\right)\sin\theta\,\dot\theta\,\dot\varphi + \frac{5}{2}mL^2\left(\frac{1}{2} - \cos^2\varphi\right)\left(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta\right)$$

$$= cL^2\left(\frac{15}{2}\sin 2\varphi - 5\cos 2\varphi\cos\theta\right)$$

$$\frac{\partial K}{\partial \dot\theta} = \frac{1}{3}mL^2\dot\theta + \frac{5}{2}mL^2\left(\frac{1}{2} - \cos^2\varphi\right)\sin\theta\,\dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot\theta}\right) = mL^2\left[\frac{1}{3}\ddot\theta + \frac{5}{2}\left(\frac{1}{2} - \cos^2\varphi\right)\left(\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi\right) + 5\cos\varphi\sin\varphi\sin\theta\,\dot\varphi^2\right]$$

$$\frac{\partial K}{\partial \theta} = mL^2\left[\left(-\frac{1}{3}\cos\theta\sin\theta + \frac{5}{2}\sin 2\varphi\sin\theta\right)\dot\varphi^2 + \frac{5}{2}\left(\frac{1}{2} - \cos^2\varphi\right)\cos\theta\,\dot\varphi\,\dot\theta\right]$$

$$EL_\theta:\quad mL^2\left[\frac{1}{3}\ddot\theta + \frac{5}{2}\left(\frac{1}{2} - \cos^2\varphi\right)\sin\theta\,\ddot\varphi + \underline{\qquad} + \frac{1}{3}\cos\theta)\sin\theta\,\dot\varphi^2\right]$$

$$= \left(\frac{mg}{2} - 3cL\right)L\cos\theta + \frac{5}{2}cL^2\sin 2\varphi\sin\theta$$

5) Poiché la sollecitazione è conservativa, le EL linearizzate (1) intorno alle configurazioni di equilibrio si scrivono

$$A\ddot{\vec{x}} + \mathcal{H}_{|\vec{q}_e}\vec{x} = \vec{0} \qquad \vec{x} = \frac{\vec{q}-\vec{q}_e}{\varepsilon}$$

dove

$$\vec{q}_e = (\varphi_e, \theta_e) = \left(0, \pm\frac{\pi}{2}\right)$$

$$A = ml^2 \begin{bmatrix} 7 & -\frac{5}{2}\sin\theta_e \\ -\frac{5}{2}\sin\theta_e & \frac{1}{3} \end{bmatrix}$$

$$\mathcal{H}_{|\vec{q}_e} = \begin{bmatrix} -15cl^2 & -5cl^2\sin\theta_e \\ -5cl^2\sin\theta_e & -\left(3cl-\frac{mg}{2}\right)l\sin\theta_e \end{bmatrix}$$

Dunque,

$$ml^2 \begin{bmatrix} 7 & -\frac{5}{2}\sin\theta_e \\ -\frac{5}{2}\sin\theta_e & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} - \begin{bmatrix} 15cl^2 & 5cl^2\sin\theta_e \\ 5cl^2\sin\theta_e & \left(3cl-\frac{mg}{2}\right)l\sin\theta_e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In $\vec{q}_e^{(1)} = \left(0, \frac{\pi}{2}\right)$, $\sin\theta_e = 1$, quindi

$$\begin{cases} ml^2\left(7\ddot{x}_1 - \frac{5}{2}\ddot{x}_2\right) - 15cl^2 x_1 - 5cl^2 x_2 = 0 \\ ml^2\left(-\frac{5}{2}\ddot{x}_1 + \frac{1}{3}\ddot{x}_2\right) - 5cl^2 x_1 - \left(3cl - \frac{mg}{2}\right)l\, x_2 = 0 \end{cases}$$

In $\vec{q}_e^{(1)} = \left(0, -\frac{\pi}{2}\right)$, si $\theta_e = -1$, quindi

$$\begin{cases} mL^2\left(7\ddot{x}_1 + \frac{5}{2}\ddot{x}_2\right) - 15cL^2 x_1 + 5cL^2 x_2 = 0 \\ mL^2\left(\frac{5}{2}\ddot{x}_1 + \frac{1}{3}\ddot{x}_2\right) + 5cL^2 x_1 + \left(3cL - \frac{mg}{2}\right)L x_2 = 0 \end{cases}$$

6) Sollecitazione reattiva dinamica in E e N

Analogamente alla Statica, la reazione dinamica è data da

$$(14.1) \quad \mathcal{L}^{(reaz)} = \left\{ \left(E, \vec{\phi}' = \phi'_y \vec{e}_y + \phi'_z \vec{e}_z\right), \left(N, \vec{\psi}' = \psi'_x \vec{e}_x + \psi'_z \vec{e}_z\right) \right\},$$

quindi abbiamo 4 incognite

$$(14.2) \quad \left(\phi'_y, \phi'_z, \psi'_x, \psi'_z\right)$$

Per determinarle, usiamo le ECD

$$\vec{R}^{(est,att)} + \vec{\phi}'_E + \psi'_N = m\,\vec{a}_G$$

$$\vec{M}_E^{(est,att)} + (N-E)\times\vec{\psi}'_N = \frac{d\vec{L}_E}{dt} + \vec{v}_E \times m\vec{v}_G$$

Calcoliamo $\vec{a}_G$.

$$\vec{a}_G = \vec{a}_E + \dot{\vec{\omega}}\times(G-E) + \vec{\omega}\times\left(\vec{\omega}\times(G-E)\right)$$

$$\vec{a}_E = \dot{\vec{v}}_E \overset{(8.2)}{=} 5L\left(-\sin\varphi\,\dot\varphi^2 + \cos\varphi\,\ddot\varphi\right)\vec{e}_x$$

$$\dot{\vec{\omega}} = \ddot\varphi\,\vec{e}_z + \ddot\theta\,\vec{e}_2 + \dot\theta\,\dot{\vec{e}}_2 = \ddot\varphi\,\vec{e}_3 + \ddot\theta\,\vec{e}_2 - \dot\varphi\dot\theta\,\vec{e}_1, \text{ poiché}$$

$$\dot{\vec{e}}_2 = \vec{\omega}\times\vec{e}_2 = \left(\dot\varphi\,\vec{e}_z + \dot\theta\,\vec{e}_2\right)\times\vec{e}_2 = \dot\varphi\,\vec{e}_3\times\vec{e}_2 = \dot\varphi\,\vec{e}_3\times\vec{e}_2 = -\dot\varphi\,\vec{e}_1$$

Quindi,

$$\dot{\vec{\omega}}\times(G-E) = \left(-\dot\varphi\dot\theta\,\vec{e}_1 + \ddot\theta\,\vec{e}_2 + \ddot\varphi\,\vec{e}_3\right)\times\frac{L}{2}\left(\vec{i} + 5\vec{j}\right) =$$

$$\overset{(1.2)}{=} \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -\dot\varphi\dot\theta & \ddot\theta & \ddot\varphi \\ \frac{1}{2}\cos\theta & \frac{5}{2} & -\frac{1}{2}\sin\theta \end{vmatrix} = \vec{e}_1\left(-\frac{1}{2}\sin\theta\,\ddot\theta - \frac{5}{2}\ddot\varphi\right) + -\vec{e}_2\left(\frac{1}{2}\sin\theta\,\dot\varphi\dot\theta - \frac{1}{2}\cos\theta\,\ddot\varphi\right) + \vec{e}_3\left(-\frac{5}{2}\dot\varphi\dot\theta - \frac{1}{2}\cos\theta\,\ddot\theta\right)$$

$$\vec{a}_G = \Big[\Big(-\tfrac{5}{2}L\sin\varphi - \tfrac{L}{2}\cos\varphi\cos\theta\Big)\dot{\varphi}^2 + \Big(\tfrac{5}{2}L\cos\varphi - \tfrac{L}{2}\sin\varphi\cos\theta\Big)\ddot{\varphi} +$$

$$+ L\sin\varphi\sin\theta\,\dot{\varphi}\,\dot{\theta} - \tfrac{L}{2}\cos\varphi\cos\theta\,\dot{\theta}^2 - \tfrac{L}{2}\cos\varphi\sin\theta\,\ddot{\theta}\Big]\vec{e}_x +$$

(15.1) $$+\Big[\Big(-\tfrac{L}{2}\sin\varphi\cos\theta - \tfrac{5}{2}L\cos\varphi\Big)\dot{\varphi}^2 - L\cos\varphi\sin\theta\,\dot{\varphi}\,\dot{\theta} + \Big(\tfrac{L}{2}\cos\varphi\cos\theta - \tfrac{5}{2}L\sin\varphi\Big)\ddot{\varphi} +$$

$$-\tfrac{L}{2}\sin\varphi\cos\theta\,\dot{\theta}^2 - \tfrac{L}{2}\sin\varphi\sin\theta\,\ddot{\theta}\Big]\vec{e}_y +$$

$$-\tfrac{L}{2}\Big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\Big)\vec{e}_z$$

$$\vec{R}^{(est,ext)} = \vec{F}_c - m\vec{g} = -c(C-H) - mg\vec{e}_z =$$

(15.2) $$= -c\Big[5L\sin\varphi\,\vec{e}_x - 3L\,\vec{e}_z + L\big(\cos\varphi\cos\theta\,\vec{e}_x + \sin\varphi\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z\big) + 4L\big(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y\big)\Big] - mg\,\vec{e}_z =$$

$$= -cL(\sin\varphi + \cos\varphi\cos\theta)\,\vec{e}_x - cL(\sin\varphi\cos\theta + 4\cos\varphi)\,\vec{e}_y +$$

$$+ \big[cL(3 + \sin\theta) - mg\big]\,\vec{e}_z$$

Proiettando la I ECD (14.3) sulla base $\mathcal{B}$, si trova

(15.3) $\vec{e}_x: \ \Psi'_x = cL(\sin\varphi + \cos\varphi\cos\theta) + m\,\vec{a}_G\cdot\vec{e}_x$ (vedi 15.1)

(15.4) $\vec{e}_y: \ \Phi'_y = cL(\sin\varphi\cos\theta + 4\cos\varphi) + m\,\vec{a}_G\cdot\vec{e}_y$ (vedi 15.1)

(15.5) $\vec{e}_z: \ \Phi'_z + \Psi'_z = -cL(3+\sin\theta) + mg + m\,\vec{a}_G\cdot\vec{e}_z$ (vedi 15.1)

Per determinare le 2 incognite (Φ'_z, Ψ'_z) utilizziamo la (15.5) e la II ECD (14.3), proiettata lungo il versore $\vec{e}_x$:

(15.6) $$\vec{M}_E^{(est,ext)}\cdot\vec{e}_x + (N-E)\times\vec{\Psi}'_N\cdot\vec{e}_x = \frac{d\vec{L}_E}{dt}\cdot\vec{e}_x + \cancel{\vec{v}_E\times m\vec{v}_G\cdot\vec{e}_x} \Leftarrow \vec{v}_E \parallel \vec{e}_x$$

Calcoliamo i termini della (13.6).

$$\vec{M}_E^{(at,est)} \cdot \vec{e}_x = \left((C-E)\times\left[-c(C-H)\right]\right)\cdot\vec{e}_x + (G-E)\times m\vec{g}\cdot\vec{e}_x$$

$$= -cL(\vec{i}+4\vec{j})\times\left[L\cancel{(\vec{i}+4\vec{j})} + \cancel{5L\sin\varphi\,\vec{e}_x} - 3L\,\vec{e}_z\right]\cdot\vec{e}_x +$$

$$+\frac{L}{2}(\vec{i}+5\vec{j})\times(-mg\,\vec{e}_z)\cdot\vec{e}_x$$

$$= -cL^2\left[\left(\cancel{\cos\varphi\cos\theta\,\vec{e}_x} + \sin\varphi\cos\theta\,\vec{e}_y - \cancel{\sin\theta\,\vec{e}_z} + 4(\cancel{-\sin\varphi\,\vec{e}_x} + \cos\varphi\,\vec{e}_y)\right)\times\right.$$

$$\left.\times(-3\,\vec{e}_z)\right]\cdot\vec{e}_x +$$

$$-mg\frac{L}{2}\left[\left(\cancel{\cos\varphi\cos\theta\,\vec{e}_x} + \sin\varphi\cos\theta\,\vec{e}_y - \cancel{\sin\theta\,\vec{e}_z} + 5(\cancel{-\sin\varphi\,\vec{e}_x} + \cos\varphi\,\vec{e}_y)\right)\times\right.$$

$$\left.\times\,\vec{e}_z\cdot\vec{e}_x\right].$$

$$= -cL^2\left[\sin\varphi\cos\theta\,\vec{e}_y\times(-3\,\vec{e}_z)\cdot\vec{e}_x + 4\cos\varphi\,\vec{e}_y\times(-3\,\vec{e}_z)\cdot\vec{e}_x\right] +$$

$$-mg\frac{L}{2}\left[\sin\varphi\cos\theta\,\vec{e}_y\times\vec{e}_z\cdot\vec{e}_x + 5\cos\varphi\,\vec{e}_y\times\vec{e}_z\cdot\vec{e}_x\right]$$

$$= -cL^2(-3\sin\varphi\cos\theta - 12\cos\varphi) - mg\frac{L}{2}(\sin\varphi\cos\theta + 5\cos\varphi)$$

$$(N-E)\times\vec{\Psi}' = 5L\,\vec{e}_2\times(\Psi'_x\,\vec{e}_x + \Psi'_z\,\vec{e}_z) = 5L(-\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)\times$$

$$\times(\Psi'_x\,\vec{e}_x + \Psi'_z\,\vec{e}_z) = 5L\left(+\sin\varphi\,\Psi'_z\,\vec{e}_y + \cos\varphi(-\Psi'_x\,\vec{e}_z + \Psi'_z\,\vec{e}_x)\right)$$

$$= 5L\left(\cos\varphi\,\Psi'_z\,\vec{e}_x + \sin\varphi\,\Psi'_z\,\vec{e}_y - \Psi'_x\cos\varphi\,\vec{e}_z\right)$$

$$(N-E)\times\vec{\Psi}'\cdot\vec{e}_x = 5L\cos\varphi\,\Psi'_z$$

Il lato destro della (15.6) si può calcolare come

$$\frac{d}{dt}\vec{L}_E\cdot\vec{e}_x = \frac{d}{dt}\left(\vec{L}_E\cdot\vec{e}_x\right),$$

dove

$$\vec{L}_E\cdot\vec{e}_x = I_E(\vec{\omega})\cdot\vec{e}_x + \cancel{(G-E)\times m\vec{v}_E\cdot\vec{e}_x} \qquad \leftarrow \vec{v}_E \parallel \vec{e}_x$$

$$= mL^2\left[-\left(\frac{7}{3}\sin\theta\,\dot\varphi+\frac{5}{4}\dot\theta\right)\vec{i}\cdot\vec{e}_x+\left(\frac{5}{4}\sin\theta\,\dot\varphi+\frac{1}{3}\dot\theta\right)\vec{j}\cdot\vec{e}_x+\frac{22}{3}\cos\theta\,\dot\varphi\,\vec{k}\cdot\vec{e}_x\right]$$

$$= mL^2\left[-\left(\frac{7}{3}\sin\theta\,\dot\varphi+\frac{5}{4}\dot\theta\right)(\cos\varphi\cos\theta)+\left(\frac{5}{4}\sin\theta\,\dot\varphi+\frac{1}{3}\dot\theta\right)(-\sin\varphi)+\frac{22}{3}\cos\theta\,\dot\varphi\cos\varphi\sin\theta\right]$$

$$= mL^2\left[\left(-\frac{5}{4}\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\dot\varphi-\left(\frac{1}{3}\sin\varphi+\frac{5}{4}\cos\varphi\cos\theta\right)\dot\theta\right]$$

Quindi,

$$\frac{d}{dt}\left(\vec{L}_E\cdot\vec{e}_x\right)=mL^2\left[\left(-\frac{5}{4}\cos\varphi\sin\theta\,\dot\varphi-\cancel{\frac{5}{4}\sin\varphi\cos\theta\,\dot\theta}-\frac{1}{6}\sin\varphi\sin 2\theta\,\dot\varphi+\frac{1}{3}\cos\varphi\cos 2\theta\,\dot\theta\right)\dot\varphi+\right.$$

$$+\left(-\frac{5}{4}\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi+$$

$$-\left(\frac{1}{3}\cos\varphi\,\dot\varphi-\cancel{\frac{5}{4}\sin\varphi\cos\theta\,\dot\varphi}-\frac{5}{4}\cos\varphi\sin\theta\,\dot\theta\right)\dot\theta$$

$$\left.-\left(\frac{1}{3}\sin\varphi+\frac{5}{4}\cos\varphi\cos\theta\right)\ddot\theta\right]$$

$$= mL^2\left[\left(-\frac{5}{4}\cos\varphi\sin\theta-\frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2\right.$$

$$+\frac{1}{3}\cos\varphi(\cos 2\theta-1)\,\dot\theta\dot\varphi$$

$$+\left(-\frac{5}{4}\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi$$

$$\left.-\left(\frac{1}{3}\sin\varphi+\frac{5}{4}\cos\varphi\cos\theta\right)\ddot\theta+\frac{5}{4}\cos\varphi\sin\theta\,\dot\theta^2\right]$$

Dunque, la (15.6) si scrive

$$+cL^2\left(3\sin\varphi\cos\theta+12\cos\varphi\right)-mg\frac{L}{2}\left(\sin\varphi\cos\theta+5\cos\varphi\right)+5L\cos\varphi\,\Psi'_z=$$

(18.1)
$$=mL^2\Big[-\left(\frac{5}{4}\cos\varphi\sin\theta+\frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}\cos\varphi\sin^2\theta\,\dot\theta\,\dot\varphi+$$
$$+\left(-\frac{5}{4}\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi-\left(\frac{1}{3}\sin\varphi+\frac{5}{4}\cos\varphi\cos\theta\right)\ddot\theta+$$
$$+\frac{5}{4}\cos\varphi\sin\theta\,\dot\theta^2\Big]$$

se $\cos\varphi\neq 0 \iff \varphi\neq\pm\frac{\pi}{2}$ (Se $\cos\varphi=0$, si può usare la II ECD lungo $\vec{e}_4$).

(18.2)
$$\Psi'_z=\frac{1}{5\cancel{L}\cos\varphi}\left[-cL^2\left(3\sin\varphi\cos\theta+12\cos\varphi\right)+mg\frac{\cancel{L}}{2}\left(\sin\varphi\cos\theta+5\cos\varphi\right)\right]+$$
$$+\frac{mL^{\cancel{2}}}{5\cancel{L}\cos\varphi}\Big[-\left(\frac{5}{4}\cos\varphi\sin\theta+\frac{1}{6}\sin\varphi\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}\cos\varphi\sin^2\theta\,\dot\theta\,\dot\varphi+$$
$$+\left(-\frac{5}{4}\sin\varphi\sin\theta+\frac{1}{6}\cos\varphi\sin 2\theta\right)\ddot\varphi-\left(\frac{1}{3}\sin\varphi+\frac{5}{4}\cos\varphi\cos\theta\right)\ddot\theta+$$
$$+\frac{5}{4}\cos\varphi\sin\theta\,\dot\theta^2\Big]$$

Sostituendo la (18.2) nella (15.5) si trova

(18.3)
$$\Phi'_z=-cL\left(3+\sin\theta\right)+mg+m\frac{L}{2}\left(\cos\theta\,\ddot\theta-\sin\theta\,\dot\theta^2\right)+$$
$$-\frac{1}{5}\Big[-cL\left(3\,\mathrm{tg}\,\varphi\cos\theta+12\right)+\frac{mg}{2}\left(\mathrm{tg}\,\varphi\cos\theta+5\right)+$$
$$-mL\left(\frac{5}{4}\sin\theta+\frac{1}{6}\,\mathrm{tg}\,\varphi\sin 2\theta\right)\dot\varphi^2-\frac{2}{3}mL\sin^2\theta\,\dot\theta\,\dot\varphi$$
$$mL\left(-\frac{5}{4}\,\mathrm{tg}\,\varphi\sin\theta+\frac{1}{6}\sin 2\theta\right)\ddot\varphi-mL\left(\frac{1}{3}\,\mathrm{tg}\,\varphi+\frac{5}{4}\cos\theta\right)\ddot\theta+$$
$$+mL\sin\theta\,\dot\theta^2\Big]$$

N.B. Si può verificare che $\vec{\Phi}'_E\big|_{\vec{q}_e^{(1)}}=\vec{\Phi}_E\big|_{\vec{q}_e^{(1)}}$, $\vec{\Psi}'_N\big|_{\vec{q}_e^{(1)}}=\vec{\Psi}_N\big|_{\vec{q}_e^{(1)}}$. Analogamente per $\vec{q}_e^{(2)}$.