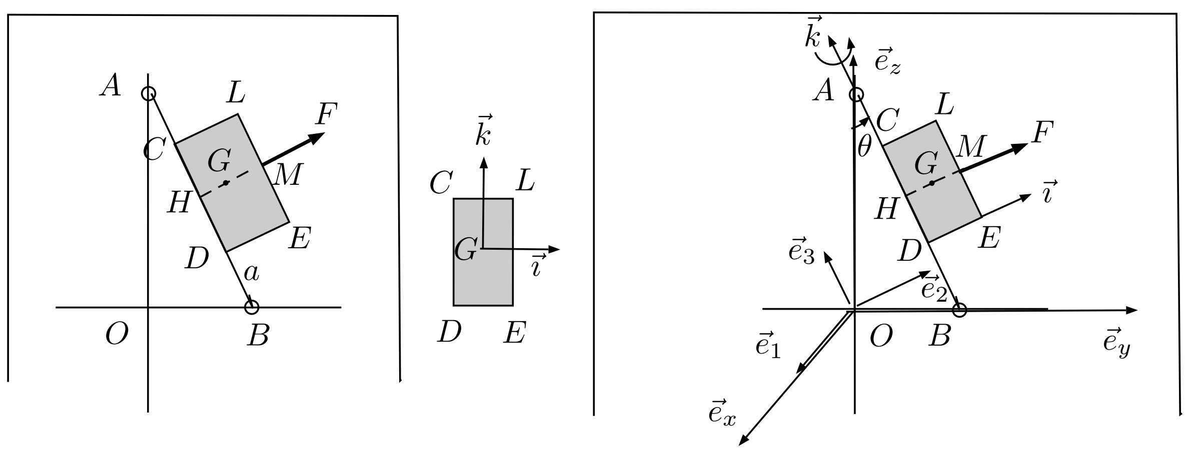
# Compito di Meccanica Razionale (9 CFU)

Trieste, 25 gennaio 2019. (G. Tondo)

Un rigido è costituito da una lamina rettangolare omogenea di massa $m$ e lati $a$ e $4a$, saldata, lungo il lato $CD$, ad un'asta $AB$ di lunghezza $6a$ e di massa trascurabile, in modo che il punto medio $H$ del lato $CD$ coincida con il punto medio di $AB$. Gli estremi dell'asta sono vincolati, come in figura, a scorrere senza attrito lungo due guide fisse ortogonali, tramite due cerniere sferiche "bucate". Sul rigido agisce il peso proprio opposto ad $\vec{e}_z$ e una forza $F$ diretta come il lato $DE$ e applicata nel punto medio $M$ del lato $EL$.




Oltre alla terna fissa $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, si suggerisce di usare anche una terna intermedia formata dai versori $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, con $\vec{e}_1 = \vec{e}_x$, $\vec{e}_3 = vers(A - D)$, $\vec{e}_2 = \vec{e}_3 \times \vec{e}_1$. Inoltre, si consiglia di prendere una terna solidale alla lamina $(\vec{\imath}, \vec{\jmath}, \vec{k})$, con il versore $\vec{\imath} = vers(E - D)$, $\vec{k} = \vec{e}_3$ e $\vec{\jmath} = \vec{k} \times \vec{\imath}$.

Sia $-\pi < \theta \leq \pi$ l'angolo compreso tra $\vec{e}_z$ e $\vec{k}$, e sia $-\pi < \psi \leq \pi$ quello compreso tra $\vec{e}_1$ e $\vec{\imath}$.

**STATICA**

Determinare:

**1)** il valore del parametro $\lambda = \frac{F}{mg}$ necessario all'equilibrio nella configurazione $\theta_e = -\frac{\pi}{6}, \psi_e = \frac{\pi}{2}$;

**2)** le reazioni vincolari esterne sul rigido in $A$, in tale configurazione di equilibrio;

**3)** le reazioni vincolari esterne sul rigido in $B$, tale configurazione di equilibrio.

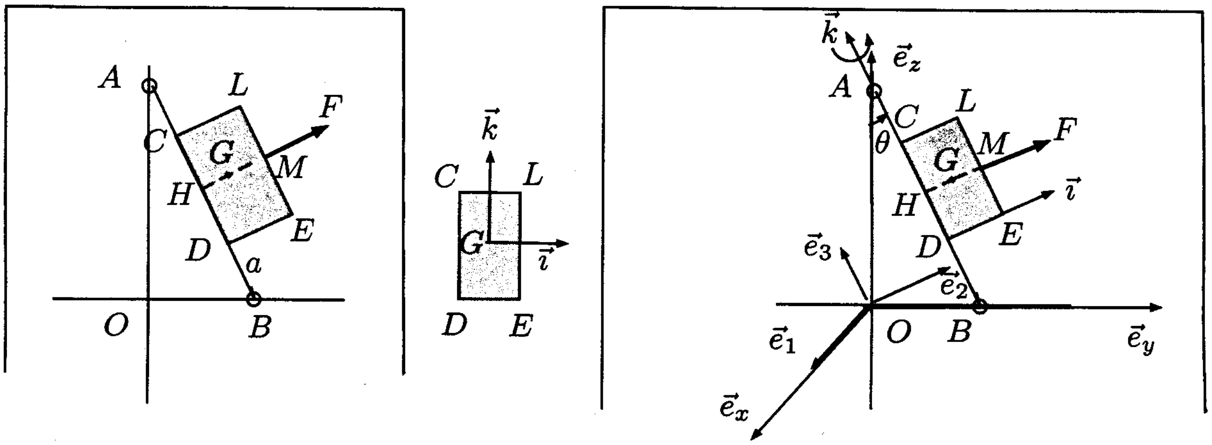
**DINAMICA**

**4)** Scrivere le equazione differenziali pure di moto;

**5)** linearizzare le equazioni di moto intorno alla configurazione di equilibrio suddetta;

**6)** calcolare le reazioni vincolare esterne nei punti $A$ e $B$, durante il moto.

Tema del 25/01/2019




Il modello è formato da un solo rigido vincolato con 2 cerniere sferiche lisce. Con il metodo dei congelamenti successivi, possiamo osservare che se blocchiamo lo scorrimento del punto B del rigido sull'asse $(O, \vec{e}_y)$, il rigido può ancora ruotare attorno all'asse $(B, \vec{k})$. Dunque, il rigido ha 2 g.l. e, come coordinate libere possiamo prendere gli angoli

$$-\pi < \theta \le \pi \quad , \quad -\pi < \psi \le \pi$$

Quindi, ogni configurazione del rigido è individuata da

$$\vec{q} = (\theta, \psi)$$

Consideriamo le 3 basi

$\mathcal{B} = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$ : "fissa"

$\mathcal{B}' = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ : "intermedia"

$\mathcal{B}'' = (\vec{\imath}, \vec{\jmath}, \vec{k})$ : solidale al rigido

Le equazioni di trasformazione sono

$$(2.1)\quad \begin{cases} \vec e_1 = \vec e_x \\ \vec e_2 = \cos\theta\, \vec e_y + \sin\theta\, \vec e_z \\ \vec e_3 = -\sin\theta\, \vec e_y + \cos\theta\, \vec e_z \end{cases}$$

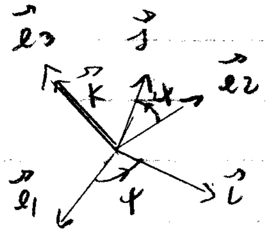
$$[\vec e_1, \vec e_2, \vec e_3] = [\vec e_x, \vec e_y, \vec e_z] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}}_{R_\theta}$$

Quindi, $\quad R_\theta R_\theta^T = \mathbb{1}$

$$[\vec e_x, \vec e_y, \vec e_z] = [\vec e_1, \vec e_2, \vec e_3]\, R_\theta^T = [\vec e_1, \vec e_2, \vec e_3] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{R_\theta^T}$$

cioè:

$$(2.2)\quad \begin{cases} \vec e_x = \vec e_1 \\ \vec e_y = \cos\theta\, \vec e_2 - \sin\theta\, \vec e_3 \\ \vec e_z = \sin\theta\, \vec e_2 + \cos\theta\, \vec e_3 \end{cases}$$




Analogamente,

$$(2.3)\quad \begin{cases} \vec\imath = \cos\psi\, \vec e_1 + \sin\psi\, \vec e_2 \\ \vec\jmath = -\sin\psi\, \vec e_1 + \cos\psi\, \vec e_2 \\ \vec k = \vec e_3 \end{cases}$$

$$[\vec\imath, \vec\jmath, \vec k] = [\vec e_1, \vec e_2, \vec e_3] \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\psi}$$

Quindi

$$(2.4)\quad [\vec e_1, \vec e_2, \vec e_3] = [\vec\imath, \vec\jmath, \vec k]\, R_\psi^T = [\vec\imath, \vec\jmath, \vec k] \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_\psi^T}$$

cioè

$$(2.5)\quad \begin{cases} \vec e_1 = \cos\psi\, \vec\imath - \sin\psi\, \vec\jmath \\ \vec e_2 = \sin\psi\, \vec\imath + \cos\psi\, \vec\jmath \\ \vec e_3 = \vec k \end{cases}$$

Componendo la trasformazioni (2.2) e (2.4) o, equivalentemente, moltiplicando le matrici $R_\theta^T$ e $R_\psi^T$, si trova

$$(3.1)\qquad [\vec e_x, \vec e_y, \vec e_z] = [\vec e_1, \vec e_2, \vec e_3]\, R_\theta^T \overset{(2.4)}{=} [\vec i, \vec j, \vec k]\, R_\psi^T R_\theta^T$$

cioè

$$(3.2)\quad \begin{cases} \vec e_x = \cos\psi\, \vec i - \sin\psi\, \vec j \\ \vec e_y = \cos\theta\,(\sin\psi\, \vec i + \cos\psi\, \vec j) - \sin\theta\, \vec k = \cos\theta \sin\psi\, \vec i + \cos\theta\cos\psi\, \vec j - \sin\theta\, \vec k \\ \vec e_z = \sin\theta\,(\sin\psi\, \vec i + \cos\psi\, \vec j) + \cos\theta\, \vec k = \sin\theta\sin\psi\, \vec i + \sin\theta\cos\psi\, \vec j + \cos\theta\, \vec k \end{cases}$$

$$(3.3)\quad \begin{cases} \vec i = \cos\psi\, \vec e_x + \cos\theta\sin\psi\, \vec e_y + \sin\theta\sin\psi\, \vec e_z \\ \vec j = -\sin\psi\, \vec e_x + \cos\theta\cos\psi\, \vec e_y + \sin\theta\cos\psi\, \vec e_z \\ \vec k = -\sin\theta\, \vec e_y + \cos\theta\, \vec e_z \end{cases}$$

$$(3.3')\quad G-H = \frac{a}{2}\vec i = \frac{a}{2}(\cos\psi\, \vec e_1 + \sin\psi\, \vec e_2) \overset{(2.1)}{=} \frac{a}{2}\left[\cos\psi\, \vec e_x + \sin\psi\,(\cos\theta\, \vec e_y + \sin\theta\, \vec e_z)\right]$$

$$(3.4)\quad H-B = 3a\, \vec k = 3a\, \vec e_3 = 3a\,(-\sin\theta\, \vec e_y + \cos\theta\, \vec e_z)$$

$$(3.5)\quad B-O = 6a\sin\theta\, \vec e_y$$

Quindi,

$$(3.6)\quad G-O = (G-H) + (H-B) + (B-O) = \frac{a}{2}\vec i + 3a\, \vec e_3 + 6a\sin\theta\, \vec e_y$$

$$= \frac{a}{2}\cos\psi\, \vec e_x + \frac{a}{2}\cos\theta\sin\psi\, \vec e_y + \frac{a}{2}\sin\theta\sin\psi\, \vec e_z +$$

$$+ 3a\,(-\sin\theta\, \vec e_y + \cos\theta\, \vec e_z) + 6a\sin\theta\, \vec e_y$$

$$= \frac{a}{2}\cos\psi\, \vec e_x + \left(\frac{a}{2}\cos\theta\sin\psi + 3a\sin\theta\right)\vec e_y + \left(\frac{a}{2}\sin\theta\sin\psi + 3a\cos\theta\right)\vec e_z$$

Statica

La sollecitazione attiva è data dal peso (conservativa) e da un carico follower. Allora, calcoliamo le forze generalizzate $(Q_\theta, Q_\psi)$. Sappiamo che, per la parte conservativa,

$$Q_\theta^{(peso)} = -\frac{\partial V^{(peso)}}{\partial\theta} \quad , \quad Q_\psi^{(peso)} = -\frac{\partial V^{(peso)}}{\partial\psi}$$

Allora, calcoliamo

$$V^{(peso)} = -m\vec{g}\cdot(G-O) = mg\vec{e}_z\cdot(G-O) = mga\left(\frac{1}{2}\sin\theta\sin\psi + 3\cos\theta\right).$$

Quindi

$$Q_\theta^{(peso)} = -mga\left(\frac{1}{2}\cos\theta\sin\psi - 3\sin\theta\right)$$

$$Q_\psi^{(peso)} = -mga\left(\frac{1}{2}\sin\theta\cos\psi\right)$$

Ora calcoliamo il contributo alle forze generalizzate dato dal carico follower

$$Q_\theta^{(foll)} \; , \quad Q_\psi^{(foll)}$$

A tale scopo, scriviamo il LV del carico follower nel rigido

$$LV^{(foll)} = \vec{F}\cdot\delta\vec{x}_H + \cancel{\vec{M}_H^{(foll)}}\cdot\vec{\epsilon}$$

$$\vec{x}_H = H-O = 3a\vec{k} + 6a\sin\theta\,\vec{e}_y \overset{(3.4)}{=} 3a\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right) + 6a\sin\theta\,\vec{e}_y$$
$$= 3a\left(\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right)$$

$$\delta\vec{x}_H = 3a\left(\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z\right)\delta\theta$$

Dunque,

$$LV^{(Roll)} = \vec{F} \cdot \delta \vec{x}_M = F\vec{\iota} \cdot 3a(\cos\theta\, \vec{e}_y - \sin\theta\, \vec{e}_z)\delta\theta =$$

(5.1)
$$= 3aF(\cos\psi\, \vec{e}_x + \cos\theta \sin\psi\, \vec{e}_y + \sin\theta\sin\psi\, \vec{e}_z)\cdot(\cos\theta\, \vec{e}_y - \sin\theta\, \vec{e}_z)\delta\theta =$$
$$= 3Fa(\cos^2\theta \sin\psi - \sin^2\theta\sin\psi)\delta\theta$$
$$= 3Fa\sin\psi(\cos^2\theta - \sin^2\theta)\delta\theta = 3Fa\sin\psi\cos 2\theta\,\delta\theta$$

Allora,

(5.2) $$Q_\theta^{(Roll)} = 3Fa\sin\psi\cos 2\theta, \qquad Q_\psi^{(Roll)} = 0$$

Quindi, le forze generalizzate sono

(5.3)
$$Q_\theta = Q_\theta^{(peso)} + Q_\theta^{(Roll)} = -mga\left(\frac{1}{2}\cos\theta\sin\psi - 3\sin\theta\right) + 3Fa\sin\psi\cos 2\theta$$
$$Q_\psi = Q_\psi^{(peso)} + \cancel{Q_\psi^{(Roll)}} = -\frac{mga}{2}\sin\theta\cos\psi$$

La configurazione assegnata $\vec{q}_e = \left(\theta_e = -\frac{\pi}{6}, \psi_e = \frac{\pi}{2}\right)$ è di equilibrio se e solo se

(5.4) $$0 = Q_\theta|_{\vec{q}_e} = -mga\left(\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{3}{2}\right) + \frac{3}{2}Fa$$

(5.5) $$0 = Q_\psi|_{\vec{q}_e} \qquad O.K.$$

Allora, la (5.4) è soddisfatta se e solo se

(5.6) $$\frac{3}{\cancel{2}}F = \frac{mg}{\cancel{2}}\left(\frac{\sqrt{3}}{2} + 3\right) \Leftrightarrow \lambda = \frac{F}{mg} = \frac{1}{3}\left(3 + \frac{\sqrt{3}}{2}\right) = 1 + \frac{1}{2\sqrt{3}} > 0$$

2) + 3) Reazioni all'equilibrio in $A$ e $B$

Le cerniere sferiche sono, per ipotesi, non dissipative e bilatere. Quindi, eserciteranno una sollecitazione reattiva data da

$$(7.1) \qquad \mathcal{L}^{(reatt)} = \{(A,\vec{\Psi}),(B,\vec{\Phi})\} \quad \text{con } \vec{\Psi}_A\cdot\vec{e}_z=0, \ \vec{\phi}_B\cdot\vec{e}_y=0$$

Per calcolare $\vec{\Psi}_A$ e $\vec{\Phi}_B$, scriviamo le ECS su tutto il rigido

$$(7.2) \qquad \begin{cases} \vec{R}^{(est,att)} + \vec{\Psi}_A + \vec{\phi}_B = 0 \\ \vec{M}_B^{(es,att)} + (A-B)\times\vec{\Psi}_A = 0 \end{cases}$$

Calcoliamo il risultante delle forze esterne attive:

$$\begin{aligned} \vec{R}^{(est,att)} &= -mg\vec{e}_z + \vec{F} = -mg\vec{e}_z + F\vec{c} = \\ &= -mg\vec{e}_z + F(\cos\varphi\,\vec{e}_x + \cos\theta\sin\varphi\,\vec{e}_y + \sin\theta\sin\varphi\,\vec{e}_z) \\ &= F\cos\varphi\,\vec{e}_x + F\cos\theta\sin\varphi\,\vec{e}_y + (F\sin\theta\sin\varphi - mg)\,\vec{e}_z \end{aligned}$$

Quindi, proiettando la I ECS (7.2) sulla base fissa $\mathcal{B}$

$$(7.3) \qquad \begin{cases} (\vec{\Psi}_A+\vec{\phi}_B)\cdot\vec{e}_x = -F\cos\varphi \\ \vec{\Psi}_A\cdot\vec{e}_y = -F\cos\theta\sin\varphi \\ \vec{\phi}_B\cdot\vec{e}_z = -(F\sin\theta\sin\varphi - mg) \end{cases}$$

Dunque, nella configurazione di equilibrio assegnata $\vec{q}_e=\left(-\frac{\pi}{6},\frac{\pi}{2}\right)$ si trova

$$(7.4) \qquad \begin{cases} (\vec{\Psi}_A+\vec{\Phi}_B)\cdot\vec{e}_x = 0 \\ \vec{\Psi}_A\cdot\vec{e}_y = -F\frac{\sqrt{3}}{2} = -mg\lambda\frac{\sqrt{3}}{2} = -mg\frac{\sqrt{3}}{2}\left(1+\frac{1}{2\sqrt{3}}\right) = -\frac{mg}{2}\left(\sqrt{3}+\frac{1}{2}\right) \\ \vec{\Phi}_B\cdot\vec{e}_z = \frac{F}{2}+mg = \frac{mg\lambda}{2}+mg = mg\left(\frac{\lambda}{2}+1\right) = \frac{mg}{2}\left(3+\frac{1}{2\sqrt{3}}\right) \end{cases}$$

Restano da determinare $\vec{\varphi}_A \cdot \vec{e}_x$ e $\vec{\phi}_B \cdot \vec{e}_x$. A tale scopo, osserviamo che, nella configurazione di equilibrio $\vec{q}_e$, il rigido giace sul piano $\Pi_x = (O, \vec{e}_y, \vec{e}_z)$ insieme con tutte le forze attive a cui è soggetto. Dunque,

$$\vec{R}^{(est,att)} \in \Pi_x \,, \quad \vec{M}_P^{(est,att)} \perp \Pi_x \quad \forall P \in \Pi_x$$

Dalle ECS, segue che

$$R^{(est,reatt)} \in \Pi_x \,, \quad \vec{M}_P^{(est,reatt)} \perp \Pi_x \quad \forall P \in \Pi_x$$

La I equivale alla I delle (7.4), la II implica che

$$\vec{M}_B^{(est,reatt)} = (A-B) \times \vec{\varphi}_A \perp \Pi_x$$

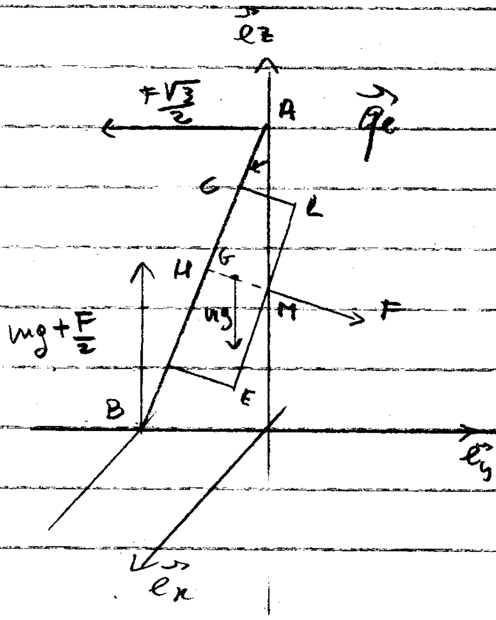
Dunque, $\vec{\varphi}_A \in \Pi_x$, quindi

$$\vec{\varphi}_A \cdot \vec{e}_x = 0 \quad et \quad \phi_B \cdot \vec{e}_x \overset{(7.4)}{=} 0$$

In conclusione,

$$\vec{\varphi}_A = -\frac{mg}{2}\left(\sqrt{3} + \frac{1}{2}\right)\vec{e}_y$$

$$\vec{\phi}_B = \frac{mg}{2}\left(3 + \frac{1}{2\sqrt{3}}\right)\vec{e}_z$$



4) Scriviamo le 2 EL relative a $\theta$ e $\varphi$. A tale scopo calcoliamo l'energia cinetica del rigido.

$$(10.1)\quad K = \frac{1}{2} m|\vec{V}_H|^2 + \frac{1}{2}\vec{\omega}\cdot I_H(\vec{\omega}) + m\vec{V}_H\cdot\vec{\omega}\times(G-H)$$

Dalle (3.4) e (3.5) segue che

$$(10.2)\quad H-O=(H-B)+(B-O) = 3a\vec{k} + 6a\sin\theta\,\vec{e}_y = 3a(\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z)$$

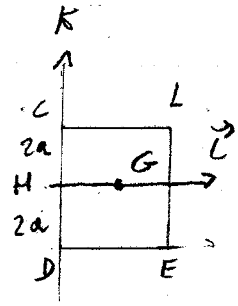
Quindi,

$$(10.3)\quad \vec{v}_H = \frac{d}{dt}(H-O) = 3a(\cos\theta\,\vec{e}_y - \sin\theta\,\vec{e}_z)\dot{\theta}$$

$$(10.4)\quad |\vec{v}_H|^2 = 9a^2(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 = 9a^2\dot{\theta}^2$$

Dal Teo. di Frisi segue che

$$(10.5)\quad \vec{\omega} = \dot{\theta}\,\vec{e}_1 + \dot{\varphi}\,\vec{K} \overset{(3.5)}{=} \dot{\theta}(\cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}) + \dot{\varphi}\,\vec{K}$$




La terna $(H, \vec{i}, \vec{j}, \vec{K})$ è una TPI(H) (perché?) quindi

$$(10.6)\quad I_H^{(B'')} = ma^2\begin{bmatrix} \frac{\cancel{16}^{4}}{\cancel{12}_{3}} & & \\ & \frac{5}{3} & \\ & & \frac{1}{3}\end{bmatrix} = \frac{ma^2}{3}\begin{bmatrix} 4 & & \\ & 5 & \\ & & 1\end{bmatrix}$$

Dunque

$$(10.7)\quad \frac{1}{2}\vec{\omega}\cdot I_H(\vec{\omega}) = \frac{1}{2}\vec{\omega}\cdot\frac{ma^2}{3}\begin{bmatrix} 4 & & \\ & 5 & \\ & & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}\cos\varphi \\ -\dot{\theta}\sin\varphi \\ \dot{\varphi}\end{bmatrix} = \frac{ma^2}{6}[\dot{\theta}\cos\varphi, -\dot{\theta}\sin\varphi, \dot{\varphi}]\begin{bmatrix}4\dot{\theta}\cos\varphi \\ -5\dot{\theta}\sin\varphi \\ \dot{\varphi}\end{bmatrix} =$$

$$= \frac{ma^2}{6}\left(4\dot{\theta}^2\cos^2\varphi + 5\dot{\theta}^2\sin^2\varphi + \dot{\varphi}^2\right) =$$

$$= \frac{ma^2}{6}\left[(4+\sin^2\varphi)\dot{\theta}^2 + \dot{\varphi}^2\right]$$

$$\vec{\omega}\times(G-H)=\left(\dot{\theta}\cos\psi\,\vec{i}-\dot{\theta}\sin\psi\,\vec{j}+\dot{\psi}\,\vec{k}\right)\times\frac{a}{2}\vec{i}=$$
$$=\frac{a}{2}\left(\dot{\theta}\sin\psi\,\vec{k}+\dot{\psi}\,\vec{j}\right)$$

Inoltre, riscrivendo $\vec{v}_H$ nella base solidale $\mathcal{B}''$, si ottiene

$$\vec{v}_H=3a\dot{\theta}\left(\cos\theta\,\vec{e}_\psi-\sin\theta\,\vec{e}_r\right)=3a\dot{\theta}\left[\cos\theta\left(\sin\psi\,\vec{i}+\cos\psi\,\vec{j}\right)-\sin\theta\,\vec{k}\right]\cos\theta+$$
$$-3a\dot{\theta}\left[\sin\theta\left(\sin\psi\,\vec{i}+\cos\psi\,\vec{j}\right)+\cos\theta\,\vec{k}\right]\sin\theta$$
$$=3a\dot{\theta}\left[\left(\cos^2\theta-\sin^2\theta\right)\left(\sin\psi\,\vec{i}+\cos\psi\,\vec{j}\right)-2\sin\theta\cos\theta\,\vec{k}\right]=$$
$$=3a\dot{\theta}\left[\cos 2\theta\left(\sin\psi\,\vec{i}+\cos\psi\,\vec{j}\right)-\sin 2\theta\,\vec{k}\right]$$

Quindi, il termine misto delle (10.1) vale

$$\vec{v}_H\cdot\vec{\omega}\times(G-H)=\frac{3}{2}a^2\dot{\theta}\left[-\dot{\theta}\sin 2\theta\sin\psi+\cos 2\theta\cos\psi\,\dot{\psi}\right]$$

Dunque,

$$K=\frac{1}{2}m\,9a^2\dot{\theta}^2+\frac{1}{6}ma^2\left[\left(4+\sin^2\psi\right)\dot{\theta}^2+\dot{\psi}^2\right]+\frac{3}{2}a^2\dot{\theta}\left[-\sin 2\theta\sin\psi\,\dot{\theta}+\cos 2\theta\cos\psi\,\dot{\psi}\right]$$
$$=ma^2\left[\left(\frac{31}{6}+\frac{1}{6}\sin^2\psi-\frac{3}{2}\sin 2\theta\sin\psi\right)\dot{\theta}^2+\frac{1}{6}\dot{\psi}^2+\frac{3}{2}\cos 2\theta\cos\psi\,\dot{\theta}\dot{\psi}\right]$$

$$=\frac{1}{2}ma^2\left[\dot{\theta},\dot{\psi}\right]\begin{bmatrix}\frac{31}{3}+\frac{1}{3}\sin^2\psi-3\sin 2\theta\sin\psi & \frac{3}{2}\cos 2\theta\cos\psi\\ \frac{3}{2}\cos 2\theta\cos\psi & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{\psi}\end{bmatrix}$$

5) Linearizzazione nelle configurazioni di eq. $\vec{q}_e = \left(\theta_e = -\frac{\pi}{6}, \psi_e = \frac{\pi}{2}\right)$

Poiché la sollecitazione non è conservativa, dobbiamo usare la formula

$$(13.1) \quad A(\vec{q}_e)\ddot{\vec{x}} + B\dot{\vec{x}} + C\vec{x} = 0 \qquad \vec{x} = \frac{\vec{q}(t) - \vec{q}_e}{\varepsilon}$$

dove $B_{ij} = -\frac{\partial Q_i}{\partial \dot{q}_j}\Big|_{\vec{q}_e} = 0$, $C_{ij} = -\frac{\partial Q_i}{\partial q_j}\Big|_{\vec{q}_e}$. Quindi

$$(13.2) \quad ma^2\begin{bmatrix} \frac{31}{3} + \frac{1}{3}\sin^2\psi_e - 3\sin 2\theta_e \sin\psi_e & \frac{3}{2}\cos 2\theta_e \cos\psi_e \\ \frac{3}{2}\cos 2\theta_e \cos\psi_e & \frac{1}{3} \end{bmatrix} = A(\vec{q}_e) = \begin{bmatrix} \frac{32}{3} + \frac{3\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} ma^2$$

$$C = -\begin{bmatrix} \frac{\partial Q_\theta}{\partial \theta} & \frac{\partial Q_\theta}{\partial \psi} \\ \frac{\partial Q_\psi}{\partial \theta} & \frac{\partial Q_\psi}{\partial \psi} \end{bmatrix}_{\vec{q}_e} = -\begin{bmatrix} +mga\left(+\frac{1}{2}\sin\theta_e \sin\psi_e + 3\cos\theta_e\right) + \\ -6Fa\sin\psi_e \sin 2\theta_e & -\frac{mga}{2}\cos\theta_e \cos\psi_e + 3Fa\cos\psi_e \cos 2\theta_e \\ -\frac{mga}{2}\cos\theta_e \cos\psi_e & \frac{mga}{2}\sin\theta_e \sin\psi_e \end{bmatrix}$$

$$= -\begin{bmatrix} +\frac{mga}{2}\left(-\frac{1}{2} + 3\sqrt{3}\right) + 3\sqrt{3}Fa & 0 \\ 0 & -\frac{mga}{4} \end{bmatrix} = mga\begin{bmatrix} \frac{1}{2}\left(\frac{1}{2} - 3\sqrt{3}\right) - 3\sqrt{3}\lambda & 0 \\ 0 & +\frac{1}{4} \end{bmatrix}$$

Dunque, tenendo conto che $\lambda = 1 + \frac{1}{2\sqrt{3}}$, si trova

$$ma^2\begin{bmatrix} \frac{32}{3} + \frac{3}{2}\sqrt{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + mga\begin{bmatrix} -\left(\frac{5}{4} + \frac{9}{2}\sqrt{3}\right) & 0 \\ 0 & +\frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Scriviamo le EL.

$$\frac{\partial K}{\partial \dot\theta} = 2ma^2\left(\left(\frac{31}{6}+\frac{1}{6}\sin^2\psi-\frac{3}{2}\sin 2\theta\sin\psi\right)\dot\theta+\frac{3}{2}\cos 2\theta\cos\psi\,\dot\psi\right)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot\theta}\right) = 2ma^2\left(\frac{1}{3}\sin\psi\cos\psi\,\dot\psi-3\cos 2\theta\sin\psi\,\dot\theta-\frac{3}{2}\sin 2\theta\cos\psi\,\dot\psi\right)\dot\theta +$$

$$+2ma^2\left(\frac{31}{6}+\frac{1}{6}\sin^2\psi-\frac{3}{2}\sin 2\theta\sin\psi\right)\ddot\theta +$$

$$+\left(-3\sin 2\theta\,\dot\theta\right)\cos\psi\,\dot\psi-\frac{3}{2}\cos 2\theta\sin\psi\,\dot\psi^2+\frac{3}{2}\cos 2\theta\cos\psi\,\ddot\psi$$

$$\frac{\partial K}{\partial\theta} = ma^2\left(-3\cos 2\theta\sin\psi\,\dot\theta^2-3\sin 2\theta\cos\psi\,\dot\theta\dot\psi\right)$$

$$EL_\theta:\ 2ma^2\left[\left(\frac{1}{3}\sin\psi-\frac{3}{2}\sin 2\theta\right)\cos\psi\,\dot\theta\dot\psi-\frac{3}{2}\cos 2\theta\sin\psi\,\dot\theta^2+\right.$$

$$\left.+\left(\frac{31}{6}+\frac{1}{6}\sin^2\psi-\frac{3}{2}\sin 2\theta\sin\psi\right)\ddot\theta-\frac{3}{4}\cos 2\theta\sin\psi\,\dot\psi^2+\frac{3}{4}\cos 2\theta\cos\psi\,\ddot\psi\right]=$$

$$=-mga\left(\frac{1}{2}\cos\theta\sin\psi-3\sin\theta\right)+3Fa\sin\psi\cos 2\theta$$

$$\frac{\partial K}{\partial\dot\psi} = ma^2\left[\frac{1}{3}\dot\psi+\frac{3}{2}\cos 2\theta\cos\psi\,\dot\theta\right]$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot\psi}\right) = ma^2\left[\frac{1}{3}\ddot\psi+\left(-3\sin 2\theta\cos\psi\,\dot\theta^2\right)+\frac{3}{2}\cos 2\theta\left(-\sin\psi\,\dot\psi\dot\theta+\cos\psi\,\ddot\theta\right)\right]$$

$$\frac{\partial K}{\partial\psi} = ma^2\left[\left(\frac{1}{3}\sin\psi\cos\psi-\frac{3}{2}\sin 2\theta\cos\psi\right)\dot\theta^2-\frac{3}{2}\cos 2\theta\sin\psi\,\dot\theta\dot\psi\right]$$

$$EL_\psi:\ ma^2\left[\frac{1}{3}\ddot\psi-\left(\frac{1}{3}\sin\psi-\frac{3}{2}\sin 2\theta\right)\cos\psi\,\dot\theta^2+\frac{3}{2}\cos 2\theta\cos\psi\,\ddot\theta\right]=-mga\left(\frac{1}{2}\sin\theta\cos\psi\right)$$

Quindi, il sistema delle EL linearizzate intorno a $\vec{q}_e=\left(-\frac{\pi}{6},\frac{\pi}{2}\right)$

$$
\begin{cases}
\cancel{m}a^{\cancel{2}}\left(\frac{32}{3}+\frac{3}{2}\sqrt{3}\right)\ddot{x}_1 - \cancel{m}g\cancel{a}\left(\frac{5}{4}+\frac{9}{2}\sqrt{3}\right)x_1 = 0 \\
\cancel{m}a^{\cancel{2}}\frac{1}{3}\ddot{x}_2 + \frac{\cancel{m}g\cancel{a}}{4}x_2 = 0
\end{cases}
$$

cioè

$$
\begin{cases}
a\left(\frac{32}{3}+\frac{3}{2}\sqrt{3}\right)\ddot{x}_1 - \frac{g}{2}\left(\frac{5}{2}+9\sqrt{3}\right)x_1 = 0 \\
\frac{a}{3}\ddot{x}_2 + \frac{g}{4}x_2 = 0
\end{cases}
$$

c) Reazioni dinamiche in A e B.

Scriviamo le 2 ECD, scegliendo come polo per la II il punto $B \in R$

$$(15.1)\quad \begin{cases} \vec{R}^{(est,att)} + \vec{\Psi}_A + \vec{\Phi}_B = m\,\vec{a}_G \\ \vec{M}_B^{(est,att)} + (A-B)\times\vec{\Psi}_A = \dfrac{d\vec{L}_B}{dt} + \vec{v}_B \times \vec{p} \end{cases}$$

Proiettando la I ECD sulla base fissa $\mathcal{B}$, possiamo ricavare subito $\Psi_A\cdot e_y$ e $\Phi_B\cdot e_z$.

$$(15.2)\quad \vec{\Psi}_A \cdot \vec{e}_x + \vec{\Phi}_B \cdot \vec{e}_x = m g\, \cancel{\vec{e}_z \cdot \vec{e}_x} - F\,\vec{i}\cdot\vec{e}_x + m\,\vec{a}_G\cdot\vec{e}_x$$

$$(15.3)\quad \vec{\Psi}_A \cdot \vec{e}_y = m g\, \cancel{\vec{e}_z \cdot \vec{e}_y} - \cancel{\vec{\Phi}_B \cdot \vec{e}_y} + m\,\vec{a}_G\cdot\vec{e}_y - F\,\vec{i}\cdot\vec{e}_y$$

$$(15.4)\quad \vec{\Phi}_B \cdot \vec{e}_z = m g\, \vec{e}_z \cdot \vec{e}_z - \cancel{\vec{\Psi}_A \cdot \vec{e}_z} + m\,\vec{a}_G\cdot\vec{e}_z - F\,\vec{i}\cdot\vec{e}_z$$

Quindi, dobbiamo calcolare le componenti di $\vec{a}_G$ in $\mathcal{B}$.
Facciamolo derivando 2 volte rispetto al tempo le (3.6)

$$\vec{v}_G = \frac{d}{dt}(G-O) = a\Big[-\frac{1}{2}\sin\varphi\,\dot\varphi\,\vec{e}_x + \Big(-\frac{1}{2}\sin\theta\sin\varphi\,\dot\theta + \frac{1}{2}\cos\theta\cos\varphi\,\dot\varphi + 3\cos\theta\,\dot\theta\Big)\vec{e}_y$$

$$(15.5)\quad + \Big(\frac{1}{2}\cos\theta\sin\varphi\,\dot\theta + \frac{1}{2}\sin\theta\cos\varphi\,\dot\varphi - 3\sin\theta\,\dot\theta\Big)\vec{e}_z\Big]$$

$$= a\Big\{-\frac{1}{2}\sin\varphi\,\dot\varphi\,\vec{e}_x + \Big[\Big(-\frac{1}{2}\sin\theta\sin\varphi + 3\cos\theta\Big)\dot\theta + \frac{1}{2}\cos\theta\cos\varphi\,\dot\varphi\Big]\vec{e}_y +$$

$$+ \Big[\Big(\frac{1}{2}\cos\theta\sin\varphi - 3\sin\theta\Big)\dot\theta + \frac{1}{2}\sin\theta\cos\varphi\,\dot\varphi\Big]\vec{e}_z\Big\}$$

$$\vec{a}_G = a\Big\{ -\frac{1}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right)\vec{e}_x +$$

$$+\Big[\Big(-\frac{1}{2}\left(\cos\theta\sin\varphi\,\dot{\theta} + \sin\theta\cos\varphi\,\dot{\varphi}\right) - 3\sin\theta\,\dot{\theta}\Big)\dot{\theta} + \left(-\frac{1}{2}\sin\theta\sin\varphi + 3\cos\theta\right)\ddot{\theta}$$

$$+\frac{1}{2}\left(-\sin\theta\cos\varphi\,\dot{\theta} + \cos\theta\sin\varphi\,\dot{\varphi}\right)\dot{\varphi} + \frac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_y$$

(16.1) $$+\Big[\Big(\frac{1}{2}\left(-\sin\theta\sin\varphi\,\dot{\theta} + \cos\theta\cos\varphi\,\dot{\varphi}\right) - 3\cos\theta\,\dot{\theta}\Big)\dot{\theta} + \left(\frac{1}{2}\cos\theta\sin\varphi - 3\sin\theta\right)\ddot{\theta} +$$

$$+\frac{1}{2}\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right)\dot{\varphi} + \frac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_z\Big\}$$

$$= a\Big\{ -\frac{1}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right)\vec{e}_x +$$

$$+\Big[-\left(\frac{1}{2}\cos\theta\sin\varphi + 3\sin\theta\right)\dot{\theta}^2 - \sin\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \left(-\frac{1}{2}\sin\theta\sin\varphi + 3\cos\theta\right)\ddot{\theta}$$

$$-\frac{1}{2}\cos\theta\sin\varphi\,\dot{\varphi}^2 + \frac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_y +$$

$$+\Big[-\left(\frac{1}{2}\sin\theta\sin\varphi + 3\cos\theta\right)\dot{\theta}^2 + \cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \left(\frac{1}{2}\cos\theta\sin\varphi - 3\sin\theta\right)\ddot{\theta}$$

$$-\frac{1}{2}\sin\theta\sin\varphi\,\dot{\varphi}^2 + \frac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big]\vec{e}_z\Big\}$$

Inoltre, proiettiamo il carico follower su $\mathcal{B}$

$$\vec{F}\cdot\vec{e}_x = F\,\vec{i}\cdot\vec{e}_x = F\cos\varphi$$

$$\vec{F}\cdot\vec{e}_y = F\,\vec{i}\cdot\vec{e}_y = F\cos\theta\sin\varphi$$

$$\vec{F}\cdot\vec{e}_z = F\,\vec{i}\cdot\vec{e}_z = F\sin\theta\sin\varphi$$

Dunque, la (15.3) e (15.4) forniscono

$$(17.1)\quad \vec{\Psi}_A \cdot \vec{e}_y = ma\Big[-\Big(\tfrac{1}{2}\cos\theta\sin\varphi + 2\sin\theta\Big)\dot{\theta}^2 - \sin\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \\ + \Big(-\tfrac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\Big)\ddot{\theta} - \tfrac{1}{2}\cos\theta\sin\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\cos\theta\cos\varphi\,\ddot{\varphi}\Big] + \\ - F\cos\theta\sin\varphi$$

$$(17.2)\quad \vec{\Phi}_B \cdot \vec{e}_z = mg + ma\Big[-\Big(\tfrac{1}{2}\sin\theta\sin\varphi + 2\cos\theta\Big)\dot{\theta}^2 + \cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \\ + \Big(\tfrac{1}{2}\cos\theta\sin\varphi - 2\sin\theta\Big)\ddot{\theta} - \tfrac{1}{2}\sin\theta\sin\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\sin\theta\cos\varphi\,\ddot{\varphi}\Big] + \\ - F\sin\theta\sin\varphi$$

Inoltre, la (15.2) fornisce la somma delle reazioni lungo $\vec{e}_x$

$$(17.3)\quad (\vec{\Psi}_A + \vec{\Phi}_B)\cdot\vec{e}_x = -\frac{ma}{2}\left(\cos\varphi\,\dot{\varphi}^2 + \sin\varphi\,\ddot{\varphi}\right) - F\cos\varphi$$

Per determinare ognuna delle 2 incognite $\vec{\Psi}_A\cdot\vec{e}_x$, $\vec{\Phi}_B\cdot\vec{e}_x$ dobbiamo usare la II ECD (15.1). Osserviamo che

$$(A-B)\times\vec{\Psi}_A = 6a\,\vec{k}\times(\Psi_x\vec{e}_x + \Psi_y\vec{e}_y) =$$

$$(17.4)\quad = 6a\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right)\times\left(\Psi_x\vec{e}_x + \Psi_y\vec{e}_y\right)$$

$$= 6a\left(\Psi_x\sin\theta\,\vec{e}_z + \Psi_x\cos\theta\,\vec{e}_y - \Psi_y\cos\theta\,\vec{e}_x\right)$$

Allora, se proiettiamo la II ECD lungo $\vec{e}_y$, otteniamo

$$(17.5)\quad 6a\cos\theta\,\Psi_x = -(G-B)\times m\vec{g}\cdot\vec{e}_y + \frac{d\vec{L}_B}{dt}\cdot\vec{e}_y + \cancel{\vec{v}_B\times\vec{p}}\cdot\vec{e}_y \qquad \vec{v}_B \parallel \vec{e}_y \\ -(M-B)\times\vec{F}\cdot\vec{e}_y$$

Il momento della forza peso e quello del carico follower sono

$$(G-B)\times m\vec{g} = \left(\frac{a}{2}\,\vec{i} + 3a\vec{k}\right)\times(-mg\vec{e}_z) = -mg\left(\frac{a}{2}\,\vec{i}\times\vec{e}_z + 3a\,\vec{k}\times\vec{e}_z\right)$$

$$= -mg\left[\frac{a}{2}\left(\cos\psi\,\vec{e}_x + \cos\theta\sin\psi\,\vec{e}_y\right)\times\vec{e}_z + 3a(-\sin\theta)\,\vec{e}_y\times\vec{e}_z\right]$$

$$= -mga\left[-\frac{1}{2}\cos\psi\,\vec{e}_y + \cos\theta\sin\psi\,\vec{e}_x - 3\sin\theta\,\vec{e}_x\right]$$

$$= -mga\left[(\cos\theta\sin\psi - 3\sin\theta)\,\vec{e}_x - \frac{1}{2}\cos\psi\,\vec{e}_y\right]$$

$$(M-B)\times\vec{F} = 3Fa\,\vec{j} = 3Fa\left(-\sin\psi\,\vec{e}_x + \cos\theta\cos\psi\,\vec{e}_y + \sin\theta\cos\psi\,\vec{e}_z\right)$$

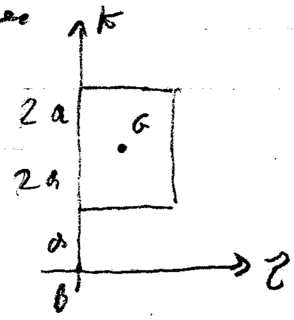
Inoltre

$$\frac{d\vec{L}_B}{dt}\cdot\vec{e}_y = \frac{d}{dt}\left(\vec{L}_B\cdot\vec{e}_y\right)$$

$$\vec{L}_B\cdot\vec{e}_y = I_B(\vec{\omega})\cdot\vec{e}_y + \cancel{(G-B)\times m\vec{v}_B\cdot\vec{e}_y} \qquad \vec{v}_B \parallel \vec{e}_y$$

Determiniamo $I_B$ con il Teo di Huygens-Steiner

$$[I_B]^{\mathcal{B}''} = [I_G]^{\mathcal{B}''} + m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$




dove con $(x,y,z)$ abbiamo indicato le coordinate di $G$ r. a $(B,\mathcal{B}'')$.

$$[I_B]^{\mathcal{B}''} = ma^2\begin{bmatrix} \frac{16}{12} & & \\ & \frac{17}{12} & \\ & & \frac{1}{12} \end{bmatrix} + m\begin{bmatrix} 9a^2 & 0 & -\frac{a}{2}3a \\ 0 & \frac{37}{4}a^2 & 0 \\ -\frac{3}{2}a^2 & 0 & \frac{a^2}{4} \end{bmatrix} = ma^2\begin{bmatrix} \frac{31}{3} & 0 & -\frac{3}{2} \\ 0 & \frac{32}{3} & 0 \\ -\frac{3}{2} & 0 & \frac{1}{3} \end{bmatrix}$$

Allora

$$I_B(\vec{\omega}) = ma^2\,[\vec{i},\vec{j},\vec{k}]\begin{bmatrix} \frac{31}{3} & 0 & -\frac{3}{2} \\ 0 & \frac{32}{3} & 0 \\ -\frac{3}{2} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \dot\theta\cos\psi \\ -\dot\theta\sin\psi \\ \dot\psi \end{bmatrix} =$$

$$= ma^2\,[\vec{i},\vec{j},\vec{k}]\begin{bmatrix} \frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi \\ -\frac{32}{3}\sin\psi\,\dot\theta \\ -\frac{3}{2}\cos\psi\,\dot\theta + \frac{1}{3}\dot\psi \end{bmatrix}$$

Quindi,

$$I_B(\vec{\omega}) = ma^2\left[\left(\frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi\right)\vec{i} - \frac{32}{3}\sin\psi\,\dot\theta\,\vec{j} + \left(\frac{1}{3}\dot\psi - \frac{3}{2}\cos\psi\,\dot\theta\right)\vec{k}\right]$$

$$\overset{(3.3)}{=} ma^2\Big[\left(\frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi\right)\left(\cos\psi\,\vec{e}_x + \cos\theta\sin\psi\,\vec{e}_y + \sin\theta\sin\psi\,\vec{e}_z\right) +$$

$$-\frac{32}{3}\sin\psi\,\dot\theta\left(-\sin\psi\,\vec{e}_x + \cos\theta\cos\psi\,\vec{e}_y + \sin\theta\cos\psi\,\vec{e}_z\right) +$$

$$+\left(\frac{1}{3}\dot\psi - \frac{3}{2}\cos\psi\,\dot\theta\right)\left(-\sin\theta\,\vec{e}_y + \cos\theta\,\vec{e}_z\right)\Big]$$

$$= ma^2\Big\{\left[\left(\frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi\right)\cos\psi + \frac{32}{3}\sin^2\psi\,\dot\theta\right]\vec{e}_x +$$

$$+\left[\left(\frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi\right)\cos\theta\sin\psi - \frac{32}{3}\dot\theta\cos\theta\sin\psi\cos\psi - \sin\theta\left(\frac{1}{3}\dot\psi - \frac{3}{2}\cos\psi\,\dot\theta\right)\right]\vec{e}_y$$

$$+\left[\left(\frac{31}{3}\cos\psi\,\dot\theta - \frac{3}{2}\dot\psi\right)\sin\theta\sin\psi - \frac{32}{3}\dot\theta\sin\theta\sin\psi\cos\psi + \left(\frac{1}{3}\dot\psi - \frac{3}{2}\cos\psi\,\dot\theta\right)\cos\theta\right]\vec{e}_z$$

$$I_B(\vec{\omega})\cdot\vec{e}_y = ma^2\left[\cos\psi\,\dot\theta\left(-\frac{1}{3}\cos\theta\sin\psi + \frac{3}{2}\sin\theta\right) - \left(\frac{1}{3}\sin\theta + \frac{3}{2}\cos\theta\sin\psi\right)\dot\psi\right]$$

Dunque,

$$\frac{d}{dt}\left(\vec{L}_B\cdot\vec{e}_y\right) = -\frac{d}{dt}\left(I_O(\vec{\omega})\cdot\vec{e}_y\right) =$$

$$= ma^2\Big[-\sin\psi\,\dot\psi\,\dot\theta\left(-\tfrac{1}{3}\cos\theta\sin\psi+\cancel{\tfrac{3}{2}\sin\theta}\right)+\cos\psi\,\ddot\theta\left(-\tfrac{1}{3}\cos\theta\sin\psi+\tfrac{3}{2}\sin\theta\right)+$$

(20.1) $$+\cos\psi\,\dot\theta\left(+\tfrac{1}{3}\sin\theta\,\dot\theta\sin\psi-\tfrac{1}{3}\cos\theta\cos\psi\,\dot\psi+\tfrac{3}{2}\cos\theta\,\dot\theta\right)+$$

$$-\left(\tfrac{1}{3}\cos\theta\,\dot\theta-\tfrac{3}{2}\sin\theta\,\cancel{\sin\psi}\,\dot\theta+\tfrac{3}{2}\cos\theta\cos\psi\,\dot\psi\right)\dot\psi-\left(\tfrac{1}{3}\sin\theta+\tfrac{3}{2}\cos\theta\sin\psi\right)\ddot\psi\Big]=$$

$$=ma^2\Big[\left(\tfrac{1}{3}\cos\theta\left(\sin^2\psi-\cos^2\psi\right)-\tfrac{1}{3}\cos\theta\right)\dot\psi\dot\theta+\cos\psi\left(\tfrac{3}{2}\sin\theta-\tfrac{1}{3}\cos\theta\sin\psi\right)\ddot\theta+\cos\psi\left(\tfrac{1}{3}\sin\theta\sin\psi+\tfrac{3}{2}\cos\theta\right)\dot\theta^2$$
$$-\tfrac{3}{2}\cos\theta\cos\psi\,\dot\psi^2-\left(\tfrac{1}{3}\sin\theta+\tfrac{3}{2}\cos\theta\sin\psi\right)\ddot\psi\Big]$$

Allora, la (17.5) si scrive:

$$6a\cos\theta\,\Psi_x=-\frac{mga}{2}\cos\psi+ma^2\Big[-\tfrac{1}{3}\cos\theta\left(\cos2\psi+1\right)\dot\psi\dot\theta+\cos\psi\left(\tfrac{3}{2}\sin\theta-\tfrac{1}{3}\cos\theta\sin\psi\right)\ddot\theta$$

(20.2) $$+\cos\psi\left(\tfrac{1}{3}\sin\theta\sin\psi+\tfrac{3}{2}\cos\theta\right)\dot\theta^2-\tfrac{3}{2}\cos\theta\cos\psi\,\dot\psi^2-\left(\tfrac{1}{3}\sin\theta+\tfrac{3}{2}\cos\theta\sin\psi\right)\ddot\psi\Big]$$

Quindi, se $\cos\theta\neq0 \iff \theta\neq\pm\pi/2$, si trova

(20.3) $$\Psi_x=-\frac{mg}{12}\frac{\cos\psi}{\cos\theta}+\frac{ma}{6\cos\theta}\Big[-\tfrac{1}{3}\cos\theta\left(1+\cos2\psi\right)\dot\theta\dot\psi+\cos\psi\left(-\tfrac{1}{3}\cos\theta\sin\psi+\tfrac{3}{2}\sin\theta\right)\ddot\theta$$
$$+\left(\tfrac{1}{6}\sin\theta\sin2\psi+\tfrac{3}{2}\cos\theta\right)\dot\theta^2-\tfrac{3}{2}\cos\theta\cos\psi\,\dot\psi^2-\left(\tfrac{1}{3}\sin\theta+\tfrac{3}{2}\cos\theta\sin\psi\right)\ddot\psi\Big]+$$
$$-\tfrac{1}{2}Fa\cos\psi.$$

Sostituendo nella (17.3), otteniamo

(20.4) $$\Phi_x=-\frac{ma}{2}\left(\cos\psi\,\dot\psi^2+\sin\psi\,\ddot\psi\right)+\frac{mg}{12}\frac{\cos\psi}{\cos\theta}+$$
$$-\frac{ma}{6}\Big[-\tfrac{2}{3}\cos^2\psi\,\dot\theta\dot\psi+\cos\psi\left(-\tfrac{1}{3}\sin\psi+\tfrac{3}{2}\operatorname{tg}\theta\right)\ddot\theta+\tfrac{1}{6}\left(\operatorname{tg}\theta\sin2\psi+\tfrac{3}{2}\right)\dot\theta^2-\tfrac{3}{2}\cos\psi\,\dot\psi^2+$$
$$-\left(\tfrac{1}{3}\operatorname{tg}\theta+\tfrac{3}{2}\sin\psi\right)\ddot\psi\Big]+\tfrac{1}{2}Fa\cos\psi$$