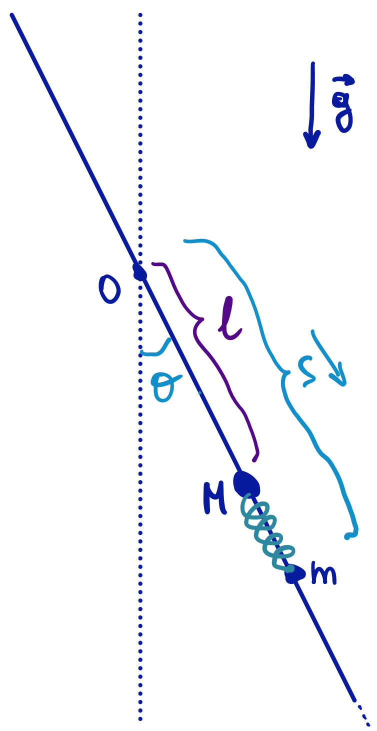
ESERCIZIO 2



$x_o = \ell \,\mathrm{sen}\theta \qquad \dot{x}_o = \ell\dot{\theta}\cos\theta$

$y_o = -\ell\cos\theta \qquad \dot{y}_o = \ell\dot{\theta}\,\mathrm{sen}\theta$

$x_p = s\,\mathrm{sen}\theta \qquad \dot{x}_p = s\dot{\theta}\cos\theta + \dot{s}\,\mathrm{sen}\theta$

$y_p = -s\cos\theta \qquad \dot{y}_p = s\dot{\theta}\,\mathrm{sen}\theta - \dot{s}\cos\theta$

$$T = \frac{M}{2}\ell^2\dot{\theta}^2 + \frac{m}{2}\left(s^2\dot{\theta}^2 + \dot{s}^2\right)$$

$$V = -Mg\ell\cos\theta - mgs\cos\theta + \frac{1}{2}k(s-\ell)^2$$

1) $$L = \frac{1}{2}\left(M\ell^2 + ms^2\right)\dot{\theta}^2 + \frac{m\dot{s}^2}{2} + Mg\ell\cos\theta + mgs\cos\theta - \frac{1}{2}k(s-\ell)^2$$

2) Eq. Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{d}{dt}\left(m\dot{s}\right) = m\ddot{s}$$

$$\frac{\partial L}{\partial s} = ms\dot{\theta}^2 + mg\cos\theta - k(s-\ell)$$

$$\ddot{s} = s\dot{\theta}^2 + g\cos\theta - \frac{k}{m}(s-\ell)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left[\left(M\ell^2 + ms^2\right)\dot{\theta}\right] = \left(M\ell^2 + ms^2\right)\ddot{\theta} + 2ms\dot{s}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -Mg\ell\,\mathrm{sen}\theta - mgs\,\mathrm{sen}\theta$$

$$\ddot{\theta} = -\frac{(Mg\ell + mgs)\,\mathrm{sen}\theta + 2ms\dot{s}\dot{\theta}}{(M\ell^2 + ms^2)}$$

3) La Lagrangiana dipende dai parametri m, M, ℓ, g, k.

Quale parametro può essere messo a zero per generare una costante del moto? Scrivere espressione della cost. del moto.

$g = 0 \to \frac{\partial L}{\partial \dot\theta} = (Ml^2 + ms^2)\dot\theta$

4) Pti di equil. (tutti param. $\neq 0$)

$$V = -Mgl\cos\theta - mgs\cos\theta + \frac{1}{2}k(s-l)^2$$

$$\partial_s V = -mg\cos\theta + k(s-l) = 0$$

$$\partial_\theta V = g(Ml + ms)\operatorname{sen}\theta = 0 \to \theta = 0, \pi \quad \text{or} \quad s = -\frac{Ml}{m}$$

$$\theta = 0, \pi \to \mp mg + k(s-l) = 0 \to s = l \pm \frac{mg}{k} = l\left(1 \pm \frac{mg}{kl}\right)$$

$$s = -\frac{Ml}{m} \to \cos\theta = -\frac{kl}{mg}\left(1 + \frac{M}{m}\right) < 0 \to \exists \text{ se } \frac{kl}{mg}\left(1 + \frac{M}{m}\right) \leq 1$$

$$\left(\text{c.m.}: \ Ml - m\left(\frac{Ml}{m}\right) = 0 \to \text{c.m. all'origine}\right)$$

Pti equil:

$$(s, \theta) = \left(l\left(1 + \frac{mg}{kl}\right), 0\right) \quad \left(l\left(1 - \frac{mg}{kl}\right), \pi\right) \quad \left(-\frac{Ml}{m}, \pm\theta^*\right)$$

$$\frac{\pi}{2} \leq \theta^* \leq \frac{3}{2}\pi$$

$$\partial^2 V = \begin{pmatrix} k & mg\operatorname{sen}\theta \\ mg\operatorname{sen}\theta & g(Ml+ms)\cos\theta \end{pmatrix}$$

$$\partial^2 V\left(l\left(1+\frac{mg}{kl}\right), 0\right) = \begin{pmatrix} k & 0 \\ 0 & g\left(Ml + ml + \frac{m^2 g}{k}\right) \end{pmatrix} \to \text{def. pos.} \qquad \underline{\text{STAB.}}$$

$$\partial^2 V\left(l\left(1-\frac{mg}{kl}\right), \pi\right) = \begin{pmatrix} k & 0 \\ 0 & -g\left(Ml + ml - \frac{m^2 g}{k}\right) \end{pmatrix} \to \text{def. pos.} \overset{\text{STAB.}}{\checkmark} \text{ se } (M+m)l < \frac{m^2 g}{k},$$

cioè se $\frac{kl}{mg}\left(1 + \frac{M}{m}\right) < 1$

altrim. INSTAB.

$$\partial^2 V\left(-\frac{Ml}{m}, \pm\theta^*\right) = \begin{pmatrix} k & \pm mg \sin\theta^* \\ \pm mg\sin\theta^* & g\left(Ml - \frac{Mlm}{m}\right)\cos\theta^* \end{pmatrix} = \begin{pmatrix} k & \pm mg\sin\theta^* \\ \pm mg \sin\theta^* & 0 \end{pmatrix}$$

$\rightarrow \det = -m^2 g^2 \sin^2\theta^* < 0$, $k > 0$ $\Rightarrow$ non-def. quindi **INSTAB.** (quando esiste)

5) Freq. piccole oscillazioni

$$T = \frac{1}{2}\left(Ml^2 + ms^2\right)\dot\theta^2 + \frac{m\dot s^2}{2} \rightarrow a = \begin{pmatrix} m & \\ & Ml^2 + ms^2 \end{pmatrix}$$

$$B = \partial^2 V\left(l\left(1+\frac{mg}{kl}\right), 0\right) = \begin{pmatrix} k & 0 \\ 0 & g\left(Ml + ml + \frac{m^2 g}{k}\right) \end{pmatrix} \quad A = a\left(l\left(1+\frac{mg}{kl}\right), 0\right) = \begin{pmatrix} m & \\ & (m+M)l^2 + \frac{m^3 g^2}{k^2} + 2\frac{m^2 g l}{k} \end{pmatrix}$$

$$Ml^2 + ml^2\left(1 + \frac{m^2 g^2}{k^2 l^2} + \frac{2mg}{kl}\right)$$

$$B - \lambda A = 0$$

$$\lambda_1 = \frac{k}{m} \qquad \lambda_2 = \frac{g\left(M + m + \frac{m^2 g}{kl}\right)}{l\left(M + m + 2\frac{m^2 g}{kl} + \frac{m^3 g^2}{k^2 l^2}\right)} =$$

$$= \frac{g/l}{1 + \frac{mg}{kl}\,\frac{1 + \frac{mg}{kl}}{1 + \frac{M}{m} + \frac{mg}{kl}}} = \frac{g/l}{1 + \frac{mg/kl}{1 + \frac{M/m}{1 + mg/kl}}}$$

6) Scrivere le eq. del moto nel sist. linearizzato e le loro soluzioni

$$s(t) = \underline{\underline{s_0}} + A_s \cos\left(\sqrt{\lambda_1}\, t + \varphi_s\right) \qquad \ddot s = -\lambda_1 s$$

$$\theta(t) = \underline{\underline{\theta_0}} + A_\theta \cos\left(\sqrt{\lambda_2}\, t + \varphi_\theta\right) \qquad \ddot\theta = -\lambda_2 \theta$$

7) Che simmetrie ha il sistema considerato (con tutti i parametri $\neq 0$ e generici)? Ci si associa una quantità conservata? Se sì, quale?

ESERCIZIO 1

6) $M_z = x p_y - y p_x$

$$\{M_z, f(x,y)\} = x\{p_y, f\} - y\{p_x, f\} =$$

$$= -x\,\partial_y f + y\,\partial_x f$$

$$f + \epsilon\{M_z, f\} = f(x,y) - \epsilon x\,\partial_y f + \epsilon y\,\partial_x f =$$

$$= f(x,y) + (\partial_x f, \partial_y f)\begin{pmatrix}\epsilon y\\ -\epsilon x\end{pmatrix} \underset{\epsilon \ll 1}{\simeq}$$

$$\simeq f(x+\epsilon y,\ y-\epsilon x) \simeq f(\underbrace{x', y'}_{\text{ruotati di angolo } \epsilon})$$

$$\begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} \sim \begin{pmatrix}1 & \epsilon\\ -\epsilon & 1\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix} = \begin{matrix}x+\epsilon y\\ y-\epsilon x\end{matrix}$$

Va bene anche dim. su vettori $\begin{pmatrix}x\\ y\end{pmatrix}$ o $\begin{pmatrix}p_x\\ p_y\end{pmatrix}$.

7) $V(\sqrt{x^2+y^2}) = g(x^2+y^2)$

$$\{M_z, g(x^2+y^2)\} = \{xP_y - yP_x, g(x^2+y^2)\} =$$

$$= -x\,\partial_y g(x^2+y^2) + y\,\partial_x g(x^2+y^2) =$$

$$= -x \cdot g' \cdot 2y + y \cdot g' \cdot 2x = 0$$