Lemma 1. *Let $\rho \in]0,1[$. Let $A \subseteq]a,\, b[$ such that for all $]\alpha,\beta[\subseteq]a,b[$,*

$$\lambda^*(A \cap]\alpha,\beta[\,) \leq \rho\,(\beta-\alpha).$$

Then $\lambda^*(A) = 0$. Consequently A is measurable and $\lambda(A) = 0$.

Proof. We recall that

$$\lambda^*(A) = \inf\{\sum_{n=1}^{+\infty}(\beta_n - \alpha_n) \;:\; A \subseteq \bigcup_{n=1}^{+\infty}]\alpha_n,\, \beta_n[\,\}.$$

Since $A \subseteq]a,\, b[$, in the previous definition it is not restrictive to suppose that, for all n, $]\alpha_n, \beta_n[\subseteq]a,b[$. Let $\varepsilon > 0$. From the definition of outer measure we have that there exists a sequence $(]\alpha_n, \beta_n[)_n$ of subintervals of $]a,b[$, such that

$$A \subseteq \bigcup_{n=1}^{+\infty}]\alpha_n,\, \beta_n[\quad \text{and} \quad \sum_{n=1}^{+\infty}(\beta_n - \alpha_n) < \lambda^*(A) + \varepsilon.$$

Recalling now that the outer measure is countably subadditive, we have

$$\lambda^*(A) \leq \sum_{n=1}^{+\infty} \lambda^*(A \cap]\alpha_n,\, \beta_n[) \leq \sum_{n=1}^{+\infty} \rho(\beta_n - \alpha_n) \leq \rho(\lambda^*(A) + \varepsilon).$$

Since this is true for all $\varepsilon > 0$, we deduce that

$$\lambda^*(A) \leq \rho\lambda^*(A)$$

and the conclusion follows.

□