Exercises 4

key words: Topological vector spaces with a topology defined by a countable number of seminorms, spaces $C^m(\Omega)$, $C^\infty(\Omega)$. Inductive limit topology, spaces $C_0^m(\Omega)$, $C_0^\infty(\Omega)$. Convolution. Mollifiers.

1) (Borel's theorem) Let $(a_n)_n$ be a sequence in $\mathbb{R}$. Let $\varphi \in \mathcal{D}(]-2,2[)$ such that $\varphi(x) = 1$ for $|x| \leq 1$.

- Show that there exists a sequence $(\lambda_k)_k$ in $\mathbb{R}$ such that, if we set

$$f_k(x) = \frac{a_k}{k!} x^k \varphi(\lambda_k x),$$

then

$$\sup_{x\in\mathbb{R}} |f_k^{(j)}(x)| \leq 2^{-k}, \quad \text{for all } 0 \leq j \leq k-1.$$

- Deduce that the series $\sum_k f_k(x)$ defines a function $f(x) \in C^\infty$ such that, for all j, $f^{(j)}(0) = a_j$.

2) Let Ω be an open set in $\mathbb{R}^n$. Let m and k be two positive integers, with $k \geq m$. Let $P(x, \partial_x) = \sum_{|\alpha|=m} a_\alpha(x)\partial_x^\alpha$ be a differential operator with $a_\alpha \in C^{k-m}(\Omega)$.

- Prove that $P(x, \partial_x)$ is continuous from $C^k(\Omega)$ to $C^{k-m}(\Omega)$.

3) Show that there exists no function $\delta \in C_0^0(\mathbb{R})$ such that $\delta * f = f$ for all $f \in C_0^0(\mathbb{R})$.

4) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n \setminus \{0\}$. For all $n \in \mathbb{N}$, we set

$$\varphi_n(x) = n(\varphi(x + \frac{h}{n}) - \varphi(x)).$$

- Prove that $(\varphi_n)_n$ converges, in the sense of $\mathcal{D}$, to a function to be determined.

5) (Poincaré inequality) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, Ω an open bounded set in $\mathbb{R}^n$.

- Prove that, for $i = 1, 2, \ldots, n$,

$$\int_{\mathbb{R}^n} |\varphi(x)|^2\, dx = -2 \int_{\mathbb{R}^n} x_i \varphi(x) \partial_{x_i} \varphi(x)\, dx.$$

- Prove that there exist $C > 0$ such that, for all $\psi \in \mathcal{D}(\Omega)$,

$$\int_\Omega |\psi(x)|^2\, dx \leq C \sum_{i=1}^n \int_\Omega |\partial_{x_i} \psi(x)|^2\, dx.$$

6) Costruct a sequence $(\varphi_k)_k$ in $\mathcal{D}(\mathbb{R})$ such that

- for each point $x \in \mathbb{R}$, the sum $\sum_k \varphi_k(x)$ is a finite sum;
- for each point $x \in \mathbb{R}$, $\sum_k \varphi_k(x) = 1$.