

BASI per il corso

FUNZIONI COMPOSITE

Prendiamo una funzione $f(x)$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$
$$x \mapsto f(x)$$

es. $f(x) = e^x, \cos x, x^2, \dots$

La sua derivata $f'(x) \equiv \frac{df(x)}{dx}$ è essa stessa una funz.

$$\frac{df}{dx}: \mathbb{R} \rightarrow \mathbb{R}$$
$$x \mapsto \frac{df(x)}{dx}$$

Posso definire funz. composte, sostituendo all'argomento x una funzione $\xi(t)$

$$\xi: \mathbb{R} \rightarrow \mathbb{R}$$
$$t \mapsto \xi(t)$$

$$\hookrightarrow g(t) \equiv f(\xi(t)) \quad g: \mathbb{R} \rightarrow \mathbb{R}$$

Facciamo la derivata di g

$$\frac{dg(t)}{dt} \equiv \dot{g}(t) = f'(\xi(t)) \dot{\xi}(t) \equiv \frac{df}{dx}(\xi(t)) \cdot \dot{\xi}(t)$$

Es: $f(x) = x^2 \quad \xi(t) = e^t \rightarrow g(t) = e^{2t}$

$$\rightarrow \dot{g}(t) = 2e^{2t}$$

$$= \frac{df}{dx} \cdot \dot{\xi} = 2\xi \cdot \dot{\xi} = 2e^t \cdot e^t = 2e^{2t}$$

\downarrow e^t

$\frac{df}{dx}(x) = 2x$

Partiamo da una funt. e due variabili:

$$f(x, y) = x^2 + y^2$$

→ c'è una mappa $f: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $(x, y) \mapsto x^2 + y^2$

Le sue derivate parziali sono ancora delle funzioni:

$$\frac{\partial f}{\partial x}: \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$(x, y) \mapsto \frac{\partial f}{\partial x}(x, y)$$

$$\frac{\partial f}{\partial y}: \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$(x, y) \mapsto \frac{\partial f}{\partial y}(x, y)$$

Prendiamo delle funzioni ξ e η

$$\xi: \mathbb{R} \rightarrow \mathbb{R}$$

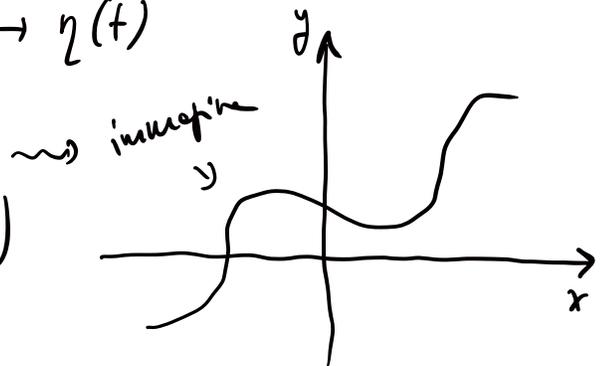
$$\eta: \mathbb{R} \rightarrow \mathbb{R}$$

$$t \mapsto \xi(t)$$

$$t \mapsto \eta(t)$$

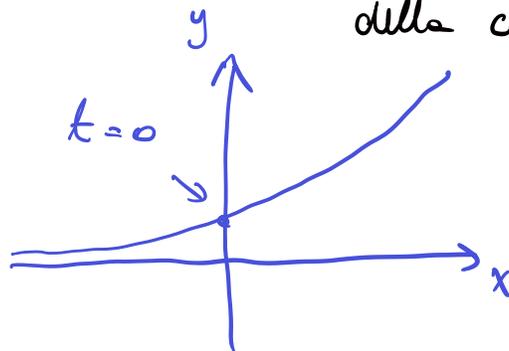
$$(\xi, \eta): \mathbb{R} \rightarrow \mathbb{R}^2$$

$$t \mapsto (\xi(t), \eta(t))$$



t è detto il parametro della curva

Es. $\xi(t) = t$
 $\eta(t) = e^t$



Def. la funt. composta

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$t \mapsto g(t) = f(\xi(t), \eta(t))$$

Es. $g(t) = f(t, e^t) = t^2 + e^{2t}$

Di qta funt. mi può interessare la derivata

$$g(t) = 2t + 2e^{2t}$$

oppure uso la regola di derivazione di funt. comp.

$$\begin{aligned} g(t) &= \frac{d}{dt} f(\xi(t), \eta(t)) = \\ &= \frac{\partial f}{\partial x}(\xi(t), \eta(t)) \cdot \dot{\xi}(t) + \frac{\partial f}{\partial y}(\xi(t), \eta(t)) \cdot \dot{\eta}(t) \\ &\quad \left| \begin{array}{l} \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y \quad f(x,y) = x^2 + y^2 \\ = 2\xi \dot{\xi} + 2\eta \dot{\eta} = 2t \cdot 1 + 2e^t \cdot e^t \\ = 2t + 2e^{2t} \end{array} \right. \end{aligned}$$

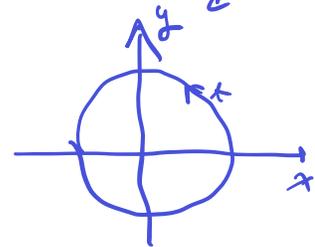
Proviamo a fare un'altra scelta per la funt. $\xi(t), \eta(t)$

$$\xi(t) = \cos t \quad \eta(t) = \sin t \quad (f = x^2 + y^2)$$

$$\hookrightarrow g(t) = \xi(t)^2 + \eta(t)^2 = \cos^2 t + \sin^2 t = 1$$

↓

$$g(t) = 0$$



$$g = \frac{\partial f}{\partial x} \dot{\xi} + \frac{\partial f}{\partial y} \dot{\eta} =$$

$$= 2\xi \dot{\xi} + 2\eta \dot{\eta} = -2 \cos t \sin t + 2 \sin t \cos t = 0$$

Altra possibilità:

$$h: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(x, y, t) \mapsto h(x, y, t)$$

$$\text{Es. } h(x, y; t) = xy^2 - \text{sech}^2 t$$

Consideriamo una curva in \mathbb{R}^2 data da

$$\begin{aligned} \xi(t) &= 1 & \eta(t) &= \cosh t \\ \dot{\xi} &= 0 & \dot{\eta} &= \text{sech} t \end{aligned}$$

$$\begin{aligned} \text{Def. } K: \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto h(\xi(t), \eta(t); t) \end{aligned}$$

$$\begin{aligned} K(t) &= h(\xi(t), \eta(t); t) = \xi(t) \eta(t)^2 - \text{sech}^2 t \\ &= 1 \cdot \cosh^2 t - \text{sech}^2 t = 1 \\ &\rightarrow \dot{K} = 0 \end{aligned}$$

Calcoliamoci \dot{K} usando la regola delle derivate di JACOBI.

$$\begin{aligned} \dot{K}(t) &= \frac{\partial h}{\partial x}(\xi(t), \eta(t); t) \dot{\xi}(t) + \frac{\partial h}{\partial y}(\xi(t), \eta(t); t) \dot{\eta}(t) + \frac{\partial h}{\partial t}(\xi(t), \eta(t); t) \\ &= \eta^2 \dot{\xi} + 2\xi \eta \dot{\eta} - 2\text{sech} t \cosh t = \\ &= (\cosh t)^2 \cdot 0 + 2 \cdot 1 \cdot \cosh t \text{sech} t - 2\text{sech} t \cosh t = 0. \end{aligned}$$

$h(x, y; t) = xy^2 - \text{sech}^2 t$

$\frac{\partial h}{\partial x} = y^2$
 $\frac{\partial h}{\partial y} = 2xy$

SPAZI VETTORIALI

Spazio vettoriale V def. su \mathbb{R} (\mathbb{C}, \dots)

è un INSIEME di oggetti che chiamiamo VETTORI

$$\vec{v}_1, \vec{v}_2, \dots \in V \quad \alpha, \beta, \dots \in \mathbb{R}$$

su cui definiamo le operazioni $+$ e "prodotto per un numero":

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &\in V \\ \alpha \vec{v}_1 &\in V \end{aligned} \quad \text{A.c.} \quad \alpha(\vec{v}_1 + \vec{v}_2) = \alpha\vec{v}_1 + \alpha\vec{v}_2$$

$\vec{v}_1, \dots, \vec{v}_m$ sono LINEARMENTE DIPENDENTI se

\exists una comb'neta. lineare nulla, cioè se

$\exists \alpha_1, \dots, \alpha_m \in \mathbb{R}$ A.c.

$$\underbrace{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_m \vec{v}_m}_{\equiv \sum_{i=1}^m \alpha_i \vec{v}_i} = 0$$

(I vett. sono lin. indep. & non sono lin. dp)

BASE per V : insieme di vettori ^{lin.} indep.

$$\{\bar{e}_1, \dots, \bar{e}_n\} \quad \text{A.c.}$$

$\forall \vec{v} \in V \quad \exists! (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ A.c.

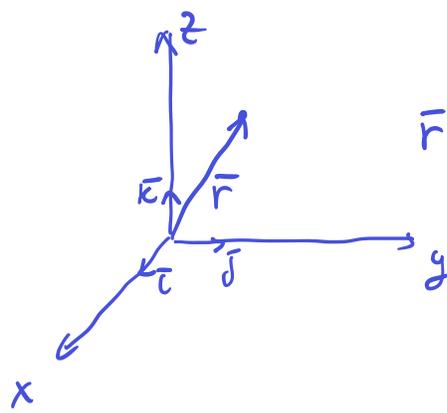
$$\vec{v} = \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \dots + \alpha_n \bar{e}_n = \sum_{i=1}^n \alpha_i \bar{e}_i$$

$$\begin{array}{c} \updownarrow \\ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \end{array}$$

componenti del vettore

Es.

$$V = \mathbb{R}^3$$



$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Prodotto scalare

$$V \times V \rightarrow \mathbb{R}$$

$$\vec{v}_1, \vec{v}_2 \mapsto \vec{v}_1 \cdot \vec{v}_2 \in \mathbb{R}$$

prodotto scalari def. positivi

$$\downarrow$$

$$\rightsquigarrow \|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$$

\rightsquigarrow Base ortonormale $\{\vec{e}_1, \dots, \vec{e}_n\}$

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$

$$= \delta_{ij}$$

$$\vec{v}_1, \vec{v}_2 \in V$$

$$\vec{v}_1 = \sum_{i=1}^n \alpha_i \vec{e}_i$$

$$\vec{v}_2 = \sum_{j=1}^n \beta_j \vec{e}_j$$

$$= \sum_{i=1}^n \alpha_i \vec{e}_i = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \dots + \alpha_n \vec{e}_n$$

$$(a+b)(c+d) = a(c+d) + b(c+d)$$

$$\vec{v}_1 \cdot \vec{v}_2 = \left(\sum_{i=1}^n \alpha_i \vec{e}_i \right) \cdot \left(\sum_{j=1}^n \beta_j \vec{e}_j \right) =$$

$$= \sum_{i=1}^n \alpha_i \left[\vec{e}_i \cdot \left(\sum_{j=1}^n \beta_j \vec{e}_j \right) \right] =$$

$$= \sum_{i=1}^n \alpha_i \sum_{j=1}^n \beta_j \vec{e}_i \cdot \vec{e}_j = \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^n \beta_j \delta_{ij} \right) =$$

$$\begin{aligned}
 \sum_{j=1}^m \beta_j \delta_{ij} &= \beta_1 \delta_{i1} + \beta_2 \delta_{i2} + \beta_3 \delta_{i3} + \dots + \beta_m \delta_{im} = \\
 &= \beta_1 \delta_{i1} + \dots + \beta_i \delta_{ii} + \dots + \beta_m \delta_{im} \\
 &= 0 + \dots + \beta_i + 0 + \dots + 0 = \beta_i
 \end{aligned}$$

$$= \sum_{i=1}^m \alpha_i \beta_i = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_n \beta_n$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

