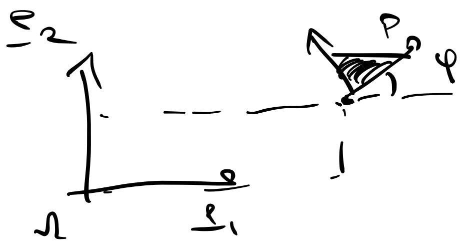
MECCANICA RAZIONALE

Corpo rigido $\longrightarrow$ 6 gradi di libertà $\mathbb{R}^3$

3 gradi di libertà $\mathbb{R}^2$



$$\underline{x}_P = \underline{x}_O + R \cdot \underline{X}_P$$

$\underline{x}_O \leftarrow (x_0, y_0)$, $R \leftarrow \varphi$

Coordinate libere

↳ vincoli (olonomi)

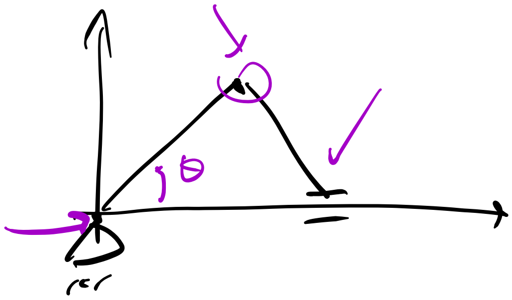
$$f(q_1, \ldots, q_n) = 0$$

$$[\, f(x_0, y_0, \varphi) = 0 \,]$$

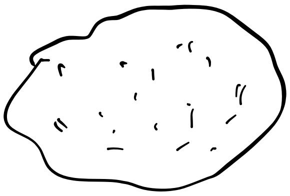
$$y_0 = \ldots\ldots_{x_0, \varphi}$$

$\rightarrow$ Sovrapposizione dei vincoli

→ Sistemi articolati



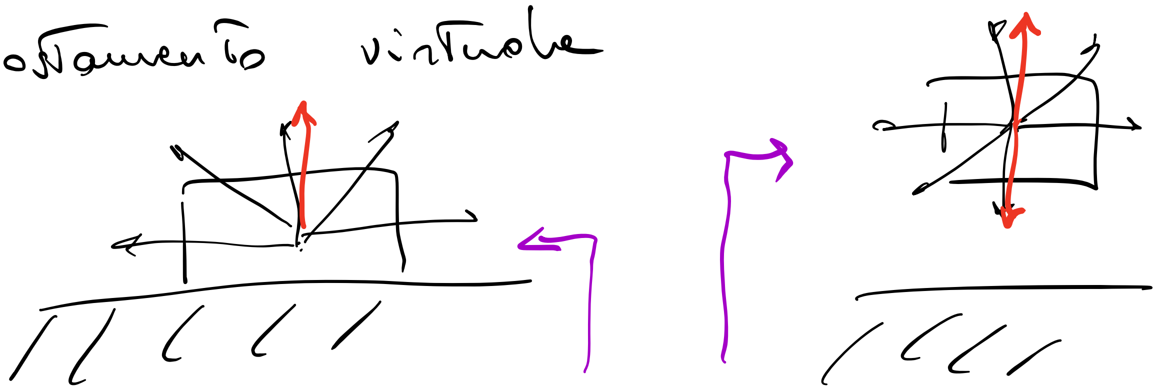
$\mathcal{R}$



$$\boxed{\underline{F} = m\underline{a}}$$

Principio dei lavori virtuali.

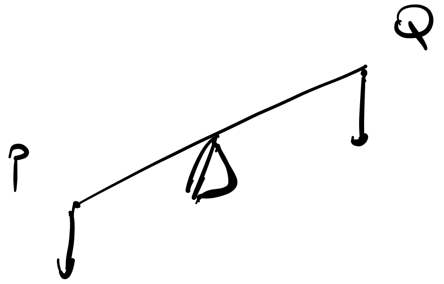
Spostamento virtuale

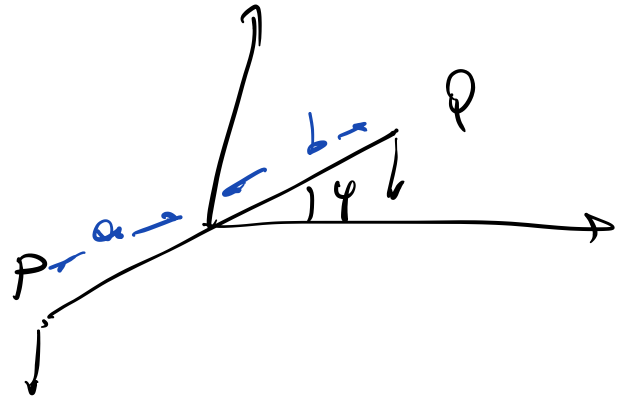


$\delta \underline{x}_B \rightarrow$ invertibili $(-\delta \underline{x}_B)$

PLV: $$LV = \sum_B \underline{F}_B \cdot \delta \underline{x}_B \leq 0$$
$$(= 0)$$

$\Longleftrightarrow$ equilibrio





$$\delta \underline{x}_Q = (-b \sin\varphi \, \underline{e}_1 + b \cos\varphi \, \underline{e}_2) \, \delta\varphi$$

$$\delta \underline{x}_P = (a \sin\varphi \, \underline{e}_1 - a \cos\varphi \, \underline{e}_2) \, \delta\varphi$$

$$LV = \underline{F}_P \cdot \delta \underline{x}_P + \underline{F}_Q \cdot \delta \underline{x}_Q = \dots$$

$$= \underbrace{(-q b \cos\varphi + p a \cos\varphi)}_{=0} \, \boxed{\delta\varphi} = 0$$

invertibili

$$p a \cos\varphi = q b \cos\varphi$$

Quindi:

S sistema olonomo

B punti di applicazione delle forze attive $\underline{F}_B^{\text{attive}}$

$\delta \underline{x}_B$ spostamenti virtuali (= spostamenti linearizzati concessi dai vincoli)

Equilibrio di S $\iff$ $\sum_{B \in S} \underline{F}_B^{\text{attive}} \cdot \delta \underline{x}_B \leq 0$

per ogni insieme di $\delta \underline{x}_B$ virtuali (invertibili)

$$\underline{x}_Q(\varphi) = b \cos\varphi \, \underline{e}_1 + b \sin\varphi \, \underline{e}_2$$

$$\underline{x}_Q(\varphi + \delta\varphi) = - - -$$

$$\delta \underline{x}_Q = \underline{x}_Q(\varphi + \delta\varphi) - \underline{x}_Q(\varphi)$$

$\delta\varphi$

$$= \cancel{\underline{x}_Q(\varphi)} + \text{primo ordine} + \text{secondo ordine} + \dots$$

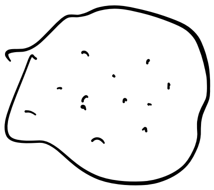
trascurabili

$$- \cancel{\underline{x}_Q(\varphi)}$$

$$\delta \underline{x}_Q = \frac{d \underline{x}_Q}{d\varphi} \delta\varphi \equiv \text{linearizzato}$$

Sistemi materiali soggetti al solo peso

S sistema punti materiali



$\mathcal{R}$ $\downarrow \underline{g}$ $\quad \forall B \quad m_B \underline{g}$

$$LV = \sum_{B \in S} m_B \underline{g} \cdot \delta \underline{x}_B =$$

$$\left[\iiint \rho(\underline{x}) \ldots \right]$$

$$\left[LV = (\ )\, \delta q_1 + (\ )\, \delta q_2 + \ldots \right.$$

$$\left. = 0 \right]$$

$$= \sum_{B \in S} m_B \underline{g} \cdot \delta \underline{x}_B = \underline{g} \cdot \left(\sum_{B \in S} m_B \, \delta \underline{x}_B \right)$$

$$= \underline{g} \cdot \left(\sum_{B \in S} \delta (m_B \underline{x}_B) \right)$$

$$= \underline{g} \cdot \delta \left(\sum_B m_B \underline{x}_B \right)$$

$$\left[\quad M = \sum_{B \in S} m_B \qquad \underline{x}_G = \frac{\sum_{B \in S} m_B \underline{x}_B}{M} \quad \right]$$

$$= \underline{g} \cdot \delta \left(M \underline{x}_G \right)$$

$$= M \, \underline{g} \cdot \delta \underline{x}_G$$

$$\Rightarrow \quad LV = \sum_{B \in S} m_B \, \underline{g} \cdot \delta \underline{x}_B = M \, \underline{g} \cdot \delta \underline{x}_G$$

<u>Principio dei lavori virtuali per sistemi ad un grado di libertà</u>

S , vincoli olonomi, bilateri, lisci

1 grado di libertà $\rightarrow$ q

$$\underline{x}_B = \underline{x}_B(q) \qquad \forall B \in S$$

$\uparrow$

funzione regolare $\Rightarrow$ $\boxed{\delta \underline{x}_B(q) := \dfrac{d\underline{x}_B(q)}{dq}\,\delta q}$

$\rightarrow$ spostamento virtuale invertibile

$\underline{F}_B^a$ $\rightarrow$ la risultante delle forze attive su B

$$LV = \sum_{B\in S}' \underline{F}_B^a \cdot \delta \underline{x}_B =$$

$$= \sum_{B\in S} \underline{F}_B^a \cdot \frac{d\underline{x}_B}{dq}\,\delta q$$

$$= \left(\sum_{B\in S} \underline{F}_B^a \cdot \frac{d\underline{x}_B}{dq} \right) \delta q$$

$$= Q\,\delta q$$

$$\boxed{Q := \sum_{B \in S} \underline{F}_B^a \cdot \frac{d\,\underline{x}_B}{dq}}$$

Forza generalizzata relativa a q

PLV $\quad LV = \sum_{B \in S} \underline{F}_B^a \cdot \delta \underline{x}_B = 0$

$$\boxed{Q\, \delta q = 0}$$

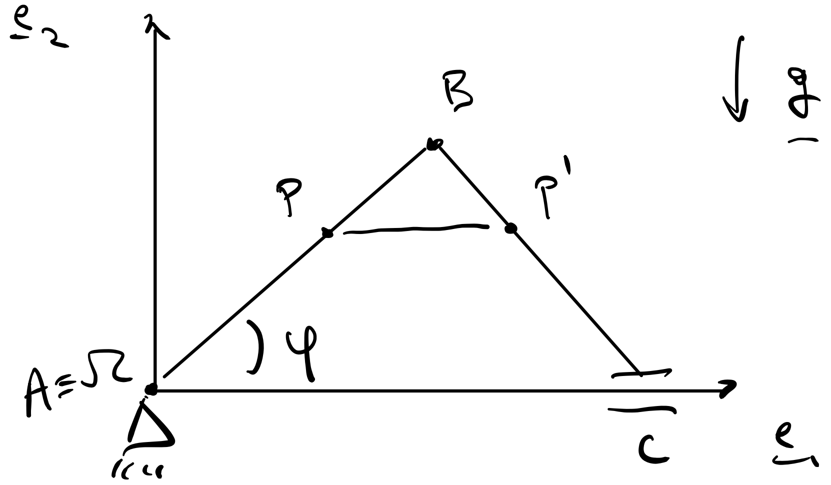
Tutti gli spostamenti virtuali dipendono da δq

Condizione di equilibrio $\Longleftrightarrow Q = 0$

$$\left[\; Q = pa \cos\varphi - qb \cos\varphi \;\right]$$

esempio sopra.

<u>Esempio</u> Problema diretto in statica:
dato l'equilibrio determinare le forze che lo consentono.



due aste rigide $\overline{AB} = \overline{BC}$ soggette al peso all' eq. $\varphi = \alpha$

Quale è la Tensione della fune per ottenere eq. per α

forza in P

$$\underline{T} = \tau \left(\underline{x}_{P'} - \underline{x}_P \right)$$

$$\tau > 0$$

Principio di azione & reazione in P' la fune esercita una forza $-\underline{T}$

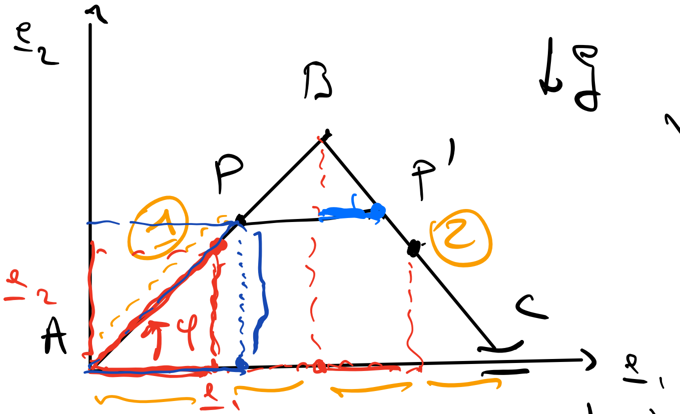
<u>Dati:</u> $\overline{AB} = \overline{BC} = \ell$ omogenee

$$m_{AB} = 2\, m_{BC} = 2\, \underline{m}$$

$$m_{BC} = \underline{m}$$

$$\overline{BP} = \overline{BP'} = \frac{2}{3}\ell \qquad B\hat{A}C = \alpha = \frac{\pi}{3}$$

$$\overline{AP} = \overline{P'C} = \frac{1}{3}\ell$$



Voglio φ per l'equilibrio:

LV totale = LV peso + LV fune

$$\begin{cases} \underline{x}_{G_1} = \frac{\ell}{2}\cos\varphi\, \underline{e}_1 + \frac{\ell}{2}\sin\varphi\, \underline{e}_2 \\ \underline{x}_{G_2} = \underbrace{\left(\ell\cos\varphi + \frac{\ell}{2}\cos\varphi\right)}_{\frac{3}{2}\ell\cos\varphi} \underline{e}_1 + \frac{\ell}{2}\sin\varphi\, \underline{e}_2 \\ \underline{x}_P = \frac{1}{3}\ell\cos\varphi\, \underline{e}_1 + \frac{1}{3}\ell\sin\varphi\, \underline{e}_2 \\ \underline{x}_{P'} = \left(\ell + \frac{2}{3}\ell\right)\cos\varphi\, \underline{e}_1 + \frac{1}{3}\ell\sin\varphi\, \underline{e}_2 \end{cases}$$

$$LV_{peso} = m_{AB}\, \underline{g} \cdot \delta \underline{x}_{G_1} + m_{BC}\, \underline{g} \cdot \delta \underline{x}_{G_2}$$

$$= m_{AB}\left(-g\, \underline{e}_2\right) \cdot \delta\left(\frac{\ell}{2}\cos\varphi\, \underline{e}_1 + \frac{\ell}{2}\sin\varphi\, \underline{e}_2\right)$$

$$+\ m_{BC}\left(-g\, \underline{e}_2\right) \cdot \delta\left(\frac{3}{2}\ell\cos\varphi\, \underline{e}_1 + \frac{\ell}{2}\sin\varphi\, \underline{e}_2\right)$$

$$\underline{e}_1 \cdot \underline{e}_2 = 0 \qquad \underline{e}_1 \cdot \underline{e}_1 = 1$$

$$\underline{e}_2 \cdot \underline{e}_2 = 1$$

$$= -g\left(m_{AB} + m_{BC}\right)\underline{e}_2 \cdot \delta\left(\frac{\ell}{2}\sin\varphi\right)\underline{e}_2$$

$$= -g\,3m\left[\frac{\ell}{2}\cos\varphi\right]\delta\varphi$$

$$LV_{peso} = -g\,\frac{3}{2}\,m\ell\cos\varphi\,\delta\varphi$$

$$LV_{fune} = \underline{T} \cdot \delta\underline{x}_p + \left(-\underline{T}\right)\cdot\delta\underline{x}_{p'}$$

$$= \underline{T} \cdot \left(\delta\underline{x}_p - \delta\underline{x}_{p'}\right) =$$

$$= \tau\left(\underline{x}_{p'} - \underline{x}_p\right)\cdot\left(\delta\underline{x}_p - \delta\underline{x}_{p'}\right)$$

$$\underline{x}_p = \frac{1}{3}\ell\cos\varphi\,\underline{e}_1 + \frac{1}{3}\ell\sin\varphi\,\underline{e}_2$$

$$\delta\underline{x}_p = -\frac{1}{3}\ell\sin\varphi\,\delta\varphi\,\underline{e}_1 + \frac{1}{3}\ell\cos\varphi\,\delta\varphi\,\underline{e}_2$$

$$\underline{x}_{p'} = \left(\ell + \frac{2}{3}\ell\right)\cos\varphi\,\underline{e}_1 + \frac{1}{3}\ell\sin\varphi\,\underline{e}_2$$

$$\delta\underline{x}_{p'} = -\frac{5}{3}\ell\sin\varphi\,\delta\varphi\,\underline{e}_1 + \frac{1}{3}\ell\cos\varphi\,\delta\varphi\,\underline{e}_2$$

$$\underline{x}_p \cdot \delta \underline{x}_{p'} = \left(-\frac{5}{9} + \frac{1}{9} \right) \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$\underline{x}_p \cdot \delta \underline{x}_p = \left(-\frac{1}{9} + \frac{1}{9} \right) \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$\underline{x}_{p'} \cdot \delta \underline{x}_{p'} = \left(-\frac{25}{9} + \frac{1}{9} \right) \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$\underline{x}_{p'} \cdot \delta \underline{x}_p = (\quad) \, \underline{e}_1 \cdot \underline{e}_1 + (\quad) \, \cancel{\underline{e}_1 \cdot \underline{e}_2} +$$

$$+ (\quad) \, \cancel{\underline{e}_2 \cdot \underline{e}_1} + (\quad) \, \underline{e}_2 \cdot \underline{e}_2$$

$$= -\frac{5}{3} \ell^2 \frac{1}{3} \cos\varphi \sin\varphi \, \delta\varphi + \frac{1}{9} \ell^2 \sin\varphi \cos\varphi \, \delta\varphi$$

$$= -\frac{4}{9} \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$\tau \left(\underline{x}_p \cdot \delta \underline{x}_p - \underline{x}_p \cdot \delta \underline{x}_{p'} - \underline{x}_p \cdot \delta \underline{x}_p + \underline{x}_p \cdot \delta \underline{x}_{p'} \right)$$

$$= (\underline{x}_{p'} - \underline{x}_p) \cdot (\delta \underline{x}_p - \delta \underline{x}_{p'}) =$$

$$= \tau \left(-\frac{4}{9} + \frac{24}{9} - \frac{4}{9} \right) \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$= \tau \frac{16}{9} \ell^2 \cos\varphi \sin\varphi \, \delta\varphi$$

$$LV_{tot} = \left(-\frac{3}{2} mg\ell \cos\varphi + \tau \frac{16}{9} \ell^2 \cos\varphi \sin\varphi \right) \delta\varphi$$

$$= Q(\varphi) \, \delta\varphi$$

$$Q(\varphi) = \left(-\frac{3}{2} mg + \frac{16}{9} \ell \tau \sin\varphi \right) \ell \cos\varphi$$

dato $\varphi = \frac{\pi}{3}$ di eq., quanto vale τ ?

$$Q\left(\frac{\pi}{3}\right) = 0 \implies$$

$$\left(-\frac{3}{2} mg + \frac{16}{9} \ell \tau \sin\frac{\pi}{3} \right) \underbrace{\ell \cos\frac{\pi}{3}}_{\neq 0} = 0$$

$$\frac{16}{9} \ell \tau \sin\frac{\pi}{3} = \frac{3}{2} mg$$

$$\boxed{\tau = \frac{3}{2} mg \, \frac{9}{16} \, \frac{1}{\ell} \, \frac{2}{\sqrt{3}}}$$

Principio dei lavori virtuali per sistemi a l gradi di libertà

$$\underline{q} = (q_1, \dots, q_l)$$

$$\underline{x}_B(\underline{q}) = \underline{x}_B(q_1, \dots, q_l)$$

$$\delta \underline{x}_B(q_1, \dots, q_l)$$

$$\underline{x}_B(\underline{q} + \delta \underline{q}) = \underline{x}_B(\underline{q}) + \sum_{i=1}^{l} \left(\frac{\partial}{\partial q_i} \underline{x}_B(\underline{q}) \right) \cdot \delta q_i$$

+ termini di ordine superiore

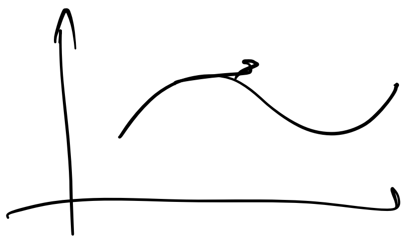
$$\underline{x}_B(\underline{q} + \delta \underline{q}) - \underline{x}_B(\underline{q}) = \sum_{i=1}^{l} \left(\frac{\partial}{\partial q_i} \underline{x}_B(\underline{q}) \right) \delta q_i$$

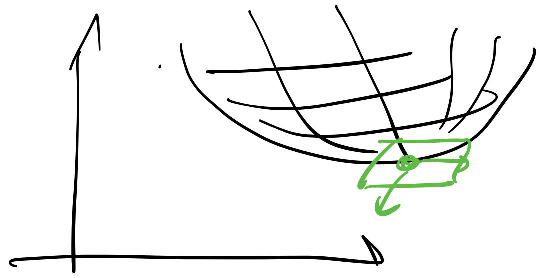
al primo ordine

ad esempio

$$\underline{x}_B(q_1 + \delta q_1, q_2, \dots, q_l) - \underline{x}_B(q_1, q_2, \dots, q_l)$$

$$= \frac{\partial \underline{x}_B}{\partial q_1} \delta q_1 + \text{ordini superiori}$$





$$LV = \sum_{B \in S} \underline{F}_B^{a} \cdot \delta \underline{x}_B = \sum_{B \in S}' \underline{F}_B^{e} \cdot \overbrace{\left(\sum_{i=1}^{l} \frac{\partial \underline{x}_B}{\partial q_i} \delta q_i \right)}^{\delta \underline{x}_B}$$

$$= \sum_{i=1}^{l} \underbrace{\left(\sum_{B \in S}' \underline{F}_B^{e} \cdot \frac{\partial \underline{x}_B}{\partial q_i} \right)}_{=: Q_i} \delta q_i$$

$$= \sum_{i=1}^{l} Q_i \, \delta q_i$$