MECCANICA RAZIONALE

Principio dei lavori virtuali

↳ equilibrio

$$LV = \sum_{B \in S} \underline{F}_B \cdot \delta \underline{x}_B$$

$\delta \underline{x}_B$: spostamento virtuale

$$\boxed{\delta \underline{x}_B = \frac{d}{dq} \underline{x}_B(q) \, \delta q}$$

1 coord. libera

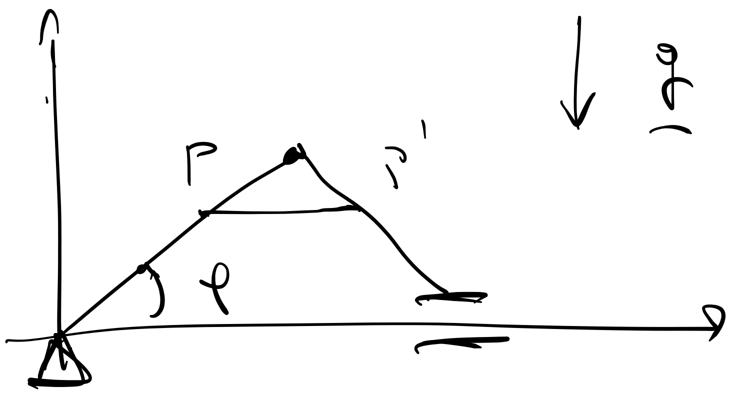
$\underline{q} = (q_1, \ldots, q_\ell)$ sistema a ℓ gradi di libertà

$$LV = \boxed{\sum_{B\in S} \underline{F}_B^e \cdot \delta \underline{x}_B} =$$

$$= \left(\sum_{B\in S} \underline{F}_B^e \cdot \frac{d\underline{x}_B(q)}{dq}\right)\delta q$$

$$= \boxed{Q(q)\,\delta q}$$

forza generalizzata



$\rightarrow \varphi$

1 coord. libera

$$LV = \underbrace{Q}_{=0}\,\delta\varphi = 0$$

PLV per l gradi di libertà

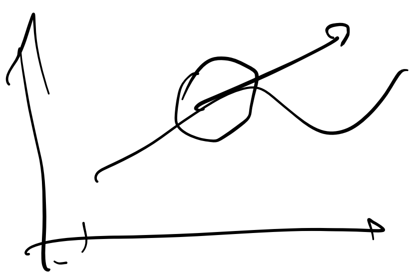
$$\underline{x}_B(\underline{q}) = \underline{x}_B(q_1, \dots, q_\ell)$$

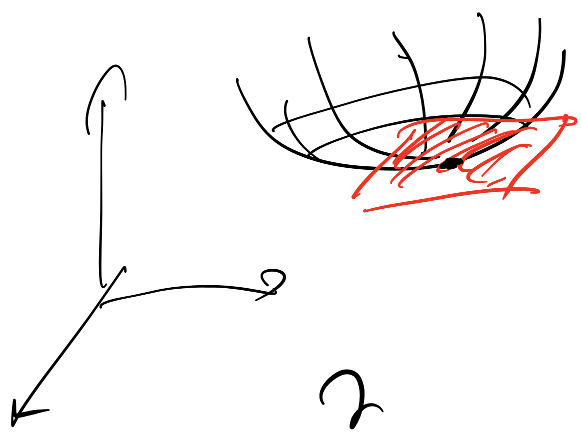
$q_1 = x_o$
$q_2 = y_o$
$q_3 = \varphi$

$$\delta \underline{x}_B(q_1, \dots, q_\ell)$$

$$\underline{x}_B(\underline{q} + \delta\underline{q}) - \underline{x}_B(\underline{q}) =$$

$$= \sum_{i=1}^{\ell} \left(\frac{\partial}{\partial q_i} \underline{x}_B(\underline{q}) \right) \delta q_i + \text{ordini più alti}$$





$$\frac{\partial}{\partial q_i}$$

$$\underline{x}_B(q_1+\delta q_1, q_2, q_3 \cdots q_l) - \underline{x}_B(q_1 \cdots q_l)$$

$$= \frac{\partial \underline{x}_B}{\partial q_1}\,\delta q_1 + \cdots$$

$$\delta \underline{x}_B(\underline{q}) = \sum_{i=1}^{l} \frac{\partial \underline{x}_B}{\partial q_i}\,\delta q_i$$

$$LV = \sum_{B\in S} \underline{F}_B^e \cdot \delta \underline{x}_B =$$

$$= \sum_{B\in S} \underline{F}_B^e \cdot \left(\sum_{i=1}^{l} \frac{\partial \underline{x}_B}{\partial q_i}\,\delta q_i \right)$$

$$= \sum_{i=1}^{l} \left(\sum_{B\in S} \underline{F}_B^e \cdot \frac{\partial}{\partial q_i}\underline{x}_B(\underline{q}) \right) \delta q_i$$

$$= \sum_{i=1}^{\ell} Q_i \, \delta q_i$$

$$Q_i := \sum_{B \in S}' \underline{F}_B^{e} \cdot \frac{\partial \underline{x}_B}{\partial q_i}$$

forze generalizzate relative a q_i

$$LV = \sum_{i=1}^{\ell}{}' \, Q_i \, \delta q_i = 0$$

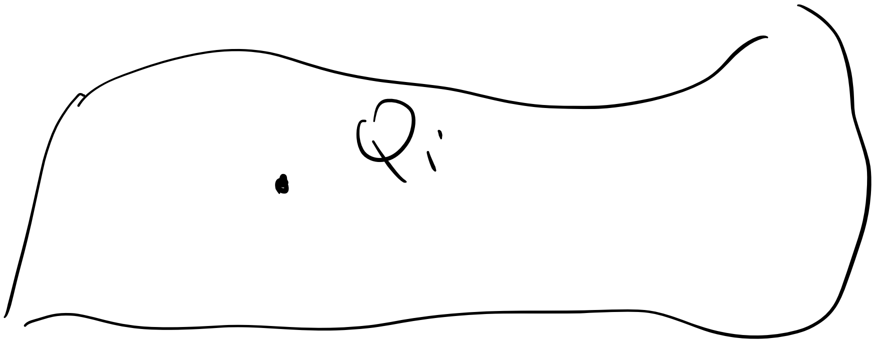
$$\Longleftrightarrow \quad Q_i = 0 \quad \forall i \in \{1, \dots, \ell\}$$

$$Q_1 \, \delta q_1 + Q_2 \, \delta q_2 = 0$$

$$Q_1 = - Q_2 \frac{\delta q_2}{\delta q_1}$$

Sistema $\to$ 1 punto nello spazio delle configurazioni

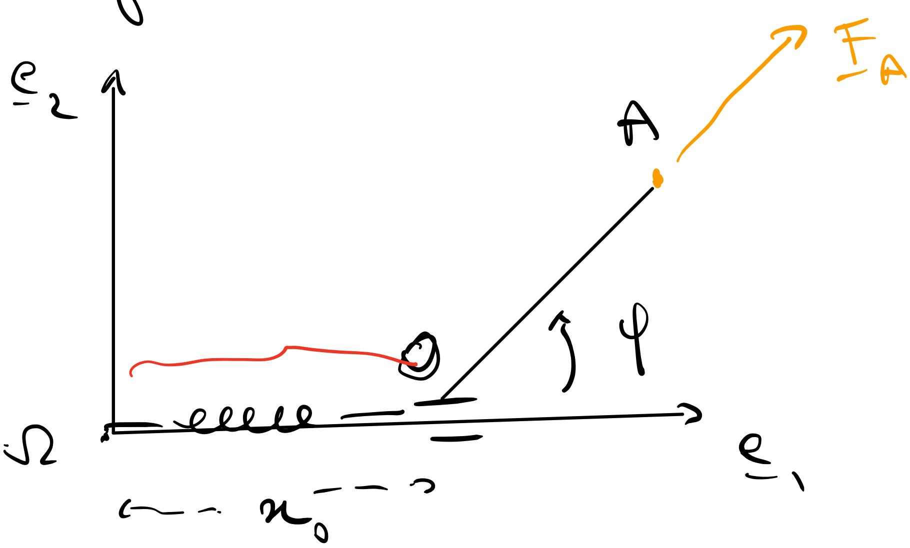
$\underline{q} = (q_1, \dots, q_\ell)$



$\underline{q} = (q_1 \dots q_\ell)$

$Q_i = 0 \qquad [\; F_i = 0 \;]$

<u>Esempio</u> Problema inverso di statica: date le forze agenti determinare l'equilibrio.



$\overline{OA} = \ell$

Forza di richiamo in O

$\underline{F}_0 = -c\, x_0\, \underline{e}_1$

Forza "FOLLOWER" $\qquad \underline{F}_A = f \, \mathrm{vers}(\underline{x}_A - \underline{x}_0)$

$$= f \frac{\underline{x}_A - \underline{x}_0}{\|\underline{x}_A - \underline{x}_0\|}$$

$$LV = \underline{F}_0 \cdot \delta \underline{x}_0 + \underline{F}_A \cdot \delta \underline{x}_A$$

$$\underline{x}_0 = \underline{x_0 \, \underline{e}_1} \qquad \delta \underline{x}_0 = \delta x_0 \, \underline{e}_1$$

$$\underline{x}_A = \underline{(x_0 + l \cos\varphi) \underline{e}_1 + l \sin\varphi \, \underline{e}_2}$$

$$\delta \underline{x}_A = \left(\delta x_0 - \underline{l \sin\varphi} \, \delta\varphi \right) \underline{e}_1 + \underline{l \cos\varphi} \, \delta\varphi \, \underline{e}_2$$

$$\delta \underline{x}_A = \frac{\partial \underline{x}_A}{\partial x_0} \delta x_0 + \frac{\partial \underline{x}_A}{\partial \varphi} \delta \varphi$$

$$LV = \underline{F}_0 \cdot \delta \underline{x}_0 + \underline{F}_A \cdot \delta \underline{x}_A =$$

$$= \left(-c\,x_0\,\underline{e}_1\right)\cdot \delta x_0\,\underline{e}_1 +$$

$$+ f\,\frac{\underline{x}_A - \underline{x}_0}{\|\underline{x}_A - \underline{x}_0\|}\cdot\Big(\delta x_0\,\underline{e}_1 +$$

$$+ \delta\varphi\left(-l\sin\varphi\,\underline{e}_1 + l\cos\varphi\,\underline{e}_2\right)\Big)$$

$$= -c\,x_0\,\delta x_0 +$$

$$+ f\,\frac{\cancel{l}\cos\varphi\,\underline{e}_1 + \cancel{l}\sin\varphi\,\underline{e}_2}{\cancel{l}}\cdot\Big(\delta x_0\,\underline{e}_1$$

$$+ \delta\varphi\left(-l\sin\varphi\,\underline{e}_1 + l\cos\varphi\,\underline{e}_2\right)\Big)$$

$$= -c\,x_0\,\delta x_0 + f\cos\varphi\,\Big(\delta x_0 +$$

$$- l\,\delta\varphi\sin\varphi\Big) + f\sin\varphi\,l\cos\varphi\,\delta\varphi$$

$$= \left(-c x_0 + f \cos\varphi\right)\delta x_0 +$$

$$+ \left(f \cos\varphi \, l \sin\varphi - f \sin\varphi \, l \cos\varphi\right)\delta\varphi$$

$$\begin{cases} Q_1 = -c x_0 + f \cos\varphi \\ Q_2 = 0 \end{cases}$$

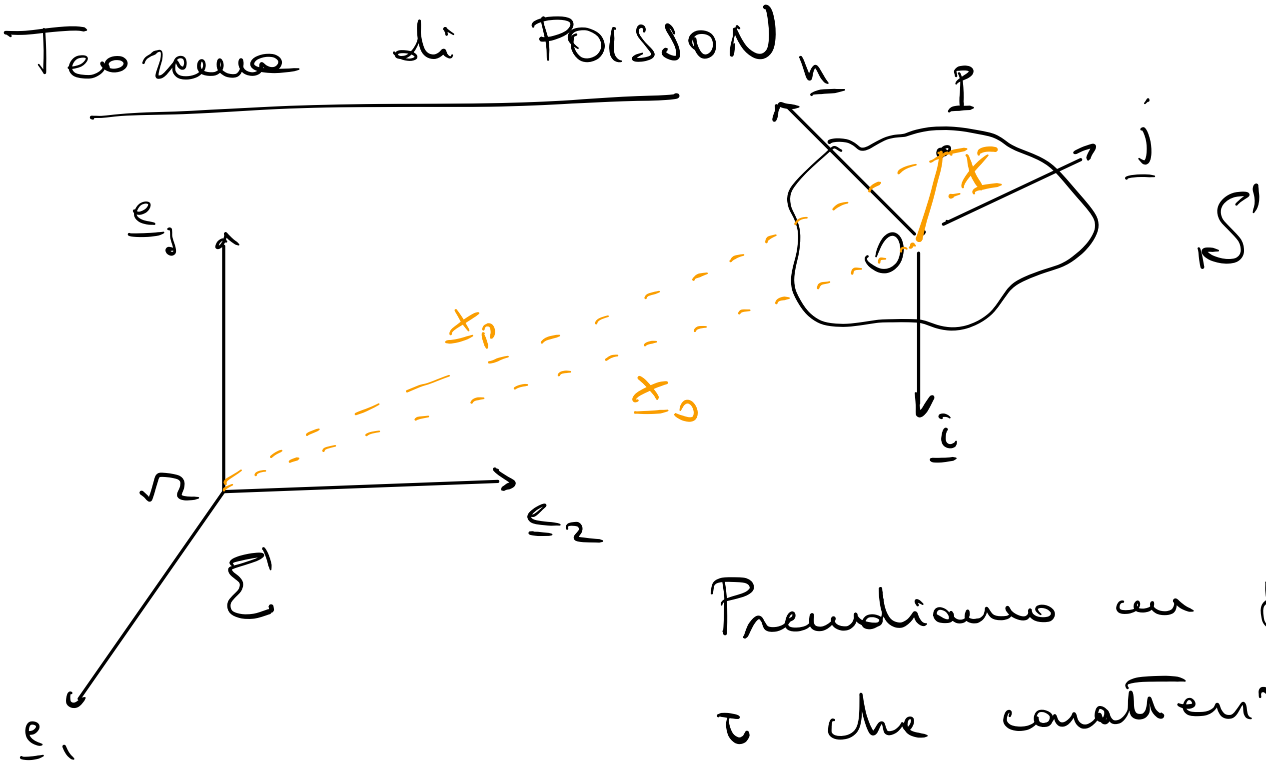
Configurazioni di equilibrio:

$$\begin{cases} x_0 = \dfrac{f}{c} \cos\alpha \\ \varphi = \alpha \end{cases} \qquad \forall \alpha$$

Grado di libertà (vincoli) $\longrightarrow$ Principio dei lavori virtuali

$\longrightarrow$ <u>CINEMATICA</u> <u>DEI RIGIDI</u>

Teorema di POISSON



Prendiamo un parametro τ che caratterizza uno spostamento

$$\tau \longrightarrow \tau + \delta\tau$$

$$\underline{x} \longrightarrow \underline{x} + \delta\underline{x}$$

Teorema: fissato uno spostamento di R, esiste un unico vettore $\underline{\omega}(\tau)$ tale che per ogni coppia di punti O e P del rigido R, vale

$$\frac{d}{d\tau}\underline{x}_P = \frac{d}{d\tau}\underline{x}_O + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$$

Dimostrazione

$$\boxed{\underline{x}_p(\tau) = \underline{x}_0(\tau) + R_{(\tau)} \cdot \underline{X}}$$

R : matrice di trasformazione tra Σ e S

$$\frac{d}{d\tau}\underline{x}_p(\tau) = \frac{d}{d\tau}\underline{x}_0(\tau) + \left(\frac{d}{d\tau}R_{(\tau)}\right)\cdot \underline{X}$$

Trasformazioni ortogonali :

$$R \cdot R^T = \mathbb{1} \qquad R^{-1} = R^T$$

ad esempio $\underline{v} \in \mathbb{R}^3$ $\quad \underline{v} \cdot \underline{v}$

$$\rightarrow v^T \cdot v \Rightarrow (Rv)^T \cdot (Rv)$$

$$= v^T R^T R v = v^T v$$

$R^T R = \mathbb{1}$

$$\underline{x}_{p(\tau)} = \underline{x}_{0(\tau)} + R_{(\tau)} \cdot \underline{X} \longrightarrow \boxed{\underline{X} = R^T \cdot (\underline{x}_p - \underline{x}_0)}$$

R^{-1}

$$\frac{d}{dt}\underline{x}_p = \frac{d}{d\tau}\underline{x}_0 + \left(\frac{d}{d\tau}R(\tau)\right)\cdot \underline{X}$$

$$= \frac{d}{d\tau}\underline{x}_0 + \left(\frac{d}{d\tau}R(\tau)\right)\cdot R^T\cdot(\underline{x}_p - \underline{x}_0)$$

Definiamo $A := \left(\frac{d}{d\tau}R(\tau)\right)R^T(\tau) = \dot{R}R^T$

si vede che A è una matrice antisimmetrica

per definizione di R, $R(\tau)R^T(\tau) = \mathbb{1}$

$$\frac{d}{d\tau}\left(R(\tau)R^T(\tau)\right) = \underbrace{\left(\frac{d}{dt}R\right)R^T}_{A} + \underbrace{R\left(\frac{d}{d\tau}R^T\right)} \qquad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$= \frac{d}{d\tau}\mathbb{1} = 0$$

$$A^T = \left(\dot{R}R^T\right)^T = R\cdot\dot{R}^T$$

quindi $A(\tau) + A^T(\tau) = 0$

la matrice A è antisimmetrica.

Risultato tecnico di algebra: se A

è una matrice 3×3 antisimmetrica allora esiste $\underline{\omega} \in \mathbb{R}^3$ tale che

$$A \underline{y} = \underline{\omega} \wedge \underline{y} \qquad \forall\, \underline{y} \in \mathbb{R}^3$$

$$A \underline{y} = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\underline{\omega} \wedge \underline{y} = \begin{Vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ y_1 & y_2 & y_3 \end{Vmatrix} = \begin{pmatrix} -\omega_3 y_2 + \omega_2 y_3 \\ \omega_3 y_1 - \omega_1 y_3 \\ -\omega_2 y_1 + \omega_1 y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} \omega_3 &= -A_{12} \\ \omega_2 &= A_{13} \\ \omega_1 &= -A_{23} \end{aligned}$$

$$\frac{d}{d\tau} \underline{x}_P = \frac{d}{d\tau} \underline{x}_0 + \left[\underbrace{\left(\frac{d}{d\tau} R(\tau) \right) \cdot R^T}_{A} \right] (\underline{x}_P - \underline{x}_0)$$

$$= \frac{d}{dt} \underline{x}_0 + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_0)$$