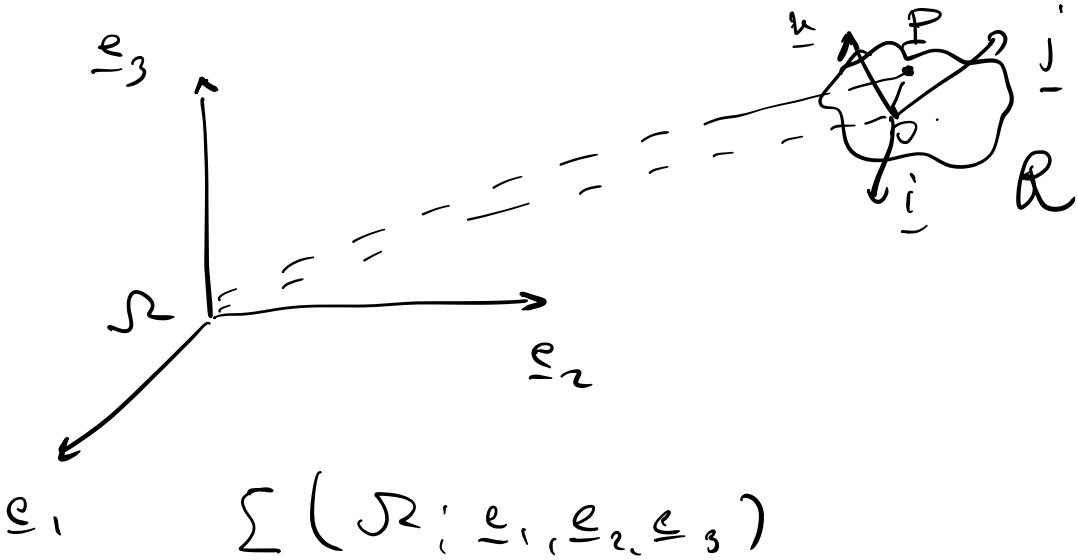
MECCANICA RAZIONALE

Coordinate libere $\longrightarrow$ PLV

$$\delta L = \sum_B \underline{F}_B \cdot \delta \underline{x}_B$$

$$= \sum_{i=1}^{l} Q_i \, \delta q_i$$

Teorema di Poisson



$\mathcal{S}(O; \underline{i}, \underline{j}, \underline{k})$

$\Sigma(\Omega; \underline{e}_1, \underline{e}_2, \underline{e}_3)$

$$\frac{d}{dt} \underline{x}_P(t) = \frac{d}{dt} \underline{x}_O(t) + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$$

$$\longrightarrow \quad \underline{x}_P(t) = \underline{x}_O(t) + R(t) \cdot \underline{X}$$

$$R R^T = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A := \dot{R} R^T \longrightarrow 3\times 3 \text{ antisimmetrica}$$

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{21} & 0 \end{pmatrix}$$

$$A\underline{y} \longleftrightarrow \underline{\omega} \wedge \underline{y} \qquad \forall\, \underline{y}$$

$$\omega_1 = -A_{23}\,, \quad \omega_2 = A_{13}\,, \quad \underline{\omega_3 = -A_{12}}$$

Significato fisico di $\underline{\omega}$

- Traslazioni

Gli assi di S non ruotano rispetto a Σ'

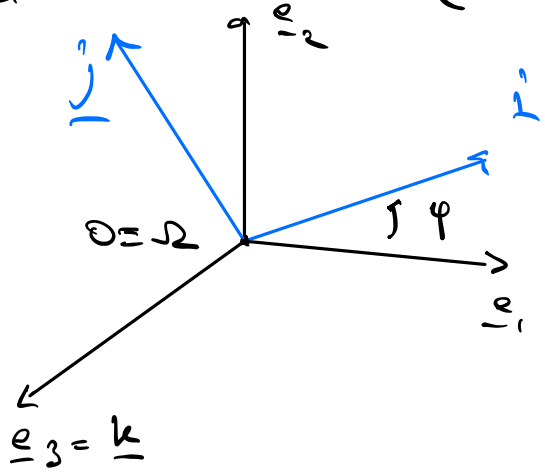
$$\Rightarrow \frac{d}{dt} R_{(t)} = 0 \quad \Rightarrow \quad A = \frac{dR}{dt} \cdot R^T = 0$$

$$\rightarrow \quad \underline{\omega} = \underline{0}$$

Rotazioni di un rigido intorno ad un asse fisso

Scegliamo Σ' e S in modo che

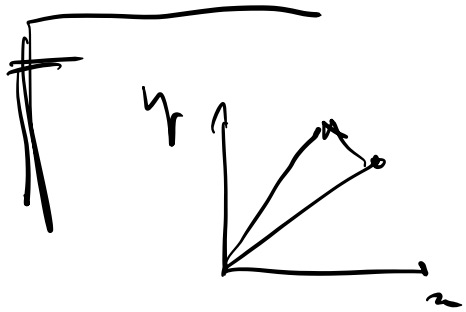
- $O = \Omega$
- $\underline{e}_3 = \underline{k}$



$\underline{x}_0 = \underline{0}$

$$\Rightarrow \quad \underline{x}_P(t) = R(\varphi(t)) \cdot \underline{X}$$

$$R(\varphi(t)) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$x = \rho\cos\alpha$$
$$y = \rho\sin\alpha$$

$$\left.\begin{aligned} x' &= \rho\cos(\alpha+\theta) = \rho(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ y' &= \rho\sin(\alpha+\theta) = \rho(\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{aligned}\right\}$$

$$\begin{cases} = x\cos\theta - y\sin\theta \\ = x\sin\theta + y\cos\theta \end{cases} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\dot{\varphi} = \frac{d}{dt}\varphi$$

$$\frac{d}{dt}\cos\varphi = (-\sin\varphi)\,\dot{\varphi}$$

$$A = \dot{R}R^{T} = \begin{pmatrix} -\dot{\varphi}\sin\varphi & -\dot{\varphi}\cos\varphi & 0 \\ \dot{\varphi}\cos\varphi & -\dot{\varphi}\sin\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

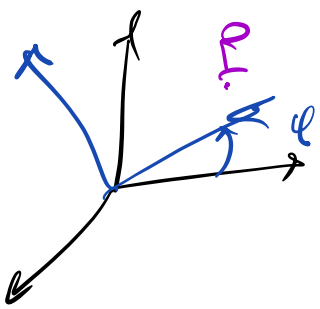
$$= \begin{pmatrix} -\dot{\varphi}\sin\varphi\cos\varphi + \dot{\varphi}\sin\varphi\cos\varphi & -\dot{\varphi}\sin^2\varphi - \dot{\varphi}\cos^2\varphi & 0 \\ \dot{\varphi}\cos^2\varphi + \dot{\varphi}\sin^2\varphi & -\dot{\varphi}\sin\varphi\cos\varphi + \dot{\varphi}\sin\varphi\cos\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \dot{\varphi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_3 = \dot{\varphi} \end{matrix}$$

$$\underline{\omega} = \dot{\varphi}\, \underline{e}_3 = \dot{\varphi}\, \underline{k}$$

$$\frac{d \underline{x}_P}{dt} = \dot{\varphi}\, \underline{e}_3 \wedge \underline{x}_P \qquad (\underline{x}_O = \underline{0})$$

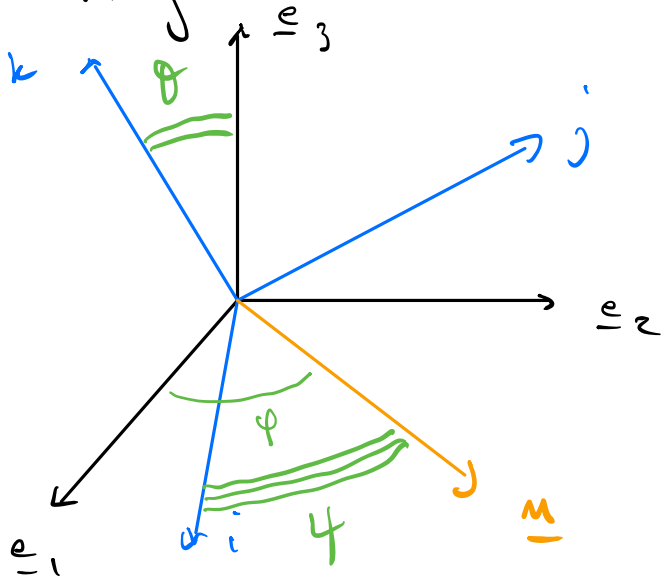


$$\frac{d \underline{x}_O}{dt} = \frac{d \underline{x}_O}{dt} + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$$

$$\dot{\varphi}\, \underline{e}_3$$

- moto con punto fisso

Angoli di Eulero $\qquad \Omega = 0$



$\underline{N} = \underline{e}_3 \wedge \underline{k}$

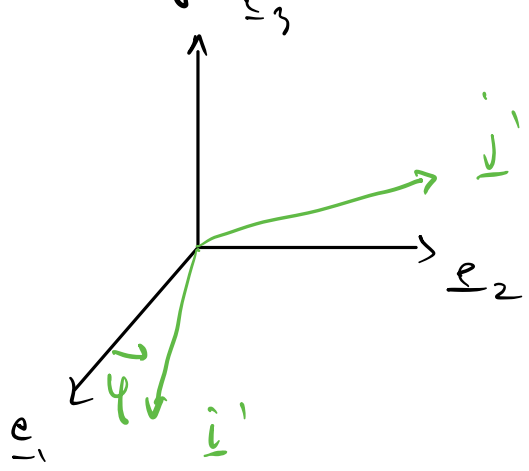
φ angolo di precessione (fra $\underline{e}_1$ e $\underline{N}$)

ϑ angolo di nutazione (fra $\underline{k}$ e $\underline{e}_3$)

ψ angolo di rotazione propria (fra $\underline{i}$ e $\underline{N}$)

Idea: costruiamo la matrice di rotazione da S a Σ come la successione di tre rotazioni indipendenti:

1. Rotazione intorno ad $\underline{e}_3$ di un angolo φ

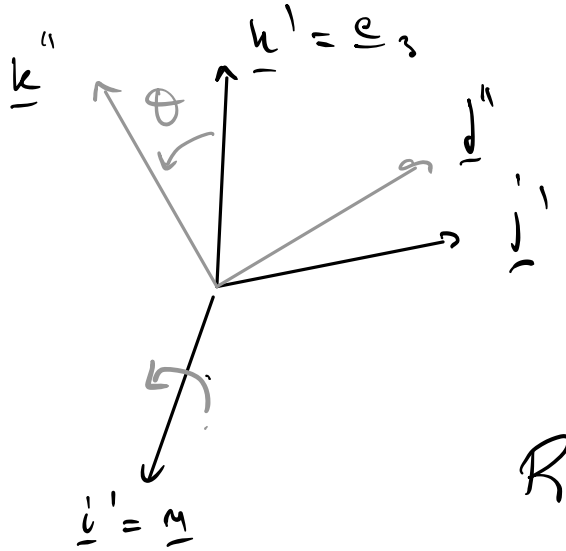


$$\Sigma \to S'\left(\Omega;\ \underline{i}',\ \underline{j}',\ \underline{k}'=\underline{e}_3\right)$$

$$(\underline{x})_\Sigma = R_1(\varphi)\,(\underline{x})_{S'}$$

$$R_1(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Rotazione intorno a $\underline{i}' = \underline{u}$ di un angolo θ

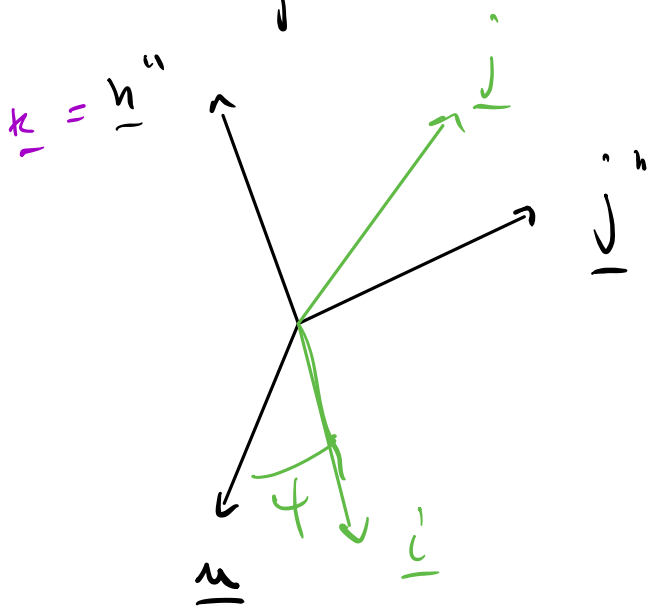


$$S' \to S''\left(\Omega;\ \underline{i}'' = \underline{u},\ \underline{j}'',\ \underline{k}''\right)$$

$$(\underline{x})_{S'} = R_2(\theta)\,(\underline{x})_{S''}$$

$$R_2(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

3. Rotazione intorno a $\underline{k}''$ di un angolo ψ



$(\underline{i}' = \underline{i}'')$

$$S'' \to S(\Omega, \underline{i}, \underline{j}, \underline{k})$$

$$(\underline{x})_{S''} = R_3(\psi)\,(\underline{x})_S$$

$$R_3(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\underline{x})_\xi = R_1(\varphi)\,(x)_{S'} = R_1(\varphi)\,R_2(\theta)\,(\underline{x})_{S''}$$

$$= \underbrace{R_1(\varphi)\,R_2(\theta)\,R_3(\psi)}\,(\underline{x})_S$$

$$\left(x_P = \cancel{x_0} + R \cdot \underline{X} \right)$$

$$R = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & -\cos\varphi\sin\psi - \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & -\cos\varphi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$R \longrightarrow A = \dot{R} R^T \longrightarrow \underline{\omega}$$

Facciamo un altro esempio:

$$\underline{x}(\tau) = \underline{x}(\varphi(\tau), \theta(\tau), \psi(\tau))$$

$$\frac{d}{d\tau}\underline{x}(\tau) = \frac{\partial \underline{x}}{\partial \varphi}\dot{\varphi} + \frac{\partial \underline{x}}{\partial \theta}\dot{\theta} + \frac{\partial \underline{x}}{\partial \psi}\dot{\psi}$$

$\frac{\partial \underline{x}}{\partial \varphi}\dot{\varphi}$: θ e ψ sono fissi

φ varia intorno ad $\underline{e}_3$

$$\frac{\partial \underline{x}}{\partial \varphi}\dot{\varphi} = \dot{\varphi}\, \underline{e}_3 \wedge \underline{x}$$

$\frac{\partial \underline{x}}{\partial \theta}\dot{\theta}$: φ e ψ fissi, θ varia intorno ad $\underline{n}$

$$\frac{\partial \underline{x}}{\partial \theta}\dot{\theta} = \dot{\theta}\, \underline{n} \wedge \underline{x}$$

$\frac{\partial \underline{x}}{\partial \psi}\dot{\psi}$: ψ e θ sono fissati, ψ varia intorno a $\underline{k}$

$$\frac{\partial \underline{x}}{\partial \psi}\dot{\psi} = \dot{\psi}\,\underline{k} \wedge \underline{x}$$

$$\frac{d}{dt}\underline{x}(t) = \dot{\psi}\,\underline{e}_3 \wedge \underline{x} + \dot{\theta}\,\underline{u} \wedge \underline{x} + \dot{\psi}\,\underline{k} \wedge \underline{x}$$

$$\frac{d}{dt}\underline{x}(t) = \underline{\omega} \wedge \underline{x} \qquad (\underline{x}_0 = \underline{0})$$

$$\boxed{\underline{\omega} = \dot{\psi}\,\underline{e}_3 + \dot{\theta}\,\underline{u} + \dot{\psi}\,\underline{k}}$$

Ricordiamo : $\mathcal{E} \to S' \to S'' \to S$

$$(\underline{u})_S = R_3^T(\psi)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\psi\\-\sin\psi\\0\end{pmatrix}$$

$$R_3 = \begin{pmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix} \qquad R_3^T = \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix}$$

$$(\underline{u})_1 = \cos\psi\, \underline{i} - \sin\psi\, \underline{j}$$

$$(\underline{e}_3) = (R_2 R_3)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \sin\psi \\ \sin\theta \cos\psi \\ \cos\theta \end{pmatrix}$$

$$= \sin\theta \sin\psi\, \underline{i} + \sin\theta \cos\psi\, \underline{j} + \cos\theta\, \underline{k}$$

$$(\underline{\omega})_S = (\dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi)\, \underline{i} +$$

$$+ (\dot\varphi \sin\theta \cos\psi - \dot\theta \sin\psi)\, \underline{j}$$

$$+ (\dot\varphi \cos\theta + \dot\psi)\, \underline{k}$$

$$\underline{\omega} = \dot\varphi\, \underline{e}_3 + \dot\theta\, \underline{u} + \dot\psi\, \underline{k}$$

PLU per un rigido

$$\delta \underline{x}_p = \frac{d\underline{x}_p}{d\tau}\,\delta\tau = \frac{d\underline{x}_0}{d\tau}\,\delta\tau + \underline{\omega}\,\delta\tau \wedge (\underline{x}_p - \underline{x}_0)$$

$\underline{x}_p(\tau)$ $\quad \tau \to \tau + \delta\tau \quad \Rightarrow \quad \begin{matrix} x \to x + \delta x \\ y \to y + \delta y \end{matrix}$ }

$$\frac{d\underline{x}_0}{d\tau}\,\delta\tau = \delta\underline{x}_0$$

$$\underline{\omega}(\tau)\,\delta\tau = \underline{\chi}$$

$$\underline{\chi} = \dot\varphi\,\delta\tau\,\underline{e}_3 + \dot\theta\,\delta\tau\,\underline{m} + \dot\psi\,\delta\tau\,\underline{k}$$

$$= \underline{\delta\varphi}\,\underline{e}_3 + \underline{\delta\theta}\,\underline{m} + \underline{\delta\psi}\,\underline{k}$$

Allora : $\quad \delta\underline{x}_p = \delta\underline{x}_0 + \underline{\chi} \wedge (\underline{x}_p - \underline{x}_0)$

Quindi :

$$LV = \sum_{P \in R}{}' \underline{F}_p^{\,e} \cdot \delta\underline{x}_p$$

$$= \sum_{P \in R} \underline{F}_P' \cdot \delta \underline{x}_0 + \sum_{P \in R} \underline{F}_P^{\circ} \cdot \underline{\chi} \wedge (\underline{x}_P - \underline{x}_0)$$

$$= \delta \underline{x}_0 \cdot \sum_{P \in R}' \underline{F}_P^{\circ} + \sum_{P \in R} (\underline{\chi} \wedge (\underline{x}_P - \underline{x}_0)) \cdot \underline{F}_P$$

IDENTITÀ $\quad \underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \wedge \underline{c}$

$$= \delta \underline{x}_0 \cdot \sum_{P \in R} \underline{F}_P^{\circ} + \underline{\chi} \cdot \left(\sum_{P \in R} (\underline{x}_P - \underline{x}_0) \wedge \underline{F}_P^{\circ} \right)$$

$$= \delta \underline{x}_0 \cdot \underline{R} + \underline{\chi} \cdot \underline{M}(O)$$

$$\underline{R} = \sum_{P \in R}' \underline{F}_P^{\circ} \qquad \underline{M}(O) = \sum_{P \in R}' (\underline{x}_P - \underline{x}_0) \wedge \underline{F}_P^{\circ}$$

risultante delle forze

momento risultante delle forze