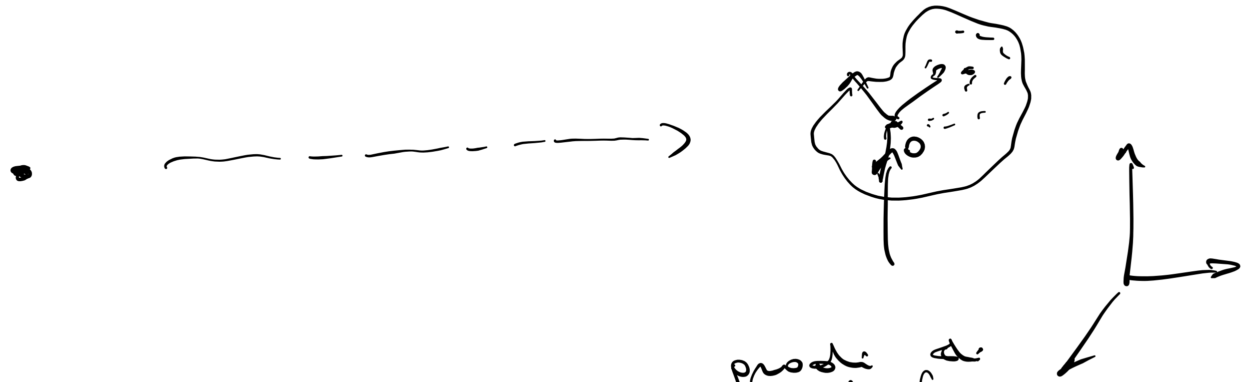
MECCANICA RAZIONALE

CONFIGURAZIONI



gradi di libertà

coordinate libere

$\underline{q} = (q_1, \dots, q_\ell)$

$$PLV = \sum_{B \in R} \underline{F}_B \cdot \delta \underline{x}_B$$

$$= \sum_{i=1}^{\ell} Q_i \, \delta q_i \le 0$$

$\boxed{= 0}$

$\le$ per $\delta \underline{x}_B$

$\le$ per $-\delta \underline{x}_B$

$$\underline{F}_B \cdot \delta \underline{x}_B \le 0 \qquad \underline{F}_B \cdot (-\delta \underline{x}_B) \le 0$$

→ δq_i ⇒ 3D rigido libero
angoli di Eulero
& Teo Poisson

$$\delta \underline{x}_P = \delta \underline{x}_O + \underline{\chi} \wedge (\underline{x}_P - \underline{x}_O)$$

$\delta\varphi \quad \underline{e}_3$

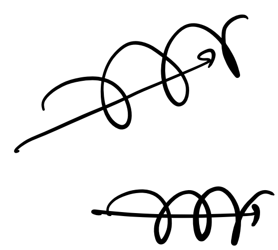
$\delta\theta \quad \underline{n}$

$\delta\psi \quad \underline{k}$

$$\frac{d\underline{x}_P}{dt} = \frac{d\underline{x}_O}{dt} + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$$

→ campo velocità

→ atto di moto



→ Forze

→ Equazioni cardinali statica

$\underline{R}$, $\underline{M}(O)$

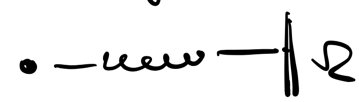
CAMPI DI FORZE &

FORZE CONSERVATIVE

Esempi

Forza elastica : $\underline{F} = -c\,(\underline{x} - \underline{x}_\Omega)$

rigida



Forza viscosa : $\underline{F} = -\lambda\,\underline{v}$

$\lambda \geq 0$

Forza di Lorentz $\underline{F} = \frac{q}{c}\,\underline{v} \wedge \underline{B}$

q carica elettrica

Forza gravitazionale

$$\underline{F}_P = -\gamma \frac{m_P\, m_\Omega}{\|\underline{x}_P - \underline{x}_\Omega\|^2} \,\mathrm{vers}\,(\underline{x}_P - \underline{x}_\Omega)$$

Le proprietà intrinseche dei corpi P e Ω.

La massa m_Ω crea un campo

di forze: ogni punto dello spazio circondante Ω ha la proprietà che se noi mettiamo un corpo di massa m_p in quel punto, questo risente di una forza $\underline{F}_p = -\gamma \frac{m_p m_\Omega}{\| \underline{x}_p - \underline{x}_\Omega \|^2} \text{vers}(\underline{x}_p - \underline{x}_\Omega)$

Campo di forze

gravitazionale
elettrico
magnetico
⋮

ad ogni punto dello spazio associa un vettore

$\underline{F}_p$

→ <u>Forze posizionali</u>: forze che dipendono dalla posizione.

<u>Forze CONSERVATIVE</u>: l'energia meccanica complessiva si conserva.

$\underline{F}$ forza posizionale, agisce su un punto P in posizione $\underline{x}$. $\underline{F}$ è conservativa se il lavoro fatto per passare da

$\underline{x}_0$ configurazione iniziale

e

$\underline{x}_1$ configurazione finale

non dipende dalla traiettoria ma solo dalle posizioni $\underline{x}_0$ e $\underline{x}_1$

Def $\underline{F}$ è conservativa se esiste una funzione $V = V(\underline{x})$ tale che il lavoro di $\underline{F}$ per andare da $\underline{x}_0$ a $\underline{x}_1$ è dato da

$$-\left(V(\underline{x}_1) - V(\underline{x}_0) \right)$$

Per il PLV:

$\underline{x}_0 = \underline{x}_E$ configurazione di equilibrio

$\underline{x}_1 = \underline{x}_E + \delta\underline{x}$ spostamento virtuale fuori dall'equilibrio

Se $\underline{F}$ è conservativa:

$$LV = -\left[V(\underline{x}_E + \delta\underline{x}) - V(\underline{x}_E) \right]$$

è ≤ 0 $\forall\ \delta\underline{x}$ virtuale

$= 0$ $\forall\ \delta\underline{x}$ invertibile

se V è una funzione regolare

$$= -\left[\cancel{V(\underline{x}_E)} + \nabla V\Big|_{\underline{x}_E} \cdot \delta\underline{x} - \cancel{V(\underline{x}_E)} \right]$$

al primo ordine

$$\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right) \rightarrow \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2 + \ldots$$

$$= - \nabla V \Big|_{\underline{x}_i} \cdot \delta \underline{x} = - \underline{dV}$$

differenziale

Per spostamenti virtuali invertibili

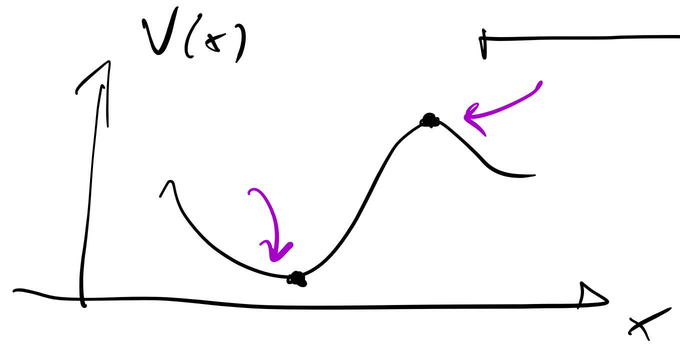
$LV = 0$

$\Rightarrow$ configurazioni di equilibrio

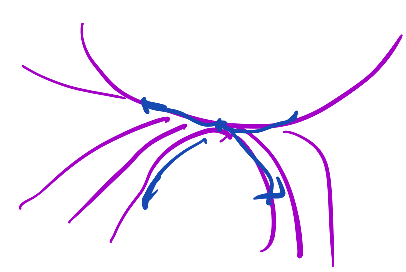
sono i punti di stazionarietà

di V $\longrightarrow$ $\dfrac{\partial V}{\partial x_i} = 0 \quad \forall i$

$PLV = 0 \longleftrightarrow dV = 0$



eq minimo di $V \longrightarrow$ eq. stabile.



↳ $\underline{F}$ conservative

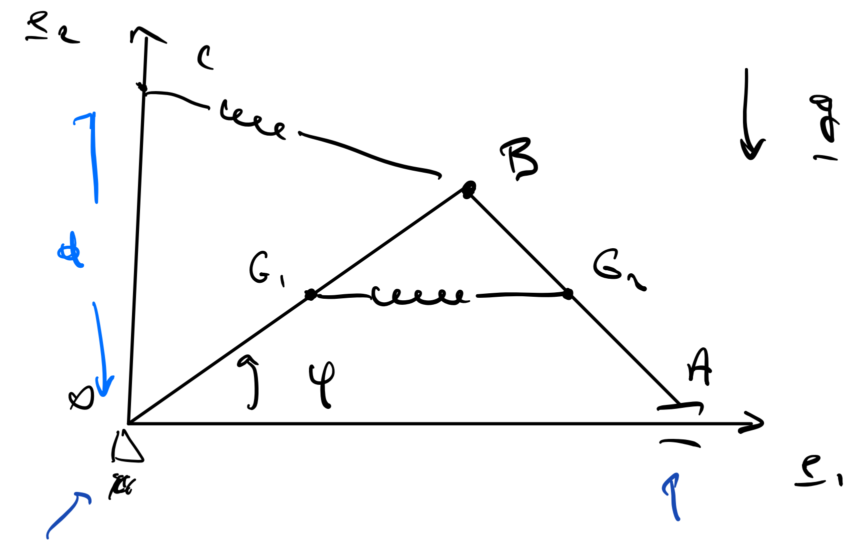
$\exists\ V$ (energia potenziale)

$$L.V. = -\nabla V\Big|_{\underline{x}_E} \cdot \delta\underline{x} = -dV$$

$$= \sum \underline{F}_\beta \cdot \delta x_\beta$$

$\boxed{dV=0}$ ← $\underline{q}$ tali che

Equilibrio di sistemi olonomi a 1 grado di libertà (con forze conservative)



$\overline{OB} = \overline{AB} = L$

omogenea

$OB \to m$

$AB \to M$

$LV = -dV$

- Peso :
$$V_1 = -m\,\underline{g}\cdot \underline{x}_{G_1} - M\,\underline{g}\cdot\underline{x}_{G_2}$$

$$LV = m\,\underline{g}\cdot\delta\underline{x}_{G_1} + M\,\underline{g}\cdot\delta\underline{x}_{G_2} = -dV_1$$

$-g\,\underline{e}_2$ $\qquad (-g\,\underline{e}_2)\cdot(\delta x_{G_1}\underline{e}_1 + \delta y_{G_1}\underline{e}_2)$

- Molla in B (esterna)

$$V_2 = \frac{c}{2}\,\|\underline{x}_B - \underline{x}_C\|^2$$

$\frac{c}{2}(x_B - x_C)^2$

$\frac{2}{2}c\,(x_B - x_C)\cdot$
$\delta(x_B - x_C)$

$$LV = -c\,(\underline{x}_B - \underline{x}_C)\cdot\delta\underline{x}_B$$

$$= -c\,(\underline{x}_B - \underline{x}_C)\cdot\delta(\underline{x}_B - \underline{x}_C)$$

$$= -dV_2$$

$\delta\underline{x}_C = 0$

- Molla in G_1 & G_2

$$V_3 = \frac{c}{2}\,\|\underline{x}_{G_1} - \underline{x}_{G_2}\|^2$$

$$LV = -c\,(\underline{x}_{G_1} - \underline{x}_{G_2})\cdot\delta\underline{x}_{G_1} +$$

$$+ c\,(\underline{x}_{G_1} - \underline{x}_{G_2})\cdot\delta\underline{x}_{G_2}$$

$$= - c\,(\underline{x}_{G_1} - \underline{x}_{G_2}) \cdot \delta(\underline{x}_{G_1} - \underline{x}_{G_2})$$

$$= - d V_3$$

Quindi $\quad V = V_1 + V_2 + V_3$

φ grado di libertà $\longrightarrow$ esprimo V

in funzione di φ.

$$V = m g y_{G_1} + M g y_{G_2} + \dots$$

$$\frac{c}{2} \| \underline{x}_B - \underline{x}_C \|^2 = \frac{c}{2} \left[(x_B \underline{e}_1 + y_B \underline{e}_2) - d \underline{e}_2 \right]^2$$

$$= \frac{c}{2} \left[x_B \underline{e}_1 + y_B \underline{e}_2 - d \underline{e}_2 \right] \cdot \left[x_B \underline{e}_1 + y_B \underline{e}_2 - d \underline{e}_2 \right]$$

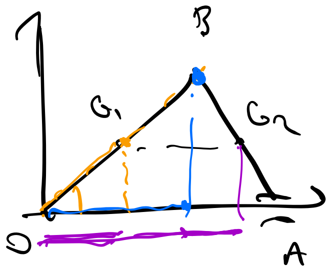
$$= \frac{c}{2} \left[x_B \underline{e}_1 + (y_B - d) \underline{e}_2 \right] \cdot \left[x_B \underline{e}_1 + (y_B - d) \underline{e}_2 \right]$$

$\underline{e}_1 \cdot \underline{e}_2 = 0 \qquad \underline{e}_1 \cdot \underline{e}_1 = 1$

$$= \frac{c}{2} \left[x_B^2 + (y_B - d)^2 \right]$$

$$V = m g\, y_{G_1} + M g\, y_{G_2} + \frac{c}{2}\left[x_B^2 + (y_B - d)^2\right]$$

$$+ \frac{c}{2}\left(x_{G_1} - x_{G_2}\right)^2$$



$$= (m+M)\, g\, \frac{L}{2}\sin\varphi + \frac{c}{2}\left[L^2\cos^2\varphi +\right.$$

$$\left. + \left(L\sin\varphi - d\right)^2\right] + \frac{c}{2} L^2 \cos^2\varphi$$

$$= (m+M)\, g\, \frac{L}{2}\sin\varphi + \left[\frac{c}{2} L^2\cos^2\varphi\right] +$$

$$+ \left[\frac{c}{2} L^2 \sin^2\varphi\right] + \left[\frac{c}{2} d^2\right] - \frac{c}{2}\, 2L\sin\varphi\, d +$$

$$+ \left[\frac{c}{2} L^2\cos^2\varphi\right]$$

$$= \left[(m+M)\, g\, \frac{L}{2} - cLd\right]\sin\varphi +$$

$$+ \frac{c}{2} L^2\cos^2\varphi + \cancel{\left[\frac{c}{2} L^2\sin^2\varphi + \frac{c}{2} d^2 + \frac{c}{2} L^2\cos^2\varphi\right]}$$

$$\sin^2\varphi + \cos^2\varphi = 1$$

$$\frac{c}{2}L^3 + \frac{c}{2}d^2$$

$$V(\varphi) = \left[(m+M)g\frac{L}{2} - cLd\right]\sin\varphi +$$
$$+ \frac{c}{2}L^2\cos^2\varphi$$

Equilibrio :

$$\frac{d}{d\varphi}V(\varphi) = \left[(m+M)g\frac{L}{2} - cLd\right]\cos\varphi +$$
$$- cL^2 \sin\varphi\cos\varphi$$

$$= cL^2\left\{\frac{1}{cL}\left[(m+M)\frac{g}{2} - cd\right] - \sin\varphi\right\}\cos\varphi$$

$$\frac{1}{cL}\left[(m+M)\frac{g}{2} - cd\right] =: \gamma$$

$$= cL^2(\gamma - \sin\varphi)\cos\varphi = 0$$

$$-\pi < \varphi \le \pi$$

1) Se $|\gamma| > 1 \Rightarrow \gamma - \sin\varphi \neq 0$

allora $\frac{dV}{d\varphi} = 0$ solo per $\cos\varphi = 0$

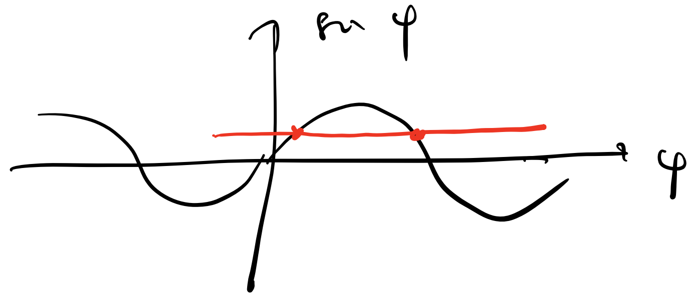
$\Rightarrow \quad \varphi = \frac{\pi}{2}, \quad \varphi = -\frac{\pi}{2}$

2) Se $|\gamma| < 1 \Rightarrow \frac{dV}{d\varphi} = 0 \quad \begin{cases} \cos\varphi = 0 \\ \sin\varphi = \gamma \end{cases}$

abbiamo 4 configurazioni di equilibrio

$\varphi = \frac{\pi}{2}, \quad \varphi = -\frac{\pi}{2} \qquad \varphi = \varphi_1 \qquad \varphi = \varphi_2$

$\sin\varphi_1 = \gamma \qquad \sin\varphi_2 = \gamma$



Stabilità:

$$\frac{d^2V}{d\varphi^2} = \frac{d}{d\varphi}\left[cL^2(\gamma - \sin\varphi)\cos\varphi\right]$$

$$= cL^2\left[-\sin\varphi(\gamma - \sin\varphi) - \cos^2\varphi\right]$$

1) $|\gamma| > 1 \rightarrow$ eq. $\varphi = \pm\frac{\pi}{2}$

$$V''\left(\frac{\pi}{2}\right) = -cL^2(\gamma - 1) = cL^2(1-\gamma)$$

$$V''\left(-\frac{\pi}{2}\right) = cL^2(1+\gamma)$$

Se $\gamma > 1$ $\quad V''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \frac{\pi}{2}$ INSTABILE

$\quad V''\left(-\frac{\pi}{2}\right) > 0 \Rightarrow -\frac{\pi}{2}$ STABILE

Se $\gamma < -1$ $\quad V''\left(\frac{\pi}{2}\right) > 0 \Rightarrow \frac{\pi}{2}$ STABILE

$\quad V''\left(-\frac{\pi}{2}\right) < 0 \Rightarrow -\frac{\pi}{2}$ INSTABILE

2) Se $|\gamma| < 1 \Rightarrow$ le configurazioni di equilibrio $\pm\frac{\pi}{2}$, φ_1, φ_2

$$V''(\varphi) = cL^2\left[-\sin\varphi(\gamma - \sin\varphi) - \cos^2\varphi\right]$$

$V''\left(\frac{\pi}{2}\right) = cL^2(1-\gamma) > 0 \rightarrow$ STABILE

$V''\left(-\frac{\pi}{2}\right) = cL^2(1+\gamma) > 0 \rightarrow$ stabile

$$V''(\varphi_1) = cL^2\left[-\sin\varphi_1(\underbrace{\gamma - \sin\varphi_1}_{\gamma}) + \right.$$

$$\left. - \cos^2\varphi_1\right] = -cL^2\cos^2\varphi_1 < 0$$

INSTABILE

$$V'(\varphi_2) = cL^2\left[-\sin\varphi_2(\gamma-\sin\varphi_2)+\right.$$
$$\left.-\cos^2\varphi_2\right] = -cL^2\cos^2\varphi_2 < 0$$

INSTABILE

Riassumiamo

$\gamma < -1$	$-1 < \gamma < 1$	$\gamma > 1$
$\frac{\pi}{2}$ stabile	$\frac{\pi}{2}$ stabile	$\frac{\pi}{2}$ instabile
$-\frac{\pi}{2}$ instabile	$-\frac{\pi}{2}$ stabile	$-\frac{\pi}{2}$ stabile
	φ_1 instabile	
	φ_2 instabile	