MECCANICA RAZIONALE

Statica & Dinamica del corpo rigido

Configurazione del nostro sistema

↳ coordinate libere ed a gli angoli di Eulero o formule di Poisson

↳ spostamenti virtuali

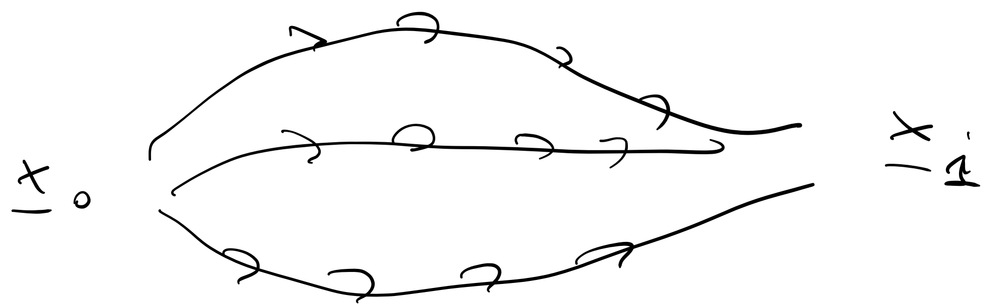
PLV → $\sum_{B \in S} \boxed{\underline{F}_B^a \cdot} \delta \underline{x}_B \leftarrow$

δq_i

$= \sum_i Q_i \, \delta q_i \leftarrow$

→ forze conservative (posizionali)

→ l'energia meccanica si conserva.



$\exists\, V$ t.c. lavoro può essere scritto calcolando $V(\underline{x})$

$$L = -\left(V(\underline{x}_1) - V(\underline{x}_0) \right)$$

$$LV = -\left(V(\underline{x}_E + \delta \underline{x}) - V(\underline{x}_E) \right)$$

$$\underline{x}_E \xrightarrow{\delta \underline{x}} \underline{x}_E + \delta \underline{x}$$

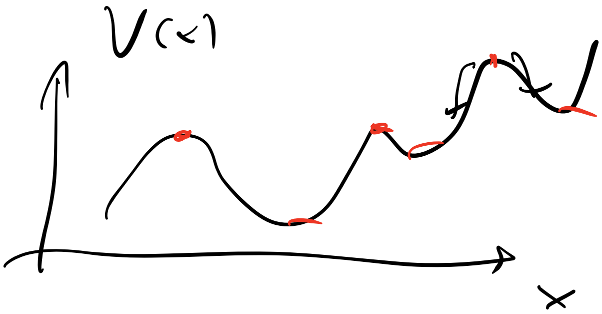
$$= -\nabla V \Big|_{\underline{x}_E} \cdot \delta \underline{x} = -dV$$

$$dV(x_1, x_2) = \frac{\partial V}{\partial x_1} dx_1 + \frac{\partial V}{\partial x_2} dx_2$$

$$\delta x_1 \qquad\qquad \delta x_2$$

Solo forze conservative $\longrightarrow V$

Equilibrio $\longrightarrow$ punti di stazionarietà di V.



q grado di libertà $\longrightarrow$ $V = V(q)$

$$LV = -dV = -\frac{dV}{dq}\,\delta q = Q\,\delta q$$

differenziale

definizione di forza generalizzata

$$Q = -\frac{dV}{dq} \qquad Q = 0 \iff \frac{dV}{dq} = 0$$

l gradi di libertà : $\underline{q} = (q_1, \dots, q_l)$

differenziale

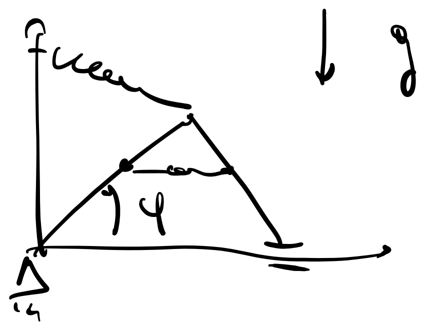
$$LV = -dV = -\left[\frac{\partial V}{\partial q_1}\delta q_1 + \dots + \frac{\partial V}{\partial q_l}\delta q_l\right]$$

def di Q

$$= Q_1\,\delta q_1 + \dots + Q_l\,\delta q_l$$

$$\hookrightarrow \boxed{Q_i = -\frac{\partial V}{\partial q_i} \qquad \forall\, i = 1,\dots,\ell}$$

Esempio



$$V = V(\varphi)$$

$$- \frac{dV}{d\varphi} = 0 \; : \; \varphi_E$$

$$- \left.\frac{d^2V}{d\varphi^2}\right|_{\varphi_E} \text{ è } \gtrless 0 \,?$$

$$\frac{dV}{d\varphi} = cL^2 \underbrace{(\gamma - \sin\varphi)}_{=0} \underbrace{\cos\varphi}_{=0} = 0$$

Tutti i φ tali che $\frac{dV}{d\varphi} = 0$ sono φ_E conf di equilibrio

- L'esistenza di configurazioni di equilibrio dipende da γ.

$\rightarrow$ Se $|\gamma| > 1$ allora $\sin\varphi = \gamma$

non ha soluzioni

Solo soluzioni per $\cos\varphi = 0$

$\varphi_E = \pm \frac{\pi}{2}$

$\rightarrow$ Se $|\gamma| < 1$ abbiamo soluzioni

$\rightarrow \cos\varphi = 0 \qquad \varphi_E = \pm \frac{\pi}{2}$

$\rightarrow \sin\varphi = \gamma \qquad \varphi_E \rightarrow \arcsin\gamma$

Abbiamo trovato le configurazioni

di equilibrio

Per trovare la stabilità:

$$\frac{d^2V}{d\varphi^2} = cL^2\left[-\sin\varphi\,(\gamma - \sin\varphi) - \cos^2\varphi\right]$$

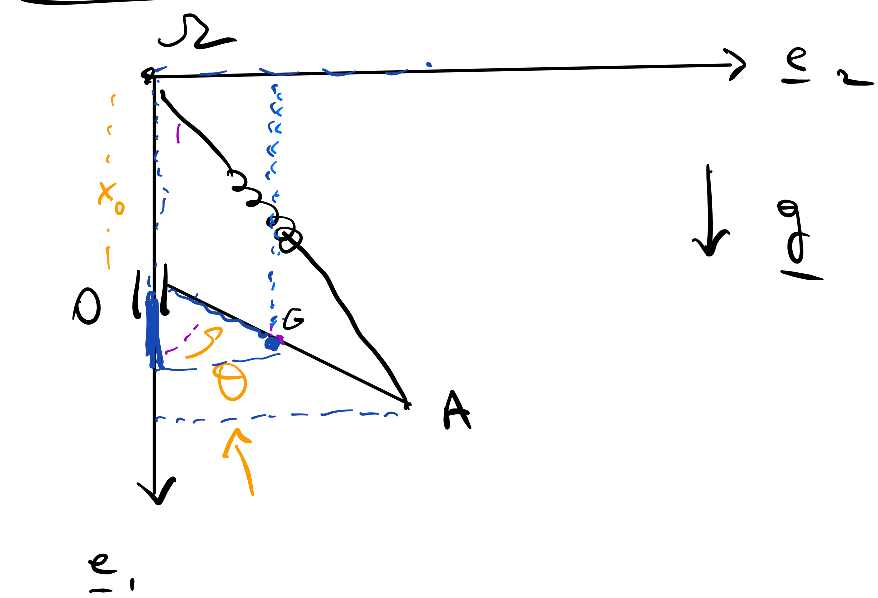
andiamo a calcolare $\frac{d^2V}{d\varphi^2}$ su φ_E

ad esempio $\varphi_E = \frac{\pi}{2}$

$$\frac{d^2V}{d\varphi^2} = cL^2\left[-\sin\frac{\pi}{2}\left(\gamma - \sin\frac{\pi}{2}\right) - \cos^2\frac{\pi}{2}\right]$$

$$= cL^2(1-\gamma) \quad \begin{matrix} > 0 \ ? \\ < 0 \ ? \end{matrix}$$

Esempio



OA = L

omogenea

massa m

Sistema a due gradi di libertà (x_0, θ)

$$V = -m\underline{g} \cdot \underline{x}_G + \frac{c}{2}\|\underline{x}_A\|^2$$

$$= -mg\left[x_0 + \frac{L}{2}\cos\theta\right] +$$

$$+ \frac{c}{2}\left[(x_0 + L\cos\theta)^2 + L^2\sin^2\theta\right]$$

$$\underline{x}_A = (x_0 + L\cos\theta)\underline{e}_1 + (L\sin\theta)\underline{e}_2$$

$$\|\underline{x}_A\|^2 = (x_0 + L\cos\theta)^2 + (L^2\sin^2\theta)$$

$$V = -mg\frac{L}{2}\cos\theta - mg\,x_0 + \frac{c}{2}\left(x_0^2 + 2Lx_0\cos\theta\right)$$
$$+\ \frac{c}{2}\left[L^2\cos^2\theta + L^2\sin^2\theta\right]$$

Forze generalizzate

$$Q_{x_0} = -\frac{\partial V}{\partial x_0} = -\left(-mg + c\,x_0 + cL\cos\theta\right)$$

$$Q_\theta = -\frac{\partial V}{\partial \theta} = -\left(\frac{mg}{2} - c\,x_0\right)L\sin\theta$$

$$\begin{cases} Q_{x_0} = 0 \\ Q_\theta = 0 \end{cases} \Rightarrow \begin{cases} -mg + c\,x_0 + cL\cos\theta = 0 \\ \left(\frac{mg}{2} - c\,x_0\right)L\sin\theta = 0 \end{cases}$$

Risolviamo: prendiamo

$$\left(\frac{mg}{2} - c\,x_0\right)L\sin\theta = 0$$

quando
$$\sin\theta = 0$$
$$x_0 = \frac{mg}{2c}$$

Sostituiamo nella prima eq $(-\pi < \theta \leq \pi)$

- se $\sin\theta = 0 \Rightarrow \theta = 0, \pi$

  nella prima eq.

  $$-mg + c x_0 + cL \cos\theta = 0$$

$$\theta = 0 \qquad -mg + c x_0 + cL = 0 \quad \rightarrow \quad x_0 = \frac{mg}{c} - L$$

$$\theta = \pi \qquad -mg + c x_0 - cL = 0 \qquad x_0 = \frac{mg}{c} + L$$

- se $x_0 = \dfrac{mg}{2c}$

$$-mg + c\left(\frac{mg}{2c}\right) + cL\cos\theta = 0$$

$$\cos\theta = \frac{mg}{2cL}$$ accettabile solo se $\dfrac{mg}{2cL} < 1$

$\forall\ \gamma \longleftrightarrow -1 < \gamma < 1$

$\rightarrow \dfrac{mg}{2cL} > 0$

Quindi :

Se $\frac{mg}{2cL} > 1 \Rightarrow$ abbiamo due configurazioni di equilibrio

$$\underline{q}_1 = \left(x_0 = \frac{mg}{c} - L \,,\ \theta = 0 \right)$$

$$\underline{q}_2 = \left(x_0 = \frac{mg}{c} + L \,,\ \theta = \pi \right)$$

Se $\frac{mg}{2cL} < 1 \rightarrow$ abbiamo 4 configurazioni di equilibrio

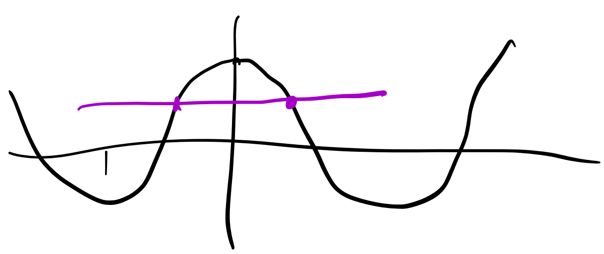
$\underline{q}_1 \,,\ \underline{q}_2$

$$\underline{q}_3 = \left(x_0 = \frac{mg}{2c} \,,\ \theta_1 = \arccos \frac{mg}{2cL} \right)$$

$\cos\theta_1 = \frac{mg}{2cL}$

$$\underline{q}_4 = \left(x_0 = \frac{mg}{2c} \,,\ \theta_2 = -\arccos \frac{mg}{2cL} \right)$$

$\cos\theta_2 = \frac{mg}{2cL}$



Funzione $V = V(x_0, \theta)$

$$\begin{cases} \dfrac{\partial V}{\partial x_0} = 0 \quad \leftarrow Q_{x_0} \\[2ex] \dfrac{\partial V}{\partial \theta} = 0 \quad \leftarrow Q_{\theta} \end{cases}$$

punti di stazionarietà

$$\text{Hess } V = \begin{pmatrix} \dfrac{\partial^2 V}{\partial x_0^2} & \dfrac{\partial^2 V}{\partial x_0 \partial \theta} \\[2ex] \dfrac{\partial^2 V}{\partial x_0 \partial \theta} & \dfrac{\partial^2 V}{\partial \theta^2} \end{pmatrix}$$

$$\frac{\partial V}{\partial x_0} = -mg + c\,\underline{x_0} + cL\,\underline{\cos\theta}$$

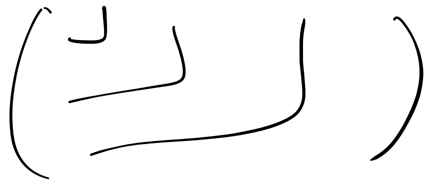
$$\frac{\partial V}{\partial \theta} = \left(\frac{mg}{2} - c x_0\right) L \sin\theta$$

$$\frac{\partial^2 V}{\partial x_0^2} = c \qquad\qquad \frac{\partial^2 V}{\partial x_0 \partial \theta} = -cL\sin\theta$$

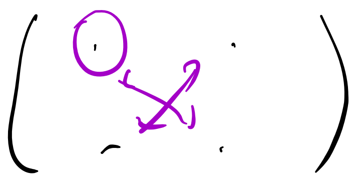
$$\frac{\partial^2 V}{\partial \theta^2} = \left(\frac{mg}{2} - c x_0\right) L \cos\theta$$

$$\text{Hess } V = \begin{pmatrix} \boxed{c} & -cL\sin\theta \\[2ex] -cL\sin\theta & \left(\dfrac{mg}{2} - c x_0\right) L\cos\theta \end{pmatrix}$$

Criterio di Sylvester: una matrice simmetrica è definita positiva se e solo se tutti i suoi minori principali sono positivi



Matrice 2×2



def + $\longrightarrow$ minimo locale
$\rightarrow$ stabile

def − $\longrightarrow$ max
indef $\longrightarrow$ sella
] instabile

$$\det \mathrm{Hess}\, V = c\left(\frac{mg}{2} - c x_0\right) L\cos\theta +$$
$$- c^2 L^2 \sin^2\theta$$

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in q_1 : $\det \mathrm{Hess}\, V\Big|_{q_1} =$

$$= cL\left(\frac{mg}{2} - c\left[\frac{mg}{c} - L\right]\right)$$

$$= cL\left(cL - \frac{mg}{2}\right) > 0 \; ?$$

$$\frac{mg}{2cL} < 1 \qquad cL - \frac{mg}{2} > 0$$

$\underline{q}_1$ è stabile solo se $\frac{mg}{2cL} < 1$.

$\underline{q}_2$: $\det \mathrm{Hess}\, V \big|_{\underline{q}_2} =$

$$= cL\left(cL + \frac{mg}{2}\right) > 0 \quad \text{sempre}$$

$\underline{q}_2$ è sempre stabile

$\underline{q}_3, \underline{q}_4$ $\det \mathrm{Hess}\, V \big|_{\underline{q}_3, \underline{q}_4} =$

$$= -c^2 L^2 \sin^2 \theta_F \quad < 0$$

θ_1, θ_2

sempre instabili