SISTEMI DINAMICI

Biforcazioni per sistemi dinamici
1-dimensionali

$\dot{x} = f(x) \quad \rightarrow \quad \dot{x} = f(x; \mu)$

$\uparrow$ parametro

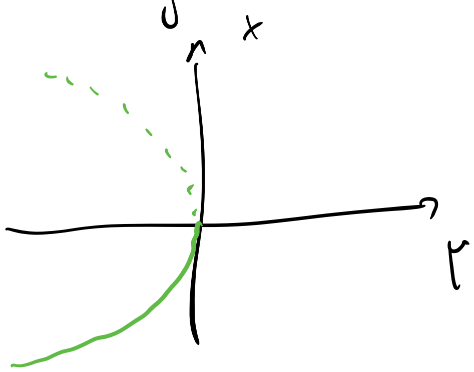
Punti critici $f(x^*; \mu^*) = 0$

Se $\left.\frac{\partial f}{\partial x}\right|(x^*; \mu^*) \neq 0 \rightarrow$ iperbolico strutturalmente stabile

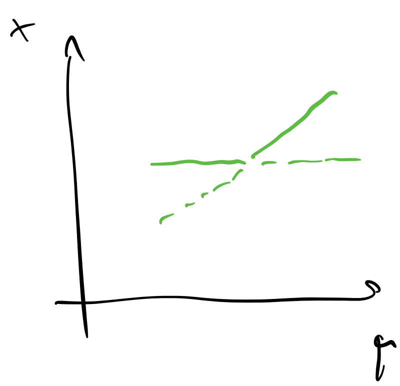
Invece se $\frac{\partial f}{\partial x} = 0$

Biforcazioni:

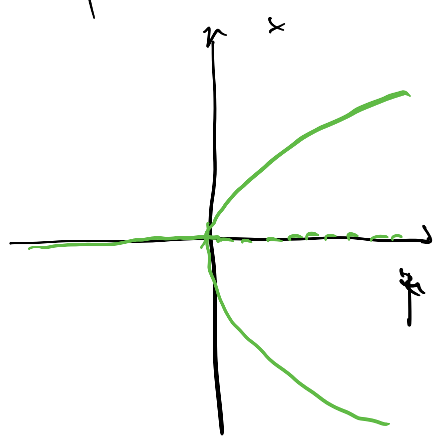
Tangente



Transcritica



pitchfork



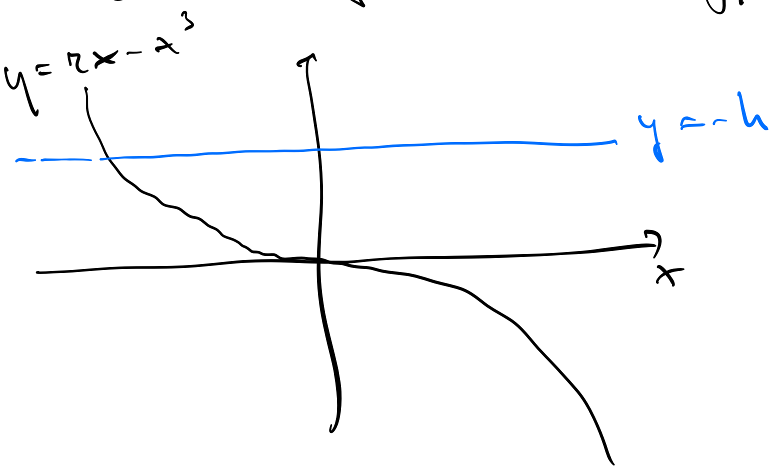
Biforcazioni imperfette

$$\dot{x} = h + r x - x^3$$

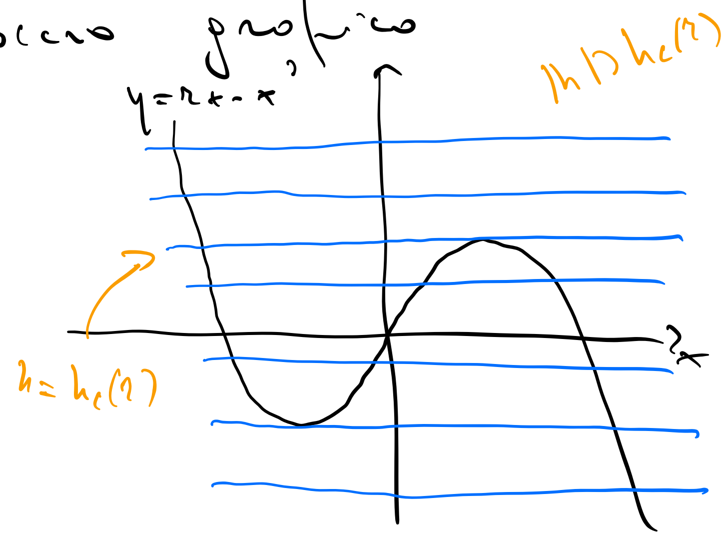
$h = 0 \rightarrow$ biforcazione pitchfork
$(x \longrightarrow -x)$

$h \neq 0 \rightarrow$ simmetria è "rotta"

Punti fissi: approccio grafico



$r \leq 0$



$r > 0$

$|h| < h_c(r)$

max della cubica:

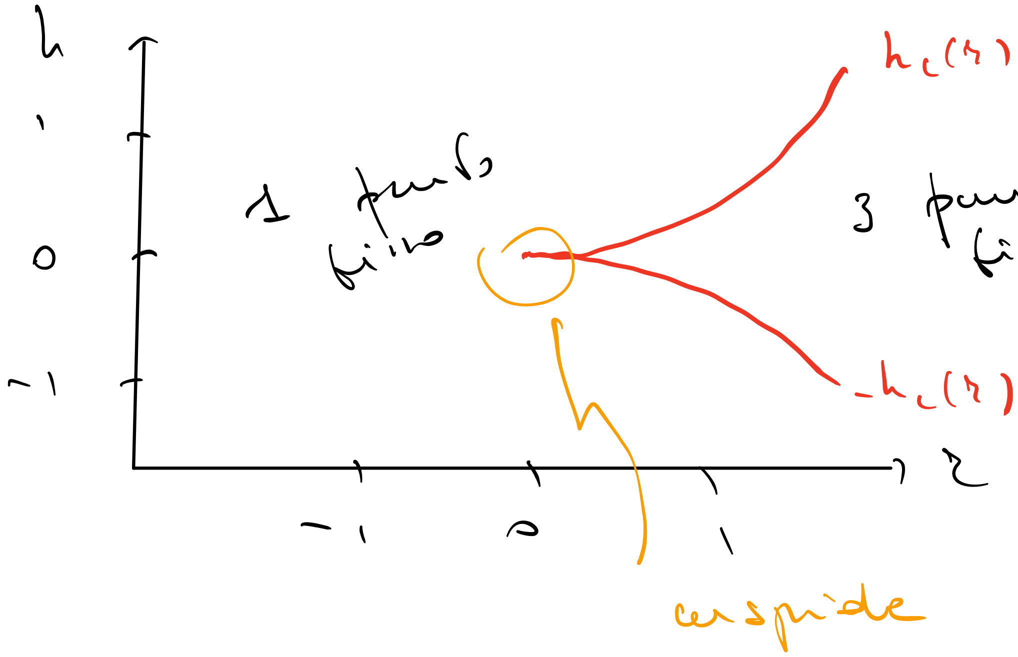
$$\frac{d}{dx}(rx - x^3) = r - 3x^2 = 0 \longrightarrow x_{max} = \sqrt{\frac{r}{3}}$$

Al max: $r x_{max} - (x_{max})^3 = r\sqrt{\frac{r}{3}} - \frac{r}{3}\sqrt{\frac{r}{3}} =$

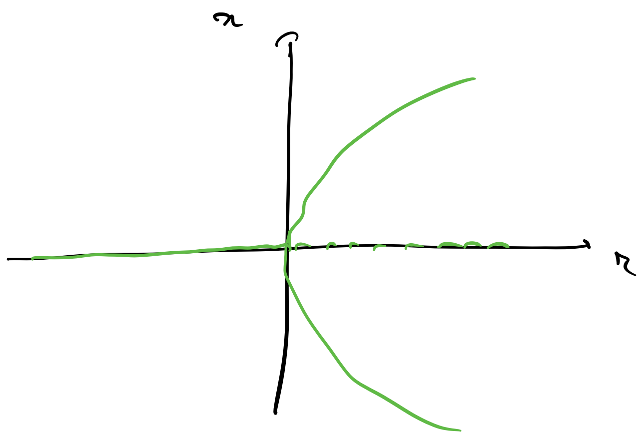
$$= \frac{2}{3} r \sqrt{\frac{r}{3}}$$

- 3 punti critici per $|h| < h_c(r)$
- 1 punto critico per $|h| > h_c(r)$

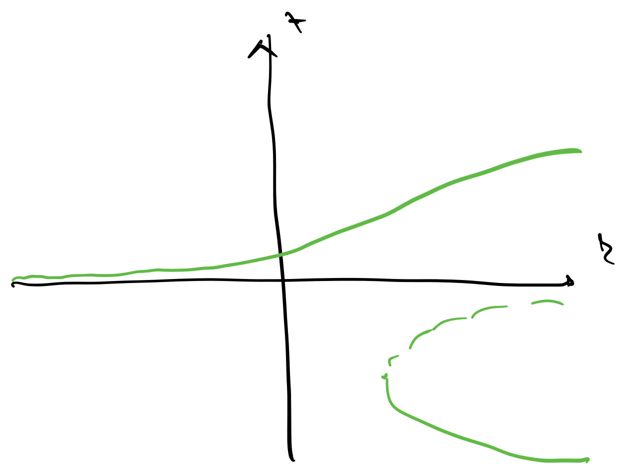
dove $h_c(r) = \frac{2}{3} r \sqrt{\frac{r}{3}}$



h = fissato

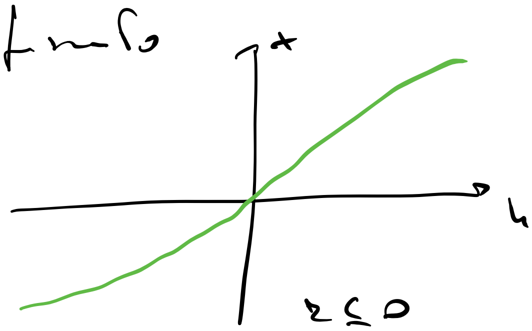


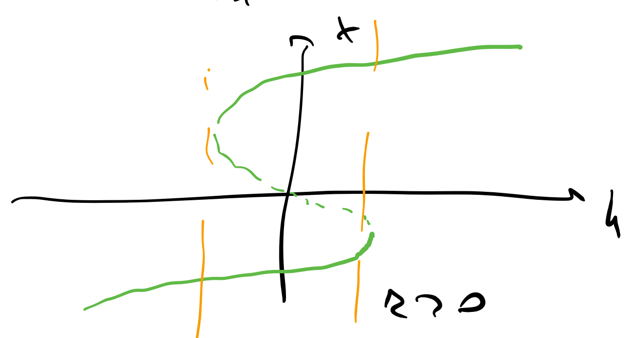
$h = 0$

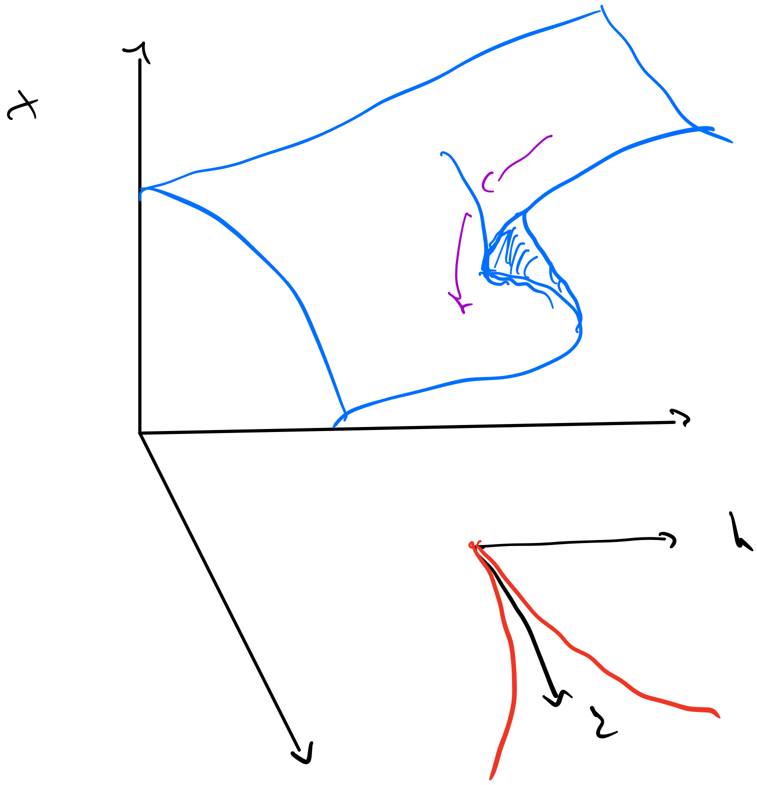


$h \neq 0$

1 punto







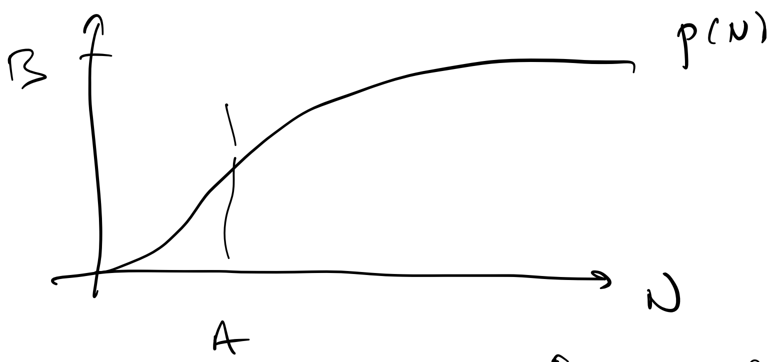
$$\dot{x} = h + rx - x^3$$

"catastrofe di tipo cuspide"

Curve di biforcazione

$$\dot{N} = \underbrace{RN\left(1 - \frac{N}{K}\right)}_{\text{modello logistico}} - p(N)$$

$\uparrow$ predatori



$$p(N) = \frac{BN^2}{A^2 + N^2}$$

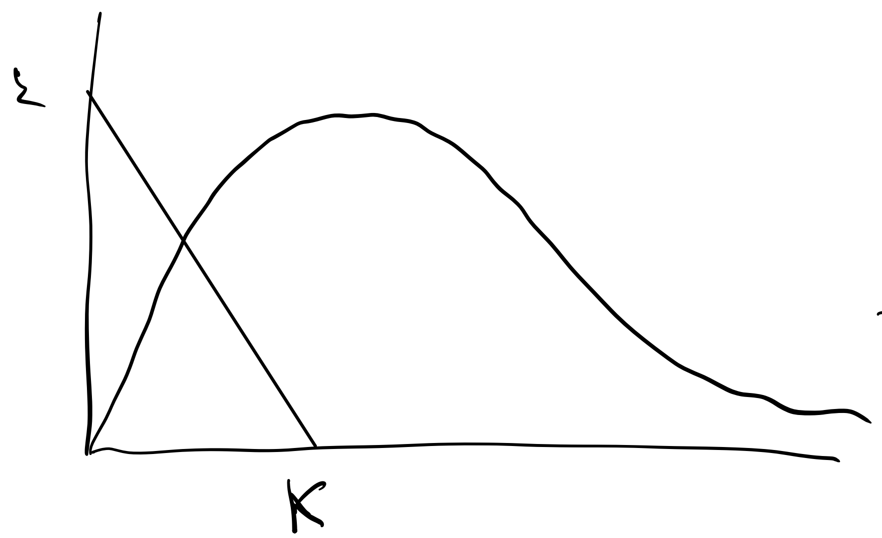
poche prede
$\rightarrow$ pochi predatori

molte prede
$\rightarrow$ molti predatori

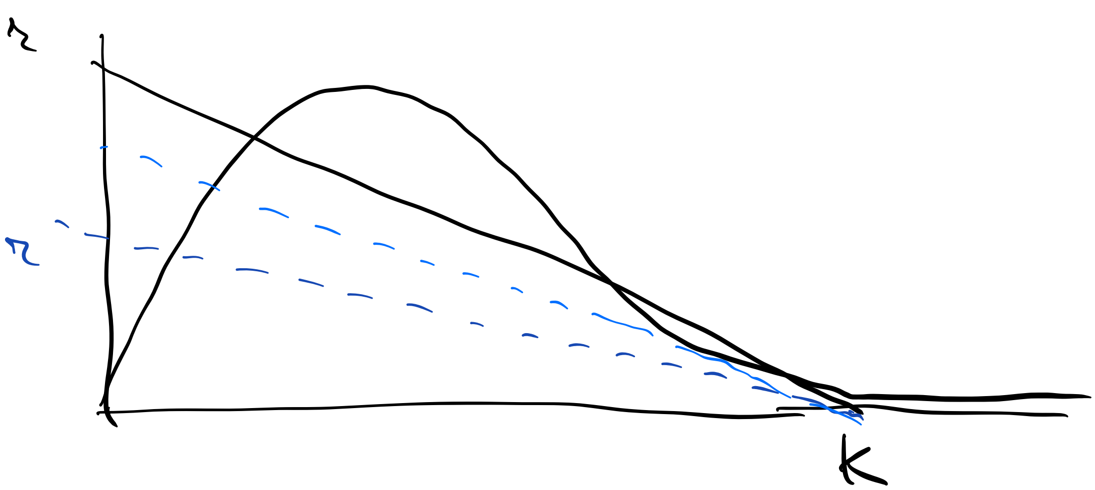
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

- punto critico $x^* = 0$

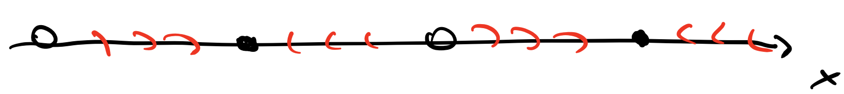
- $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$



K piccolo $\longrightarrow$ solo un punto critico



Nel caso in cui obbiamo 3 punti fissi



lontani dalla biforcazione tangente

Curva di biforcazione:

cerchiamo le condizioni per cui

$r\left(1-\frac{x}{K}\right)$ interseca $\frac{x}{1+x^2}$

tangenzialmente

1. $$r\left(1-\frac{x}{K}\right)=\frac{x}{1+x^2}$$

2. $$\frac{d}{dx}\left[r\left(1-\frac{x}{K}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right]$$

2. $\Rightarrow$ $$-\frac{r}{K}=\frac{1}{1+x^2}-\frac{x(2x)}{(1+x^2)^2}$$

$$=\frac{1+x^2-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

1. $\Rightarrow$ $$r=\frac{r}{K}x+\frac{x}{1+x^2}$$

$$=\frac{x^2-1}{(1+x^2)^2}x+\frac{x}{1+x^2}=$$

$$= \frac{(x^2-1)\,x + x\,(x^2+1)}{(1+x^2)^2} = \frac{2x^3}{(1+x^2)^2}$$

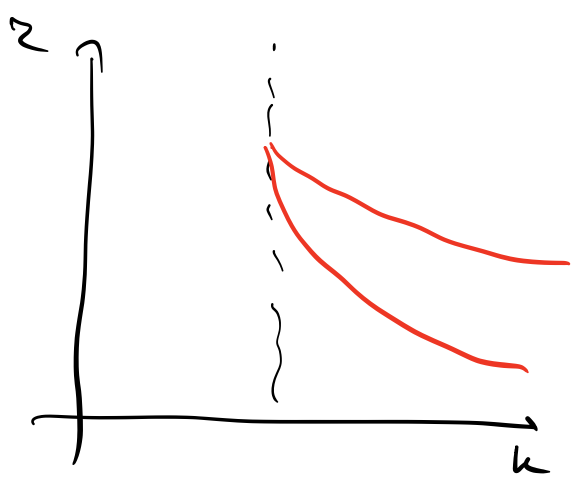
$$\hookrightarrow \quad r = \frac{2x^3}{(1+x^2)^2}$$

$$\frac{1}{k} = -\frac{1}{r}\,\frac{1-x^2}{(1+x^2)^2} = -\frac{(1+x^2)^2}{2x^3}\,\frac{1-x^2}{(1+x^2)^2}$$

$$\hookrightarrow \quad k = \frac{2x^3}{x^2-1}$$

Descrizione parametrica delle curve di biforcazione $\big(k(x),\ r(x)\big)$

$$\begin{cases} k(x) = \dfrac{2x^3}{x^2-1} \\[2ex] r(x) = \dfrac{2x^3}{(1+x^2)^2} \end{cases}$$



$x > 1$

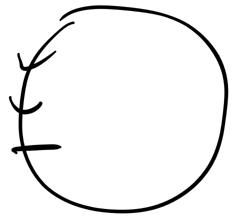
$k > 0$

Flussi sul cerchio

$\dot{x} = f(x)$ su $\mathbb{R}$ $\longrightarrow$ $\dot{\theta} = f(\theta)$

θ valori in S^1

$f(\theta + 2\pi) = f(\theta)$

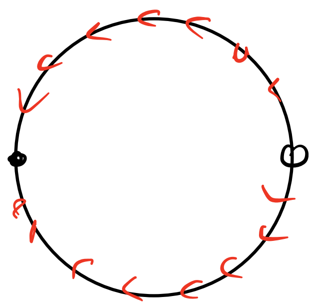


$$\dot{\theta} = \sin\theta$$

$\theta^* = 0$

$\theta^* = \pi$

$\theta^* = \pi$



$\theta^* = 0$

semicerchio sup $\sin\theta > 0$

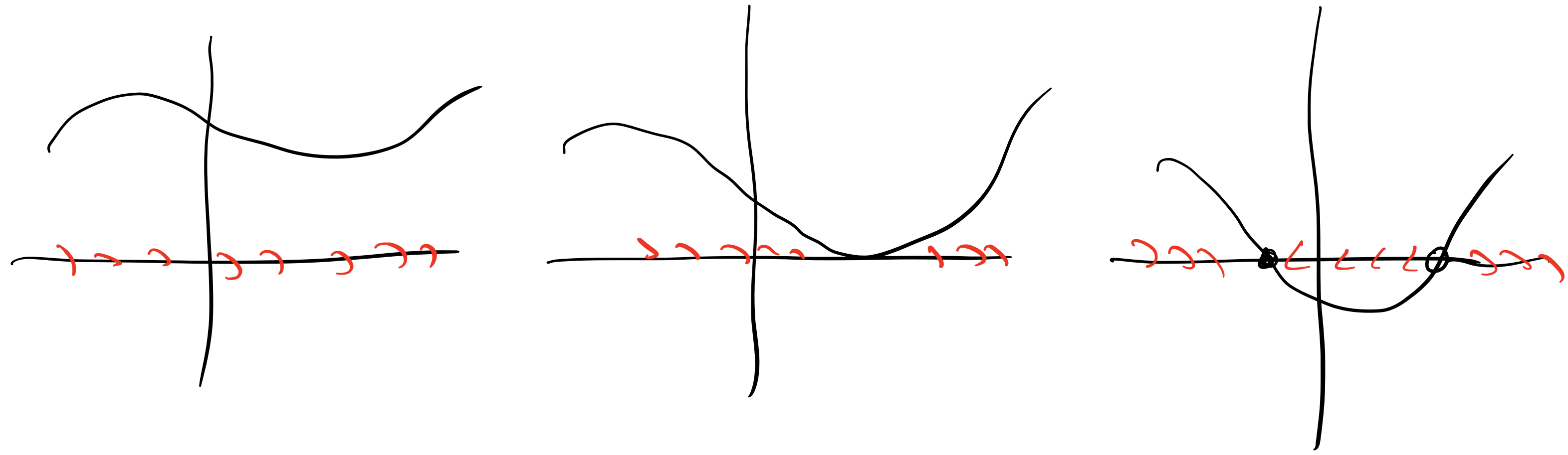
semicerchio inf $\sin\theta < 0$

differenza $\longrightarrow$ possono esistere soluzioni periodiche

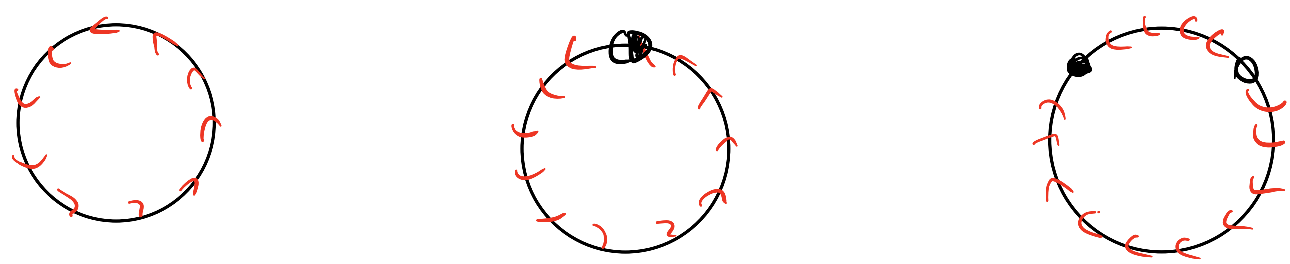
Esempio: $\dot{\theta} = \omega - a\sin\theta$

$\uparrow$ costante

$a = 0$ $\qquad \theta(t) = \omega t + \theta_0$



$a < \omega$ $\qquad$ $a = \omega$ $\qquad$ $a > \omega$



↓

Per $a < \omega$ ci sono oscillazioni

di periodo

$$T = \int d\tau = \int_0^{2\pi} \frac{d\tau}{d\theta}\, d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a \sin\theta}$$

$$= \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

$\Gamma \qquad z = e^{i\theta} \qquad \displaystyle\int_{|z|=1} \frac{1}{1 - *\left(z - \frac{1}{z}\right)} \frac{dz}{z}$

$\longrightarrow$ prendere il residuo nel dominio

SISTEMI DINAMICI 1D DISCRETI

$f : \mathbb{R} \to \mathbb{R}$, $f^n = \underbrace{f \circ \dots \circ f}_{n \text{ volte}}$

iterazione n-esima

f un processo che prende uno stato iniziale $x_0 \in \mathbb{R}$ e lo trasforma in un nuovo stato $x_1 = f(x_0) \in \mathbb{R}$

Diciamo $x_n = f^n(x_0) = \underbrace{f(f(\dots f(x_0)))}_{n \text{ volte}}$

Orbita in avanti : $O^+ = \{x_n\}_{n \in \mathbb{N}}$

Se f è invertibile : orbita completa

$O = \{x_n\}_{n \in \mathbb{Z}}$

Il punto x_0 è detto seme (seed) dell'orbita.

Esempio : $f(x) = x^2 + 1$

$x_0 = 0 \qquad x_1 = f(0) = 1 \qquad x_2 = f(1) = 2$

$x_3 = 5 \qquad x_4 = 26 \quad \ldots$

Un punto x_0 è fisso se $f(x_0) = x_0$

(orbita è $x_0, x_0, x_0, x_0 \ldots$)

$f^n(x_0) = x_0$ $\qquad$ n-ciclo

(punto periodico di periodo n se n è il numero minimo t.c. $f^n(x_0) = x_0$)

Orbite : $x_0, x_1, \ldots, x_n$ e si ripete

Esempio : $f(x) = -x^3$

- punto fisso $f(0) = 0$
- $f(\pm 1) = -(\pm 1) = \mp 1$

$\Rightarrow f^2(\pm 1) = \pm 1$ $\qquad$ punti periodici periodo 2

Iterazione grafica