MECCANICA RAZIONALE

Problema della dinamica

$$\underline{F}_P - \frac{d}{dt}\underline{P}_P = \underline{0}$$

$\longrightarrow$ { PLV $\longrightarrow$ Eq. Lagrange
ECS ECD

Non - conservativo $$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}_i} - \frac{\partial k}{\partial q_i} = Q_i$$

Conservativo $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$$\mathcal{L} = k - V$$

$$\underline{R}^{(e)} = \frac{d}{dt}\underline{P} \quad , \quad \underline{M}^{e}(O) = \frac{d}{dt}\underline{L}(O) + \underline{v}_o \wedge \underline{P}$$

$$L(O) = \sum_{B\in S}{}' (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B = \dots$$

$$K = \frac{1}{2} M v_O^2 + \underline{v}_O \cdot \underline{\omega} \wedge M(\underline{x}_G - \underline{x}_O)$$

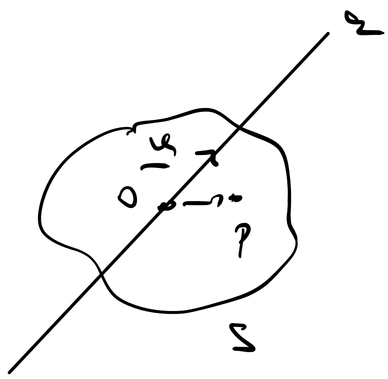
$$+ \frac{1}{2} \underline{\omega} \cdot \sum_{B\in S}{}' m_B (\underline{x}_B - \underline{x}_O) \wedge [\underline{\omega} \wedge (\underline{x}_B - \underline{x}_O)]$$

$\underline{u}$

$$\underline{F} = m \underline{a} \rightsquigarrow M \quad I_O$$

Trasformazione di inerzia

Consideriamo un sistema S



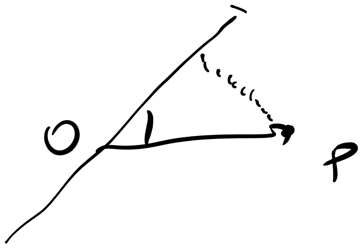
→ momento di inerzia

P punto di massa m_P

$\underline{u} := \text{vers}(\underline{r})$

distanza del punto P

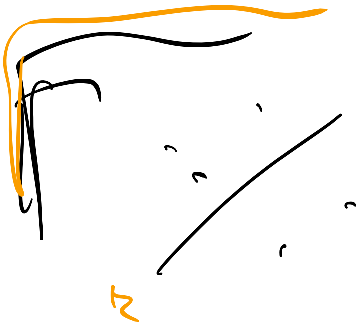
alla retta r $= \| \underline{u} \wedge (\underline{x}_P - \underline{x}_O) \|$



$$\left(\underline{a} \wedge \underline{b} = \hat{\underline{u}}\, ab \sin\theta\right)$$

Momento di Inerzia : $$\boxed{I_z = \sum_{P \in S}{}' \, m_P \left\| \underline{u} \wedge (\underline{x}_P - \underline{x}_0) \right\|^2}$$

$$\left(\text{"} I_z = \sum m_i r_i^2 \text{"} \right)$$

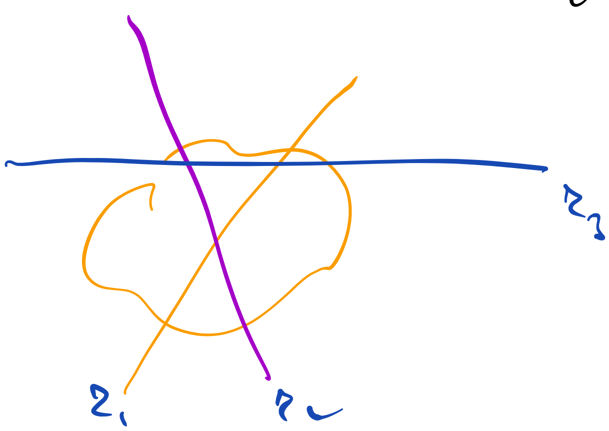


$$I_z = \sum_i m_i r_i^2$$

$r_i = \omega d_i$

$$L_z = \sum_i r_i m_i v_i \longrightarrow \omega I_z$$

$$E = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} I_z \omega^2$$



Dalla definizione

$$I_z = \sum_{P \in R}{}' \, m_P \left[\underline{u} \wedge (\underline{x}_P - \underline{x}_0) \right] \cdot \left[\underline{u} \wedge (\underline{x}_P - \underline{x}_0) \right]$$

$$\underline{a} \wedge \underline{b} \cdot \underline{c}$$

usiamo $\underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \wedge \underline{c}$

$$= \underline{u} \cdot \underbrace{\sum_{P \in R}' m_p (\underline{x}_p - \underline{x}_0) \wedge [\underline{u} \wedge (\underline{x}_p - \underline{x}_0)]}$$

$$= \underline{u} \cdot I_0(\underline{u})$$

Trasformazione di inerzia (relativa ad O).

$$I_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$I_0(\underline{y}) = \sum_{P \in R}' m_p (\underline{x}_p - \underline{x}_0) \wedge [\underline{y} \wedge (\underline{x}_p - \underline{x}_0)]$$

"Tensore di inerzia"

<u>Proposizione</u>: la trasformazione di inerzia è lineare e simmetrica

$$I_0(\underline{y}) = \sum_p m_p (\underline{x}_p - \underline{x}_0) \wedge [\underline{y} \wedge (\underline{x}_p - \underline{x}_0)]$$

$$I_0(\lambda \underline{y} + \mu \underline{z}) \overset{?}{=} \lambda I_0(\underline{y}) + \mu I_0(\underline{z})$$

$$\underline{y}, \underline{z} \in \mathbb{R}^3$$

$$\lambda, \mu \in \mathbb{R}$$

ovvio dalla definizione

$$I_0(\lambda \underline{y} + \mu \underline{z}) = \sum_p m_p (\underline{x}_p - \underline{x}_0) \wedge \left[(\lambda \underline{y} + \mu \underline{z}) \wedge (\underline{x}_p - \underline{x}_0)\right]$$

simmetria: $\underline{y} \cdot I_0(\underline{z}) \overset{?}{=} \underline{z} \cdot I_0(\underline{y})$

Proprietà: $\underline{a} \wedge [\underline{b} \wedge \underline{c}] = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c}$

$$I_0(\underline{y}) = \sum_p m_p \underbrace{(\underline{x}_p - \underline{x}_0)}_{a} \wedge [\underbrace{\underline{y}}_{b} \wedge \underbrace{(\underline{x}_p - \underline{x}_0)}_{c}]$$

$$= \sum_{p \in \mathcal{R}} m_p \left[\underbrace{\|\underline{x}_p - \underline{x}_0\|^2}_{(a \cdot c)} \underline{y} - \underbrace{\left((\underline{x}_p - \underline{x}_0) \cdot \underline{y}\right)}_{(a \cdot b)} (\underline{x}_p - \underline{x}_0) \right]$$

$$\underline{z} \cdot I_0(\underline{y}) =$$

$$= \sum_{p \in R}' m_p \left[\|\underline{x}_p - \underline{x}_0\|^2 \, \underline{z} \cdot \underline{y} - \left((\underline{x}_p - \underline{x}_0) \cdot \underline{y} \right) \left((\underline{x}_p - \underline{x}_0) \cdot \underline{z} \right) \right]$$

$$= \underline{y} \cdot I_0(\underline{z}) \,.$$

Lineare & Simmetrica:

Lineare $\longrightarrow$ matrice

Simmetrica $\longrightarrow$ matrice simmetrica

I_0 può essere rappresentato da una matrice simmetrica 3×3

$$I_0(\underline{y}) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Componenti:

$$I_{jk} = \underline{u}_j \cdot I_0 \cdot \underline{u}_k \qquad j,k = 1,2,3$$

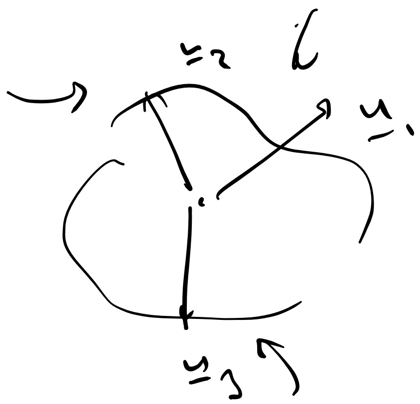
versori base

$u_1 = (1, 0, 0) \qquad \underline{u}_2 = (0, 1, 0) \qquad \underline{u}_3 = (0, 0, 1)$

$$I_{jj} = \underline{u}_j \cdot I_O \, \underline{u}_j = I_{\underline{u}_j}$$

momento di inerzia rispetto all'asse $\underline{u}_j$

sulla diagonale

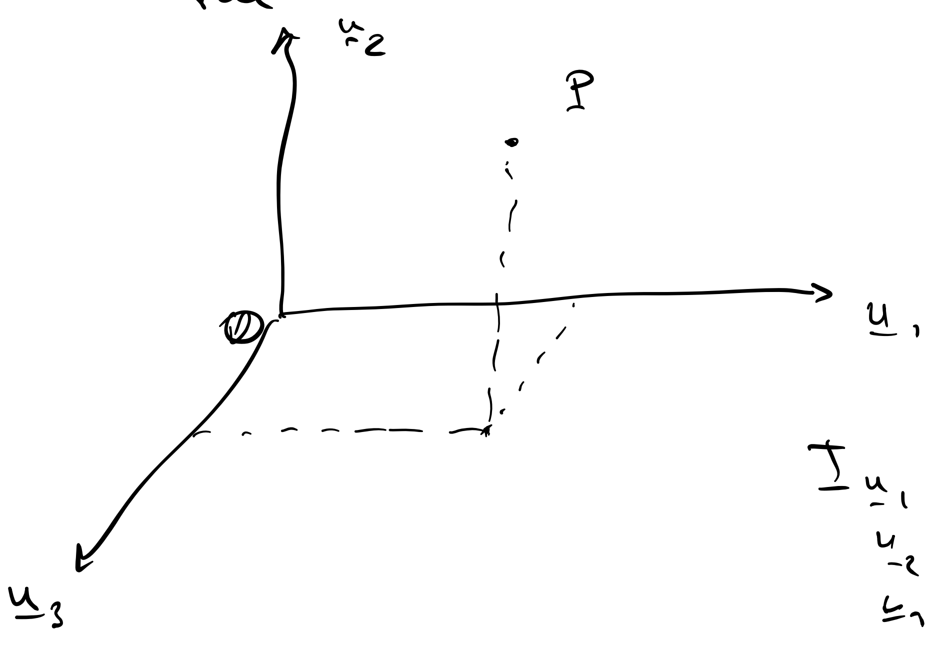


$$\begin{pmatrix} \bigcirc & - & - \\ \cdot & \bigcirc & - \\ - & - & \bigcirc \end{pmatrix}$$

$$\underline{u} \longrightarrow \underline{u} \cdot I_O(\underline{u}) = I_{\underline{u}}$$

$$\underline{u}' \cdot I_O(\underline{u}') = I_{\underline{u}'}$$

Fissiamo poi un punto P le sue coordinate $x_{P,1}$, $x_{P,2}$, $x_{P,3}$



$$\underline{x}_P = x_{P,1}\,\underline{u}_1 + x_{P,2}\,\underline{u}_2 + x_{P,3}\,\underline{u}_3$$

$$I_{\underline{u}_1} = \sum_P m_P \, \|\underline{u}_1 \wedge \underline{x}_P\|^2$$

$\underline{u}_2$

$\underline{u}_3$

distanza dall'asse $\underline{u}_1$

$$I_{11} = \sum_p{}' m_p \left(x_{p,2}^2 + \underline{x_{p,3}^2} \right)$$

$$I_{22} = \sum_p{}' m_p \left(x_{p,1}^2 + \underline{x_{p,3}^2} \right)$$

$$I_{33} = \sum_p m_p \left(x_{p,1}^2 + x_{p,2}^2 \right)$$

$$\underline{u}_1 \wedge \left(x_{p,1} \underline{u}_1 + x_{p,2} \underline{u}_2 + x_{p,3} \underline{u}_3 \right) =$$

$$= x_{p,1} \underbrace{\underline{u}_1 \wedge \underline{u}_1}_{0} + x_{p,2} \underbrace{\underline{u}_1 \wedge \underline{u}_2}_{\underline{u}_3} + x_{p,3} \underbrace{\underline{u}_1 \wedge \underline{u}_3}_{-\underline{u}_2}$$

$$(\quad)^2 = x_{p,2}^2 + x_{p,3}^2$$

Andiamo a vedere gli elementi fuori dalla diagonale.

Ricordiamo : $\underline{z} \cdot I_0(\underline{y}) =$

$$= \sum_{PER}{}' m_p \left[\|\underline{x}_p - \underline{x}_0\|^2 (\underline{z} \cdot \underline{y}) - \left((\underline{x}_p - \underline{x}_0) \cdot \underline{y} \right) (\underline{x}_p - \underline{x}_0) \cdot \underline{z} \right]$$

elementi fuori dalla diagonale :

$\underline{u} \cdot I_0(\underline{v})$ dove $\underline{u} \perp \underline{v}$

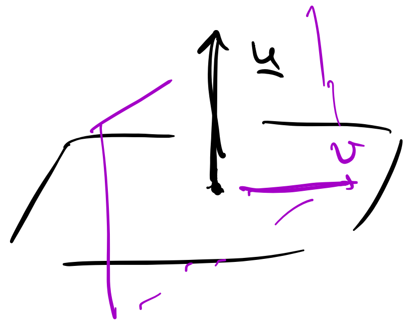
quindi per $\underline{u} \perp \underline{v}$:

$$\underline{u} \cdot \underline{\underline{I}}_O(\underline{v}) = - \sum_{P \in \mathcal{R}}' m_P \left[(\underline{x}_P - \underline{x}_O) \cdot \underline{u} \right] \left[(\underline{x}_P - \underline{x}_O) \cdot \underline{v} \right]$$

$\underline{u} \cdot \underline{v}$

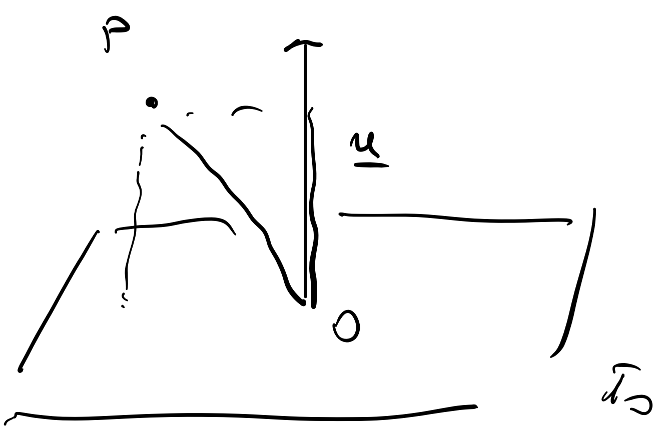
$\underline{u}_1 , \underline{u}_2 , \underline{u}_3$

Prendiamo π_O : piano passante per O e ortogonale a $\underline{u}$



π_O' : piano passante per O e ortogonale a $\underline{v}$

$\underline{u} \cdot \underline{\underline{I}}_O(\underline{v})$ = momento DEVIATORE rispetto alla coppia di piani π_O, π_O'.



$\underline{u} \cdot (\underline{x}_P - \underline{x}_O)$ = la distanza (con il segno) del punto P dal piano π_O

$$I_{12} = - \sum_P m_P \left[\underline{x}_P \cdot \underline{u}_1 \right] \left[\underline{x}_P \cdot \underline{u}_2 \right] =$$

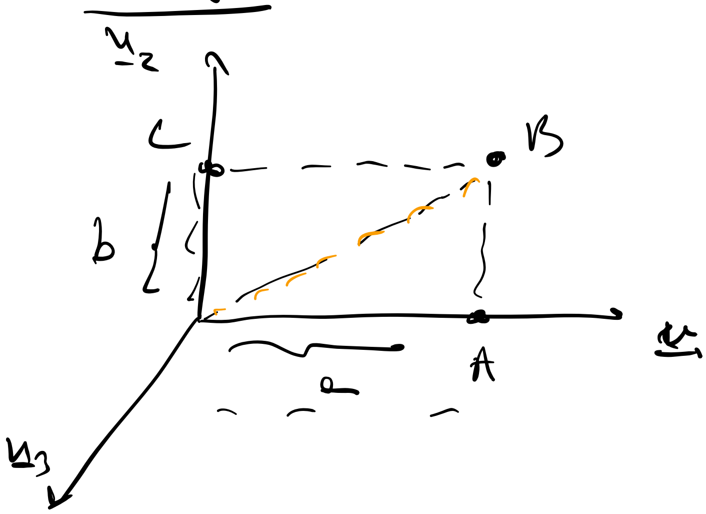
$$= - \sum_P m_P \, x_{P,1} \, x_{P,2}$$

$$I_{13} = -\sum_p m_p \, x_{p,1} \, x_{p,3}$$

$$I_{23} = -\sum_p m_p \, x_{p,2} \, x_{p,3}$$

$$I_O = \begin{pmatrix} \sum_p m_p (x_{p,2}^2 + x_{p,3}^2) & -\sum_p m_p x_{p,1} x_{p,2} & -\sum_p m_p x_{p,1} x_{p,3} \\ -\sum_p m_p x_{p,1} x_{p,2} & \sum_p m_p (x_{p,1}^2 + x_{p,3}^2) & -\sum_p m_p x_{p,2} x_{p,3} \\ -\sum_p m_p x_{p,1} x_{p,3} & -\sum_p m_p x_{p,2} x_{p,3} & \sum_p m_p (x_{p,1}^2 + x_{p,2}^2) \end{pmatrix}$$

Esempio Trovatune inde formule



- moto della asse è trascurabile rispetto alla massa dei nodi A, B, C

$\to m_A, m_B, m_C$

$$I_{11} = m_A (0) + m_B b^2 + m_C b^2$$

$$I_{22} = m_A a^2 + m_B a^2 + m_C (0)$$

$$I_{33} = m_A a^2 + m_B(a^2+b^2) + m_C b^2$$

$$= I_{11} + I_{22}$$

$$I_{12} = -\sum_P m_P x_{P,1} x_{P,2}$$

$$= -\left(m_A (a \cdot 0) + m_B (a \cdot b) + m_C (b \cdot 0) \right)$$

$$= - m_B a b$$

$$I_{13} = I_{23} = 0$$

$$I = \begin{pmatrix} m_B b^2 + m_C b^2 & -m_B ab & 0 \\ -m_B ab & m_A a^2 + m_B a^2 & 0 \\ 0 & 0 & m_A a^2 + m_C b^2 + m_B(a^2+b^2) \end{pmatrix}$$

Rigido piano : $\underline{u}_3$ ortogonale al piano (giace su $(\underline{u}_1, \underline{u}_2)$)

$$I_{13} = I_{23} = 0$$

$$I_{33} = I_{11} + I_{22}$$

Rigido piano

$$I_O = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{11}+I_{22} \end{pmatrix}$$

Esercizio Nell'esempio precedente

$Q = (a, a, a)$ calcoliamo il

momento di inerzia rispetto a $\overline{OQ}$

$\rightarrow$ versore $\underline{u} = \dfrac{\underline{x}_Q - \underline{x}_O}{\| \underline{x}_Q - \underline{x}_O \|^2} =$

$$= \frac{(a, a, a)}{\sqrt{a^2 + a^2 + a^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$I_{\overline{OQ}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$\underline{u} \cdot I_O \cdot \underline{u}$

$$= \frac{1}{3} (1, 1, 1) \begin{pmatrix} I_{11} + I_{12} \\ I_{12} + I_{22} \\ I_{33} \end{pmatrix}$$

$$= \frac{1}{3}\left(I_{11} + I_{12} + I_{12} + I_{22} + I_{33} \right)$$

$$= \frac{1}{3}\left(I_{11} + 2 I_{12} + I_{22} + I_{33} \right)$$

$$= \frac{1}{3}\Big(m_B b^2 + m_C b^2 - 2 m_B a b +$$

$$+ m_A a^2 + m_B a^2 + m_B b^2 + m_C b^2 +$$

$$+ m_A a^2 + m_B a^2 \Big)$$

$$= \frac{1}{3}\Big[b^2 (2m_B + 2m_C) + a^2 (2m_A + 2m_B)$$

$$- 2 m_B a b \Big]$$