MECCANICA RAZIONALE

Trasformazione di inerzia

$$I_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\underline{\omega} \longmapsto I_0(\underline{\omega})$$

$$I_0(\underline{\omega}) = \sum_{P \in S}' m_P (\underline{x}_P - \underline{x}_0) \wedge [\underline{\omega} \wedge (\underline{x}_P - \underline{x}_0)]$$

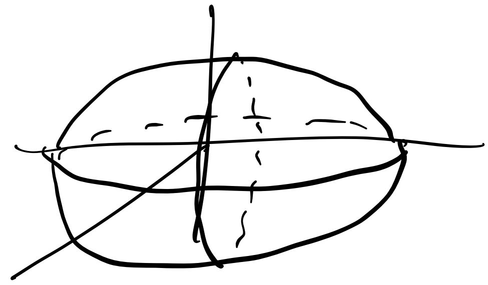
$$\underline{u} \cdot I_0(\underline{u}) = I_{\underline{u}} \leftarrow \text{"} \sum m r^2 \text{"}$$

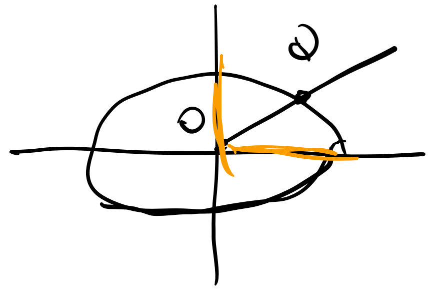
$\hookrightarrow$ lineare simmetrica $\Rightarrow$ $\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$

Ellissoide di inerzia

$$f(\underline{x}) = \underline{x} \cdot I_0(\underline{x}) = C \quad (=1)$$

$$(- \; -)\begin{pmatrix} - & - \\ - & - \end{pmatrix}\begin{pmatrix} | \\ | \end{pmatrix}$$





$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

I_O diventa diagonale

Assi principali di inerzia

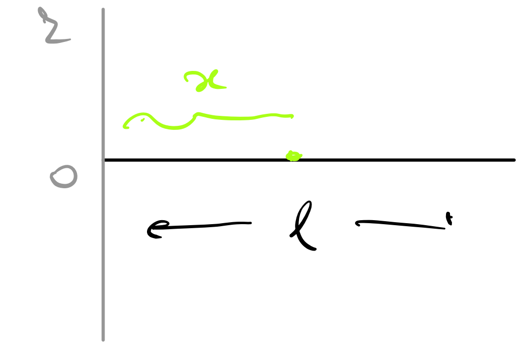
$$\det\left(I_O - \lambda \mathbb{1}_{3\times3}\right) = 0$$

$$\left(I_O - \lambda \mathbb{1}_{3\times3}\right)\underline{x} = \underline{0}$$

<u>Calcolo di alcuni elementi di inerzia</u>

ESEMPI :

Asta



Calcoliamo il momento di inerzia rispetto alla retta z passante per O.

Densità $\rho(x)$

$$I_z = \int_0^\ell dx\, \rho(x)\, x^2$$

ρ costante :

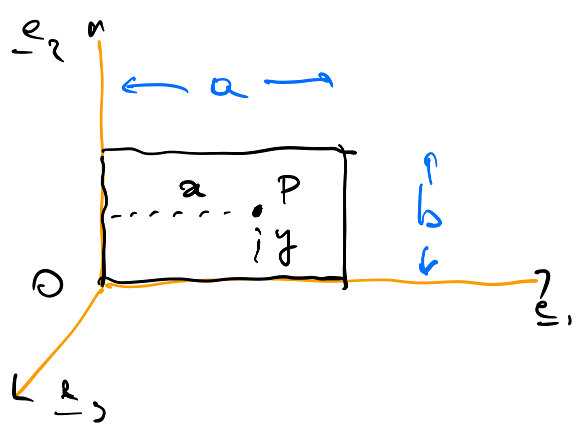
$$M = \int_0^\ell \rho(x)\, dx = \rho \ell$$

$$\rho = \frac{M}{\ell}$$

Allora

$$\boxed{I_z = \int_0^\ell dx\, \frac{M}{\ell}\, x^2 = \frac{M}{\ell} \left.\frac{x^3}{3}\right|_0^\ell = \frac{M \ell^2}{3}}$$

Rettangolo



$$I_O = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{11} + I_{22} \end{pmatrix}$$

$$\rho(x, y)$$

$$I_{11} = \int_0^a \int_0^b dx\,dy\, \rho(x,y)\, y^2$$

se omogeneo : $\rho = \frac{M}{ab}$ $\quad M = \int_0^a\int_0^b \rho\, dx\,dy = \rho\, ab$

$$= \frac{M}{ab} \int_0^a \int_0^b dx\,dy\, y^2 = \frac{M}{ab}\, a\, \frac{b^3}{3} = \boxed{\frac{Mb^2}{3}}$$

$$\left.\frac{y^3}{3}\right|_0^b$$

$$I_{22} = \int_0^a \int_0^b dx\,dy\, \rho(x,y)\, x^2$$

$$= \int_0^a \int_0^b \frac{M}{ab}\, dx\,dy\, x^2 = \boxed{\frac{Ma^2}{3}}$$

omogeneo

$$\left.\frac{x^3}{3}\right|_0^a$$

$$I_{12} = -\sum_p^4 w_p\, x_{p,1}\, x_{p,2}$$

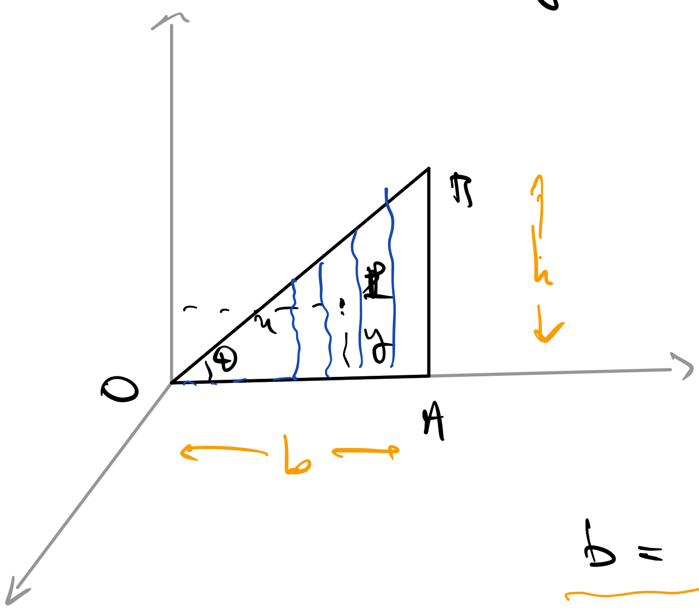
$$= -\int_0^a \int_0^b dx\,dy\, \rho(x,y)\, x\, y$$

$$= -\frac{M}{ab}\, \frac{a^2}{2}\, \frac{b^2}{2} = -\frac{M}{4}\, ab.$$

omogeneo

$$I_o = \begin{pmatrix} M\frac{b^2}{3} & -M\frac{ab}{4} & 0 \\ -M\frac{ab}{4} & M\frac{a^2}{3} & 0 \\ 0 & 0 & M\frac{a^2+b^2}{3} \end{pmatrix}$$

Lamina Triangolare



$$\rho = \rho(x, y)$$

$$I_{11} = \int_0^b \int_0^{\frac{h}{b}x} dx\, dy\, \rho(x,y)\, y^2$$

$$b = \underline{OB}\cos\theta \qquad h = \underline{OB}\sin\theta$$

$$x = L\cos\theta \qquad y = L\sin\theta$$

$$L = \frac{x}{\cos\theta} \Rightarrow y = x\frac{\sin\theta}{\cos\theta}$$

$$= x\frac{h}{b}$$

$$\frac{b}{\cos\theta} = \frac{h}{\sin\theta}$$

$$I_{11} = \int_0^b dx \int_0^{\frac{h}{b}x} dy\, \rho(x,y)\, y^2 = \rho \int_0^b dx \left(\int_0^{\frac{h}{b}x} dy\, y^2\right)$$

omogeneo

$$= \rho \int_0^b dx \left(\frac{h}{b} x \right)^3 \frac{1}{3} = \frac{\rho}{3} \frac{h^3}{b^3} \int_0^b x^3 \, dx$$

$$= \frac{\rho}{3} \frac{h^3}{b^3} \frac{b^4}{4} = \rho \frac{1}{12} h^3 b =$$

$$= \rho \underbrace{\frac{hb}{2}}_{\text{Area}} \frac{h^2}{6} = M \frac{h^2}{6}$$

omogenea

$$I_{22} = \int_0^b dx \, x^2 \int_0^{\frac{h}{b}x} \rho(x,y) \, dy =$$

$$= \rho \int_0^b dx \, x^2 \frac{h}{b} x = \rho \frac{h}{b} \frac{b^4}{4} =$$

$$= \rho \frac{hb}{2} \frac{b^2}{2} = M \frac{b^2}{2}$$

omogeneo

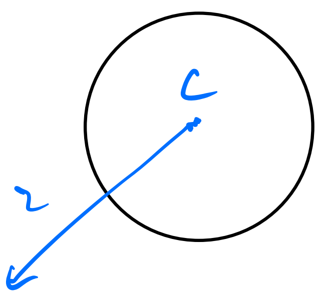
$$I_{12} = - \int_0^b dx \, x \int_0^{\frac{h}{b}x} dy \, y \, \rho(x,y) =$$

$$= - \rho \int_0^b dx \, x \, \frac{1}{2} \left(\frac{h}{b} x \right)^2 =$$

$$= -\rho \frac{1}{2}\frac{h^2}{b^2}\frac{b^4}{4} = -\rho\frac{hb}{2}\frac{bh}{4} = -M\frac{bh}{4}$$

$$I_0 = \begin{pmatrix} M\frac{h^2}{6} & -M\frac{bh}{4} & 0 \\ -M\frac{bh}{4} & M\frac{b^2}{2} & 0 \\ 0 & 0 & M\left(\frac{h^2}{6}+\frac{b^2}{2}\right) \end{pmatrix}$$

Circonferenza / Cerchio



Circonferenza: rispetto ad un asse z passante per il centro C e ortogonale al piano della circonferenza

$$I_z = \int_0^{2\pi} \rho(\theta)\, R^2\, d\theta = R^2 \int_0^{2\pi} \rho(\theta)\, d\theta = R^2 M$$

Cerchio (lamina)

$$I_z = \int_0^R \int_0^{2\pi} \rho(r,\theta)\, r^2 \; r\, dr\, d\theta$$

$dx\,dy$
$\downarrow$
$r\,dr\,d\theta$

se ad esempio $\rho = \rho(r)$

$$= 2\pi \int_0^R \rho(r)\, r^3\, dr$$

se ad esempio $\rho =$ costante $= \dfrac{M}{\pi R^2}$

$$= 2\pi \rho \frac{R^4}{4} = \frac{M R^2}{2}$$

Variazione di I_O con il polo

$$I_O(\underline{u}) = \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_O)]$$

consideriamo un altro polo O'.

$$(\underline{x}_B - \underline{x}_O) = (\underline{x}_B - \underline{x}_{O'}) + (\underline{x}_{O'} - \underline{x}_O)$$

$$I_O(\underline{u}) = \sum_{B \in R}' m_B \left[(\underline{x}_B - \underline{x}_{O'}) + (\underline{x}_{O'} - \underline{x}_O) \right] \wedge$$

$$\wedge \left[\underline{u} \wedge \left[(\underline{x}_B - \underline{x}_{O'}) + (\underline{x}_{O'} - \underline{x}_O) \right] \right]$$

$$= \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_{O'}) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_{O'})]$$

$\hookrightarrow I_{O'}$

$$+ \sum_{B \in R} m_B (\underline{x}_B - \underline{x}_{O'}) \wedge [\underline{u} \wedge (\underline{x}_{O'} - \underline{x}_O)]$$

$$+ \sum_{B \in R} m_B (\underline{x}_{O'} - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_{O'})]$$

$$+ \underbrace{\left(\sum_{B \in R} m_B \right)}_{M} (\underline{x}_{O'} - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_{O'} - \underline{x}_O)]$$

Prendiamo $O' = G$ centro di massa

$$\sum_{B \in R} m_B (\underline{x}_B - \underline{x}_G) = \underbrace{\underbrace{\sum_{B \in R} m_B \underline{x}_B}_{M \underline{x}_G} - \underbrace{\left(\sum_{B \in R} m_B \right)}_{M} \underline{x}_G}_{= 0}$$

$$\underline{x}_G := \frac{\sum m_B \underline{x}_B}{M}$$

La formula di Trasporto assume la forma semplice

$$I_O(\underline{u}) = I_G(\underline{u}) + M (\underline{x}_G - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_G - \underline{x}_O)]$$

Teorema di Huygens-Steiner

Siano r_G e r_O due rette parallele, per G e per O

$$I_{r_O} = I_{r_G} + M \| \underline{u} \wedge (\underline{x}_G - \underline{x}_O) \|^2$$

$$= I_{r_G} + M \left(\begin{array}{c} \text{distanza} \\ \text{tra } r_O \text{ e } r_G \end{array} \right)^2$$

Dimostrazione moltiplichiamo per $\underline{u}$

$$\underline{u} \cdot I_O(\underline{u}) = \underline{u} \cdot I_G(\underline{u}) +$$

I_{r_O}

$$+ M \, \underline{u} \cdot (\underline{x}_G - \underline{x}_O) \wedge \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_O) \right]$$

$$\underline{a} \cdot \underline{b} \wedge \underline{c}$$

$$\underline{a} \wedge \underline{b} \cdot \underline{c}$$

$$= I_{r_G} + M \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_O) \right] \cdot \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_O) \right]$$

Teorema di trasporto dei momenti deviatori

π_0, π_G 2 piani ortogonali ad un vettore $\underline{u}$

π'_0, π'_G 2 piani ortogonali ad un vettore $\underline{v}$

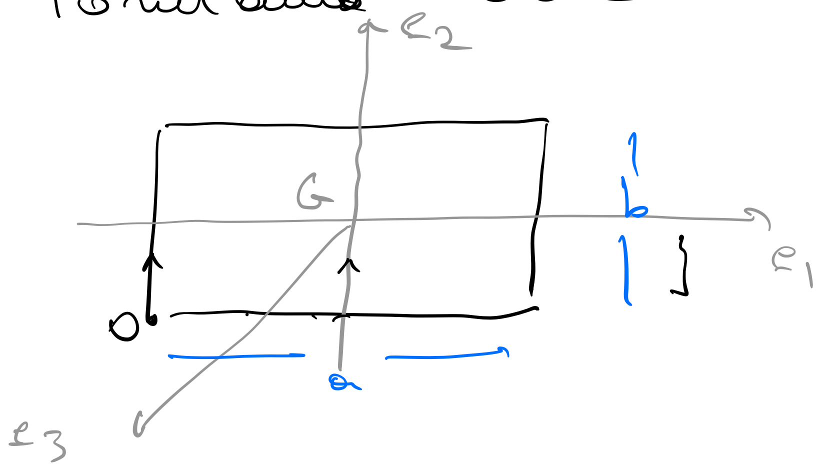
$\longrightarrow$ passanti per O e per G

$$\underline{u} \cdot \underline{v} = 0$$

$$I_{\pi_0 \pi'_0} = I_{\pi_G \pi'_G} + M \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_0)\right] \cdot \left[\underline{v} \wedge (\underline{x}_G - \underline{x}_0)\right]$$

Si dimostra come prima.

Torniamo alla lamina rettangolare.



questo è una terna principale di inerzia (omogenea)

$$I_O = \begin{pmatrix} M\frac{b^2}{3} & -M\frac{ab}{4} & 0 \\ -M\frac{ab}{4} & M\frac{a^2}{3} & 0 \\ 0 & 0 & M\frac{a^2+b^2}{3} \end{pmatrix}$$

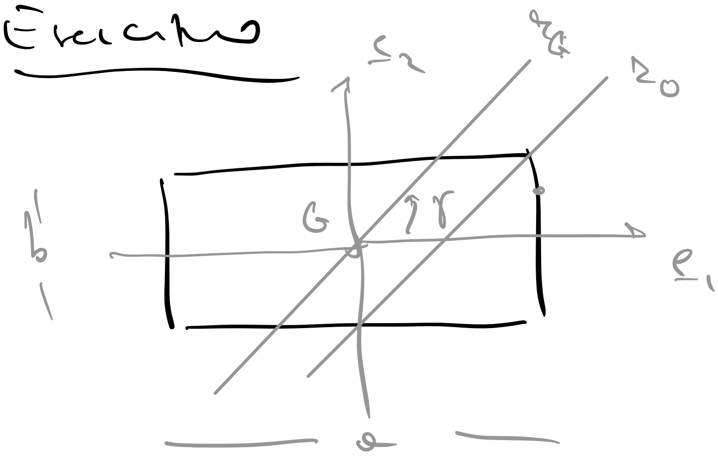
$$I_{G,11} = I_{O,11} - M(\text{distanza fra } z_O \text{ e } z_G)^2$$

$$= M\frac{b^2}{3} - M\left(\frac{b}{2}\right)^2 = M\frac{b^2}{12}$$

$$I_{G,22} = M\frac{a^2}{3} - M\left(\frac{a}{2}\right)^2 = M\frac{a^2}{12}$$

$$I_G = \begin{pmatrix} M\frac{b^2}{12} & 0 & 0 \\ 0 & M\frac{a^2}{12} & 0 \\ 0 & 0 & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$

<u>Esercizio</u>



$$O = \left(\frac{a}{2}, \frac{b}{2}\right)$$

$$\tilde{I}_G = \begin{pmatrix} M\frac{b^2}{12} & 0 \\ 0 & M\frac{a^2}{12} \end{pmatrix}$$

$$I_{z_G} = \underline{u} \cdot I_G \cdot \underline{u} = \begin{pmatrix} \cos\gamma & \sin\gamma \end{pmatrix} \begin{pmatrix} \frac{Mb^2}{12} & 0 \\ 0 & \frac{Ma^2}{12} \end{pmatrix} \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$$

$$= M \frac{b^2}{12} \cos^2\gamma + M \frac{a^2}{12} \sin^2\gamma$$

$$\left\| \left(\cos\gamma \, \underline{e}_1 + \sin\gamma \, \underline{e}_2 \right) \wedge \left(\frac{a}{2} \underline{e}_1 + \frac{b}{2} \underline{e}_2 \right) \right\|^2$$

$$= \left\| \left(\frac{b}{2} \cos\gamma - \frac{a}{2} \sin\gamma \right) \underline{e}_3 \right\|^2$$

$$= \left(\frac{b}{2} \cos\gamma - \frac{a}{2} \sin\gamma \right)^2$$

$$I_{z_O} = I_{z_G} + M \left(\frac{b}{2} \cos\gamma - \frac{a}{2} \sin\gamma \right)^2$$

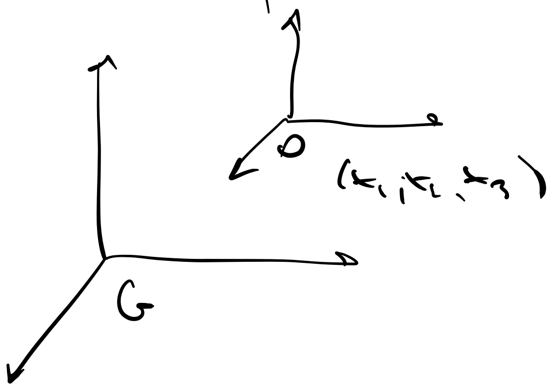
$$= \frac{Mb^2}{12} \cos^2\gamma + \frac{Ma^2}{12} \sin^2\gamma + M \frac{b^2}{4} \cos^2\gamma$$

$$+ M \frac{a^2}{4} \sin^2\gamma - \frac{Mab}{2} \cos\gamma \sin\gamma$$

$$= M \left(\frac{a^2}{3} \sin^2\gamma + \frac{b^2}{3} \cos^2\gamma - \frac{ab}{2} \cos\gamma \sin\gamma \right)$$

Proprietà degli assi principali di inerzia relativi al centro di massa

Prendiamo una terna principale di inerzia per G.
assi principali centrali



Vogliamo la matrice di inerzia in O in una terna con assi paralleli agli assi principali di inerzia

$$I_G = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \Longrightarrow$$

$$I_O = \begin{pmatrix} J_1 + M(x_2^2 + x_3^2) & -Mx_1x_2 & -Mx_1x_3 \\ -Mx_1x_2 & J_2 + M(x_1^2+x_3^2) & -Mx_2x_3 \\ -Mx_1x_3 & -Mx_2x_3 & J_3 + M(x_2^2+x_1^2) \end{pmatrix}$$

Quindi :

•) Se $O \in API(G)$ $\Rightarrow$ la terna principale per O è parallela a quella centrale.

•) usiamo la formula di trasporto:

$$\underbrace{I_O(\underline{u})}_{\lambda \underline{u}} = I_G(\underline{u}) + M(\underline{x}_G - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_G - \underline{x}_O)]$$

$$= \underbrace{I_G(\underline{u})}_{\lambda(\underline{u})} + M \| \underline{x}_G - \underline{x}_O \|^2 \underline{u} +$$

$$- \underline{M(\underline{u} \cdot (\underline{x}_G - \underline{x}_O))(\underline{x}_G - \underline{x}_O)} .$$

$$\underline{a} \wedge [\underline{b} \wedge \underline{c}] = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

Se $\underline{u}$ è autovettore di I_G $(I_G(\underline{u}) = \lambda \underline{u})$ questo implica che è autovettore di I_O $(I_O(\underline{u}) = \lambda \underline{u})$ <u>solamente</u> se

$\left(\underline{u}\cdot(\underline{x}_G-\underline{x}_0)\right)\cdot(\underline{x}_G-\underline{x}_0)$ è $\begin{cases} \text{zero} \\ \text{oppure} \\ \text{parallelo a } \underline{u} \end{cases}$

e cioè

— $\underline{u}\cdot(\underline{x}_G-\underline{x}_0)=0$

$O \in$ piano per G ortogonale a $\underline{u}$

OPPURE

— $(\underline{x}_G-\underline{x}_0) \parallel \underline{u}$

$O \in$ retta per G di direzione $\underline{u}$

Torniamo alla dinamica:

Equazioni di Lagrange $\quad \dfrac{d}{dt}\dfrac{\partial k}{\partial \dot q_i}-\dfrac{\partial k}{\partial q_i}=Q_i$

ECD: $\begin{cases} \underline{R}^e=\dfrac{d}{dt}\underline{P} \\ \underline{M}^e(O)=\dfrac{d}{dt}\underline{L}(O)+\underline{v}_0\wedge\underline{P} \end{cases}$

$$K = \frac{1}{2} \sum_\beta m_\beta \|\underline{v}_\beta\|^2 =$$

$$= \frac{1}{2} \sum_\beta m_\beta \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_\beta - \underline{x}_0)\right) \cdot \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_\beta - \underline{x}_0)\right)$$

$\underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \wedge \underline{c}$

$$= \frac{1}{2} M v_0^2 + \frac{1}{2} \underline{\omega} \cdot I_0(\underline{\omega}) + \underline{v}_0 \cdot \underline{\omega} \wedge M(\underline{x}_G - \underline{x}_0)$$

$$\underline{\omega} = 0 \longrightarrow \frac{1}{2} M v_0^2$$

$$\underline{v}_0 = 0 \longrightarrow \frac{1}{2} \underline{\omega} \cdot I_0(\underline{\omega})$$

$$\underline{\omega} = \omega \underline{u} \longrightarrow \frac{1}{2} \omega^2 \underline{u} \cdot I_0(\underline{u})$$

$$= \frac{1}{2} \omega^2 I_z$$

Riassumendo:

- moto rigido generico

$$K = \frac{1}{2} M v_G^2 + \frac{1}{2} \underline{\omega} \cdot I_G(\underline{\omega})$$

- moto rigido con punto fisso O

$$K = \frac{1}{2} \underline{\omega} \cdot I_0(\underline{\omega})$$

- moto rigido con asse fisso z

$$K = \frac{1}{2} I_z \omega^2$$

- moto rigido di traslazione

$$K = \frac{1}{2} M v_G^2$$

CASO PIANO $\qquad \underline{\omega} = \omega \underline{e}_3$

$$\underline{\omega} \cdot I_O(\underline{\omega}) = \omega^2 \underline{e}_3 \cdot I_O(\underline{e}_3) = \omega^2 \underbrace{I_{3,O}}_{I_3}$$

- moto rigido generico

$$K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3,G} \, \omega^2$$

- moto rigido piano con punto fisso O

$$K = \frac{1}{2} I_{3,O} \, \omega^2$$

- moto traslatorio

$$K = \frac{1}{2} M v_G^2$$