MECCANICA RAZIONALE

Dinamica : principio di D'Alembert

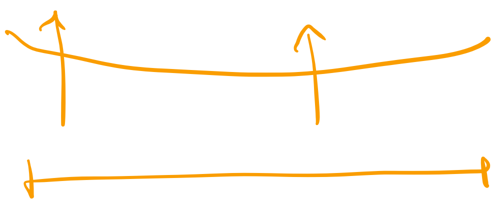
+

Statica

$$\underline{F}_P - \frac{d}{dt}\underline{p}_P = 0$$

P punto materiale

$$\underline{F} = m\,\ddot{\underline{x}}$$



$$\sum_P \underline{F}_P \cdot \delta \underline{x}_P = 0$$

$$\underline{F}_P = \underline{0} \longrightarrow$$

PLV

ECS

} corpo rigido

$$(\underline{F}_P = \underline{0} \quad \forall P)$$

$$\sum_P \underline{F}_P \cdot \delta \underline{x}_P \rightsquigarrow \sum_i Q_i \, \delta q_i$$

$$\sum_P \left(\underline{F}_P - \frac{d}{dt}\underline{p}_P\right) \delta \underline{x}_P$$

$$\underline{q}(\tau) = (q_1(\tau), \ldots, q_\ell(\tau))$$

$$\hookrightarrow \quad \frac{d}{d\tau}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i$$

$$\frac{d}{d\tau}\underline{P}_p = \underline{F}_p$$

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad \mathcal{L} = K - V$$

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Dinamica è contenuta in $\mathcal{L}$

Nel caso piano:

- $K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3G}\, \omega^2$ ← G
- $K = \frac{1}{2} I_{3,O}\, \omega^2$ ←
- $K = \frac{1}{2} M v_G^2$ ←

$$I_O(\underline{\omega}) = \sum_p m_p (\underline{x}_p - \underline{x}_o) \wedge [\underline{\omega} \wedge (\underline{x}_p - \underline{x}_o)]$$

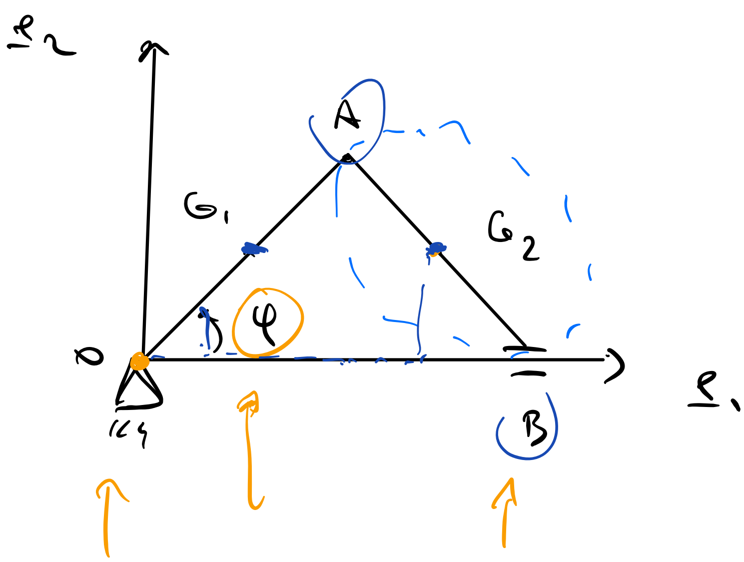
$$\hookrightarrow \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & \boxed{I_{33}} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$\uparrow$ " $m\, r^2$ "

$I_{33} = I_{11} + I_{22}$ per sistemi piani

$$\begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

Esempio



$\overline{OA} = \overline{AB} = \ell$

e uno no m

$$K = \sum_p \frac{1}{2} m_p v_p^2$$

$$= K_{OA} + K_{AB}$$

$$\cdot\ K_{OA} = \frac{1}{2} I_{3,0}\, \dot{\varphi}^2 = \frac{1}{2}\, \frac{m \ell^2}{3}\, \dot{\varphi}^{2}$$

$$\cdot\ K_{AB} = \frac{1}{2} m\, v_{G_2}^2 + \frac{1}{2}\, \frac{m \ell^2}{12}\, \dot{\varphi}^{2}$$

G_2

$$\rightarrow \frac{1}{2} m v_P^2 + \frac{1}{2} I_P\, \omega^2 + m\, v_P \wedge - - - -$$

$$\underline{x}_{G_2}(t) = \frac{3}{2} \ell \cos\varphi(t)\, \underline{e}_1 + \frac{\ell}{2} \sin\varphi(t)\, \underline{e}_2$$

$$\downarrow$$

$$\underline{v}_{G_2}(t) = \frac{d}{dt}\, \underline{x}_{G_2}(t) =$$

$$= -\frac{3}{2} \ell \sin\varphi(t)\, \dot{\varphi}(t)\, \underline{e}_1 + \frac{\ell}{2}\, \dot{\varphi}(t) \cos\varphi(t)\, \underline{e}_2$$

$$\frac{d}{dt} \cos\varphi = -\dot{\varphi} \sin\varphi$$

$$\varphi = \varphi(t)$$

$$v^2 = \| \underline{v}_G \|^2 = \underline{v}_G \cdot \underline{v}_G =$$

$$= \frac{\ell^2}{4}\, \dot{\varphi}^2 \left(9 \sin^2\varphi + \cos^2\varphi \right)$$

$$= \frac{\ell^2}{4}\dot{\varphi}^2 \left(8 \sin^2\varphi + 1 \right)$$

$$K = \frac{1}{2}\frac{m\ell^2}{3}\dot{\varphi}^2 + \frac{1}{2} m \left[\frac{\ell^2}{4}\dot{\varphi}^2 \left(8 \sin^2\varphi + 1 \right) \right]$$

$$+ \frac{1}{2}\frac{m\ell^2}{12}\dot{\varphi}^2$$

$$= \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left[2 \sin^2\varphi + \frac{2}{3} \right]$$

Forze : Q_i $\qquad \frac{d}{dt}\frac{\partial K}{\partial \dot{\varphi}} - \frac{\partial K}{\partial \varphi} = Q_i$

$V \longrightarrow \mathcal{L} = K - V$, $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$

ECS $\longrightarrow$ ECD

$$\underline{P} = M \underline{v}_G$$

$$\begin{cases} \underline{R}^e = \frac{d}{dt}\underline{P} \\ \underline{M}^e(O) = \frac{d}{dt}\underline{L}(O) + \underline{v}_O \wedge \underline{P} \end{cases}$$

$$\underline{L}(O) = \sum_{B \in S} (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B$$

A

$$\sum_{R_1} + \sum_{R_2}$$

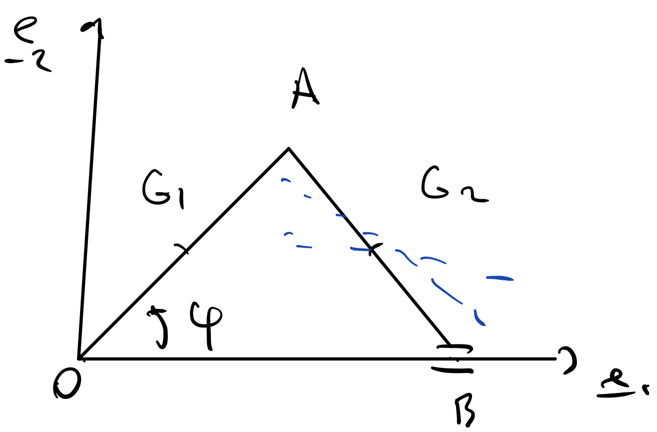
$O \in R$

$$\underline{L}(O) = M (\underline{x}_G - \underline{x}_O) \wedge \underline{v}_O + \underline{\underline{I}}_O (\underline{\omega})$$

$$\left(\begin{array}{ll} O \text{ fisso} & \underline{L}(O) = I_O(\underline{\omega}) \\ O \equiv G & \underline{L}(G) = I_G(\underline{\omega}) \end{array} \right)$$

$O \notin R$, scegli $C \in R$

$$\underline{L}(O) = \underline{L}(C) + \underline{(\underline{x}_C - \underline{x}_O) \wedge M \underline{v}_G}$$



$$\underline{L}(O) = \underline{L}_{OA}(O) + \underline{L}_{AB}(O)$$

$O \in OA$ $\quad$ $O \notin AB$

$$\underline{L}_{OA}(O) = I_O(\underline{\omega}) = \frac{m\ell^2}{3}\dot\varphi\,\underline{e}_3$$

$$\uparrow \quad \dot\varphi\,\underline{e}_3$$

$$\underline{L}_{AB}(O) = \underline{L}_{AB}(G_2) + (\underline{x}_{G_2} - \underline{x}_O) \wedge m\,\underline{v}_{G_2}$$

$$\downarrow$$

$$I_G(\underline{\omega}) = -\frac{m\ell^2}{12}\dot\varphi\,\underline{e}_3$$

$$\hookrightarrow \; -\dot\varphi\,\underline{e}_3$$

$$(\underline{x}_{G_2} - \cancel{\underline{x}_O}^{\,O}) \wedge m\,\underline{v}_{G_2} =$$

$$\left(\begin{aligned} \underline{x}_{G_2} &= \frac{3}{2}\ell\cos\varphi\,\underline{e}_1 + \frac{\ell}{2}\sin\varphi\,\underline{e}_2 \\ \underline{v}_G &= \frac{\ell}{2}(-3\sin\varphi\,\underline{e}_1 + \cos\varphi\,\underline{e}_2)\dot\varphi \end{aligned}\right)$$

$$= \left(\frac{3}{2}\ell\cos\varphi\,\underline{e}_1 + \frac{\ell}{2}\sin\varphi\,\underline{e}_2\right) \wedge$$

$$\wedge\, m\frac{\ell}{2}\left[-3\sin\varphi\,\underline{e}_1 + \cos\varphi\,\underline{e}_2\right)\dot\varphi$$

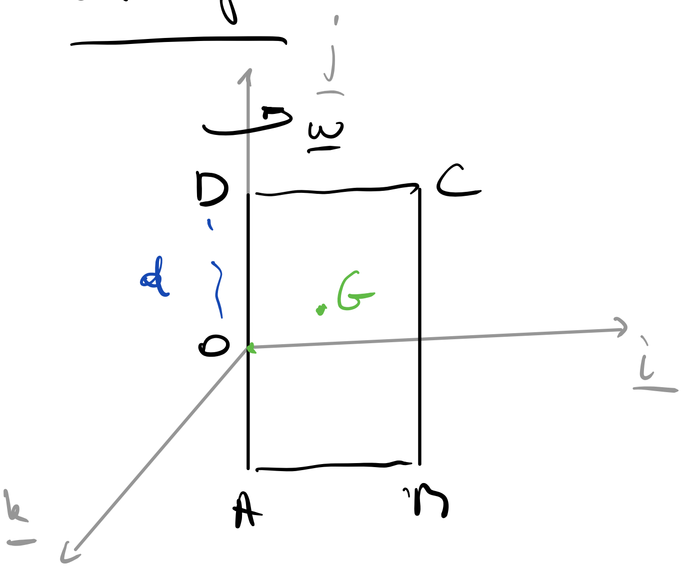
$$= m\frac{\ell}{2}\Big[-3\frac{\ell}{2}\sin^2\varphi\,\underbrace{\underline{e}_2\wedge\underline{e}_1}_{-\underline{e}_3} + \frac{3}{2}\ell\cos^2\varphi\,\underbrace{\underline{e}_1\wedge\underline{e}_2}_{\underline{e}_3}\Big]\dot\varphi$$

$$= m \ell^2 \frac{3}{4} \dot\varphi \, \underline{e}_3$$

$$\underline{L}(O) = \frac{m\ell^2}{3} \dot\varphi \, \underline{e}_3 + \left(- \frac{m\ell^2}{12} \dot\varphi \, \underline{e}_3 + \right.$$

$$\left. + m \ell^2 \frac{3}{4} \dot\varphi \, \underline{e}_3 \right)$$

Esempio



Lamina rettangolare omogenea, in rotazione intorno ad AD, con una cerniera cilindrica in O

$$\underline{\omega} = \dot\varphi \, \underline{j}$$

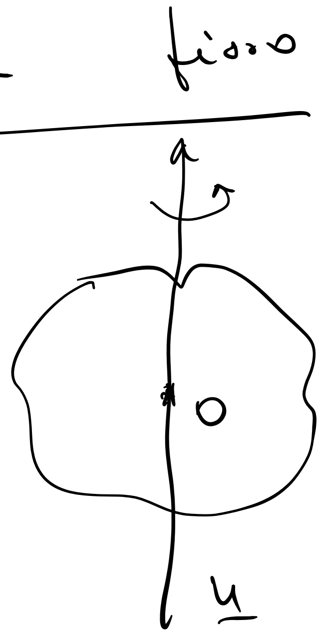
$$\underline{L}(O) = I_O(\underline{\omega}) = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \dot\varphi \\ 0 \end{pmatrix} \begin{array}{c} \underline{i} \\ \underline{j} \\ \underline{k} \end{array}$$

$$= I_{12} \dot\varphi \, \underline{i} + I_{22} \dot\varphi \, \underline{j}$$

Se $\underline{j}$ è asse principale di inerzia $\Rightarrow I_{12} = 0 \Rightarrow \underline{L} \parallel \underline{\omega}$

Se $\underline{j}$ non è asse principale di inerzia $\Rightarrow I_{12} \neq 0 \Rightarrow \underline{L} \nparallel \underline{\omega}$

Rigido in rotazione attorno ad un asse fisso



cerniera cilindrica in O

φ coordinata libera

ECD:

$$\underline{R}^{e} = \underline{R}^{e,\,attiva} + \underline{F}_O^{r} = \underline{\dot{P}}$$

$\uparrow$ 1 nota $\uparrow$ 3 incognite

$$\underline{M}^{e}(O) = \underline{M}^{e,\,attiv}(O) + \underline{\mu}^{2}_{O} = \frac{d}{dt}\,\underline{L}(O)$$

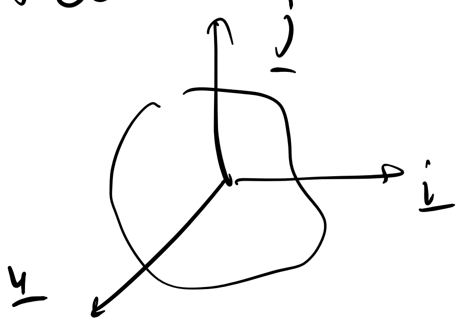
↑ momento forza di reazione

$$\underline{\mu}^{2} \cdot \underline{u} = 0$$

niente attrito

$(\longrightarrow$ 2 incognite $)$

Terna solidale $\underline{i}, \underline{j}, \underline{k}$



$$\underline{j} = \underline{u}$$

$$\underline{L}(O) = I_O(\underline{\omega})$$

$$= \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\varphi} \\ 0 \end{pmatrix}$$

$$= \dot{\varphi}\left(I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k} \right)$$

Vogliamo $\dfrac{d}{dt}\,\underline{L}(O)$

Supponiamo $\underline{r}$ vettore fisso in S e dipendente dal tempo.

$\underline{r} = \underline{x}_A - \underline{x}_B$ e usiamo

Poisson utilizzando B come polo

$$\frac{d}{dt}\underline{x}_A = \frac{d}{dt}\underline{x}_B + \underline{\omega} \wedge (\underline{x}_A - \underline{x}_B)$$

$$\frac{d}{dt}\underbrace{(\underline{x}_A - \underline{x}_B)}_{\underline{r}} = \underline{\omega} \wedge \underbrace{(\underline{x}_A - \underline{x}_B)}_{\underline{r}}$$

$$\boxed{\frac{d}{dt}\underline{r} = \underline{\omega} \wedge \underline{r}}$$

$$\underline{L}(O) = \dot{\varphi}\, I_O(\underline{u}) = \dot{\varphi}\left[I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k} \right]$$

$$\frac{d}{dt}\underline{L}(O) = \frac{d}{dt}\left[\dot{\varphi}\, I_O(\underline{u})\right] =$$

$$= \left[\frac{d}{d\tau}\dot{\varphi}\right] I_o(\underline{u}) + \dot{\varphi}\frac{d}{d\tau} I_o(\underline{u})$$

$$= \ddot{\varphi}\, I_o(\underline{u}) + \dot{\varphi}\left[\underline{\omega} \wedge I_o(\underline{u})\right]$$

$$= \ddot{\varphi}\, I_o(\underline{u}) + \dot{\varphi}^2\, \underline{u} \wedge I_o(\underline{u})$$

$$\underline{\omega} = \dot{\varphi}\,\underline{u}$$
$$= \dot{\varphi}\,\underline{j}$$

$$= I_o(\dot{\underline{\omega}}) + \underline{\omega} \wedge I_o(\underline{\omega})$$

$$\boxed{\frac{d}{d\tau}\underline{L}(O) = I_o(\dot{\underline{\omega}}) + \underline{\omega} \wedge I_o(\underline{\omega})}$$

Nel nostro caso :

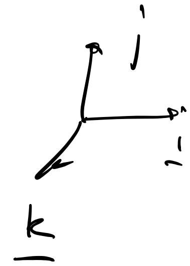
$$\frac{d}{d\tau}\underline{L}(O) = \ddot{\varphi}\, I_o(\underline{u}) + \dot{\varphi}^2\, \underline{u} \wedge I_o(\underline{u})$$

$$= \ddot{\varphi}\left(I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k}\right) +$$

$$+ \dot{\varphi}^2 \, \underline{j} \wedge \left(I_{12} \, \underline{i} + I_{22} \, \underline{j} + I_{23} \, \underline{k} \right)$$

$$= \ddot{\varphi} \left(I_{12} \, \underline{i} + I_{22} \, \underline{j} + I_{23} \, \underline{k} \right) +$$

$$+ \dot{\varphi}^2 \left(- I_{12} \, \underline{k} + I_{23} \, \underline{i} \right)$$



$$= \left(I_{12} \ddot{\varphi} + \dot{\varphi}^2 I_{23} \right) \underline{i} + \ddot{\varphi} \, I_{22} \, \underline{j}$$

$$+ \left(\ddot{\varphi} \, I_{23} - \dot{\varphi}^2 I_{12} \right) \underline{k}$$

ECD dei momenti

$$\underline{i} \qquad \underline{M}^{e,e}(O) \cdot \underline{i} + \underline{\Gamma}_O^{?} \cdot \underline{i} = I_{12} \ddot{\varphi} + \dot{\varphi}^2 I_{23}$$

$$\underline{j} \qquad \underline{M}^{e,e}(O) \cdot \underline{j} = I_{22} \ddot{\varphi} \qquad (\ast)$$

$$\underline{k} \qquad \underline{M}^{e,o}(O) \cdot \underline{k} + \underline{\Gamma}_O^{?} \cdot \underline{k} = I_{23} \ddot{\varphi} - I_{12} \dot{\varphi}^2$$

Dall' eq per $\underline{j}$ $\longrightarrow$ troviamo φ

e dalle altre due $\underline{\mu}^2$

Se $\underline{u}$ asse principale di inerzia

$$I_o(\underline{\omega}) = \dot{\varphi}\, I_o(\underline{u})$$

$$\frac{d}{dt}\underline{L}(O) = \ddot{\varphi}\, I_2\, \underline{u}$$

(perché $I_o(\underline{u}) = I_2\, \underline{u}$ $\Rightarrow$ $\underline{u} \wedge I_o(\underline{u}) = \underline{0}$)

def asse principale

$$\Rightarrow \underline{M}^{e,a}(O) \cdot \underline{u} = I_2\, \ddot{\varphi}$$

e $\underline{\mu}^2$ si ricavano da (*)

e ha solo parte statica.

Sistemi ad un grado di libertà

con forze conservative

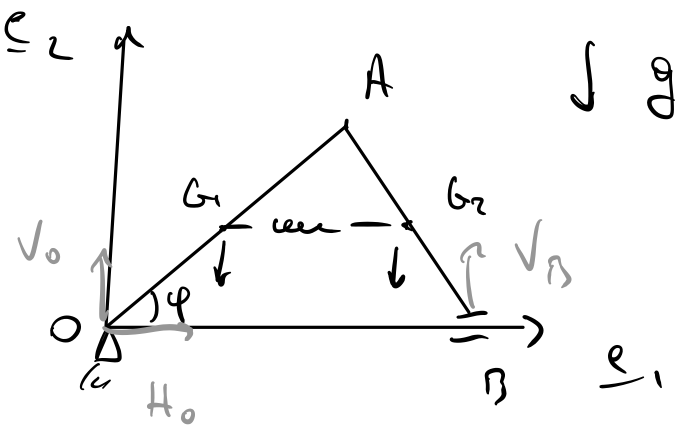
$V \longrightarrow$ energia potenziale

$$\mathcal{L} = K - V \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Notiamo che: forze conservative

$$(K+V)_t = E = (K+V)\big|_{t=0}$$

E si conserva



$OA = AB = \ell$

omogenee

massa m

- $K = K_{OA} + K_{AB}$

$$K = \frac{1}{2} m \ell^2 \dot\varphi^2 \left(\frac{2}{3} + 2\sin^2\varphi\right)$$

- $V = V_{peso} + V_{molla} =$

$$V_{pers} = 2\, m g \frac{\ell}{2} \sin\varphi$$

$$V_{molla} = \frac{c}{2} \ell^2 \cos^2\varphi$$

$$\left(\frac{c}{2} \| \underline{x}_{G_1} - \underline{x}_{G_2} \|^2 \right)$$

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot\varphi^2 \left(\frac{2}{3} + 2 \sin^2\varphi \right) +$$

$$- \left(m g \ell \sin\varphi + \frac{c}{2} \ell^2 \cos^2\varphi \right)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$= \frac{d}{dt} \left[m \ell^2 \dot\varphi \left(\frac{2}{3} + 2 \sin^2\varphi \right) \right] +$$

$$- \left[2 m \ell^2 \dot\varphi^2 \sin\varphi \cos\varphi - m g \ell \cos\varphi + \right.$$

$$\left. + \frac{c}{2} \ell^2\, 2 \cos\varphi \sin\varphi \right]$$

$$= m \ell^2 \ddot\varphi \left(\frac{2}{3} + 2 \sin^2\varphi \right) +$$

$$+ m l^2 \dot\varphi \, 4 \sin\varphi \cos\varphi \, \dot\varphi +$$

$$- 2 m l^2 \dot\varphi^2 \sin\varphi \cos\varphi + mgl \cos\varphi$$

$$- \frac{c}{2} l^2 \, 2 \cos\varphi \sin\varphi$$

$$= \boxed{m l^2 \ddot\varphi \left(\frac{2}{3} + 2 \sin^2\varphi \right) + 2 m l^2 \dot\varphi^2 \sin\varphi \cos\varphi + mgl \cos\varphi - c l^2 \cos\varphi \sin\varphi = 0}$$

equazione del moto

Conservazione dell' energia:

$$K + V = \frac{1}{2} m l^2 \dot\varphi^2 \left(\frac{2}{3} + 2 \sin^2\varphi \right) + \left(mgl \sin\varphi + \frac{c}{2} l^2 \cos^2\varphi \right)$$

$$= E = K + V \Big|_{t=0}$$

Immaginiamo che al tempo $t=0$

ci siano le condizioni iniziali

$$\begin{cases} \varphi(T=0) = \varphi_0 \\ \dot{\varphi}(T=0) = \omega_0 \end{cases}$$

Calcoliamo l'energia:

$$\underline{\underline{E}} = \frac{1}{2} m \ell^2 \omega_0^2 \left(\frac{2}{3} + 2 \sin^2 \varphi_0 \right) + $$

$$+ mg\ell \sin \varphi_0 + \frac{c}{2} \ell^2 \cos^2 \varphi_0$$

$$= \underline{\frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right)} + \underline{V(\varphi)}$$

dove $V(\varphi) = mg\ell \sin \varphi + \frac{c}{2} \ell^2 \cos \varphi$

$$\underline{\underline{\dot{\varphi}^2}} = \frac{2}{m\ell^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right)} \left(E - V(\varphi) \right) =: \underline{\underline{f(\varphi)}}$$

possiamo ricavare $\dot{\varphi}^2$ in funzione di φ.

Allora le equazioni del moto

$$m\ell^2 \ddot{\varphi}\left(\frac{2}{3} + 2\sin^2\varphi\right) + 2m\ell\,\dot{\varphi}^2 \sin\varphi\cos\varphi$$

$$+\; mg\ell\cos\varphi - c\ell^2 \sin\varphi\cos\varphi = 0$$

le possiamo riscrivere come

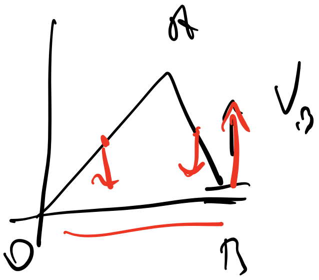
$$\ddot{\varphi} = \frac{1}{m\ell^2\left(\frac{2}{3} + 2\sin^2\varphi\right)}\Big[-2m\ell^2 f(\varphi)\sin\varphi\cos\varphi$$

$$-\, mg\ell\cos\varphi + c\ell^2 \sin\varphi\cos\varphi\Big] =: g(\varphi)$$

Riassumendo : $\underline{\ddot{\varphi} = g(\varphi)}$

Andiamo a calcolare la reazione

V_3



$$\boxed{\underline{M}^e(O) = \frac{d}{dt}\underline{L}(O)}$$

$$\underline{L}(O) = m\ell^2\dot{\varphi}\,\underline{e}_3$$

$$\underline{M}(O) = V_B \, 2\ell \cos\varphi - mg \frac{\ell}{2} \cos\varphi +$$

$$- mg \frac{3\ell}{2} \cos\varphi$$

$$V_B = \frac{m\ell}{2\cos\varphi} \ddot{\varphi} + \underbrace{\frac{1}{2\ell\cos\varphi} \left(mg \, 2\ell \cos\varphi \right)}_{\text{statica}}$$

dinamica

$$= \frac{m\ell}{2\cos\varphi} g(\varphi) + \frac{1}{2\ell\cos\varphi} \left(mg \, 2\ell \cos\varphi \right)$$

V_B esplicitamente in funzione di φ, senza risolvere le eq. del moto.