$c_n(f') = i n\, c_n(f)$

Derivata e serie di Fourier

$\omega = \frac{2\pi}{T}$

$f : \mathbb{R} \to \mathbb{C}$ $\quad f$ T-continua $\quad f' \in L^2([-T/2, T/2])$

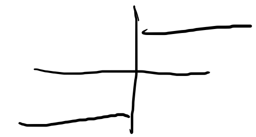
esiste la serie di Fourier di f' $\quad \sum c_n(f')\, e^{in\omega x}$

$$c_n(f') = \frac{1}{T} \int_{-T/2}^{T/2} f'(x)\, e^{-in\omega x}\, dx = \frac{1}{T} \cdot \left(\underbrace{\left[f(x)\, e^{-in\omega x} \right]_{-T/2}^{T/2}}_{=0} + T\, i n\omega\, c_n(f) \right)$$

$= i n\omega\, c_n(f)$

Per induzione si prova che $\boxed{c_n(f^{(k)}) = (in\omega)^k\, c_n(f)}$

Quindi $\frac{d}{dx}\sum_{n=-\infty}^{+\infty} c_n e^{inwx} = \sum_{n=-\infty}^{+\infty} c_n \frac{d}{dx} e^{inwx}$



vale anche per la rappresentazione reale

Es: $f(x) = |x|$ su $]-\pi,\pi]$

$$\hat{f}(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}\cos\left((2n-1)x\right)$$

$f'(x) = sgn(x)$
$\in L^2$

$$sgn(x) = -\frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{-(2n-1)}{(2n-1)^2}\,sen\left((2n-1)x\right) = \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{1}{2n-1}\,sen\left((2n-1)x\right)$$

$\sum \frac{1}{2n-1}$ diverge!

Integrale

OSS: $f : \mathbb{R} \to \mathbb{C}$ primitivabile e sia $F' = f$; supponiamo che F sia T-periodica

allora $0 = F(T/2) - F(-T/2) = \displaystyle\int_{-T/2}^{T/2} \underbrace{F'(x)}_{= f(x)}\, dx$

F è periodica se e solo se f è a media nulla $\displaystyle\frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx = 0$

$$F(x) = \int_{-T/2}^{x} f(t)\,dt$$

[f periodica e a media nulla

$$F(x+T) = \int_{-T/2}^{x+T} f(t)\,dt = \underbrace{\cancel{\int_{-T/2}^{T/2} f(t)\,dt}}_{=0} + \int_{T/2}^{x+T} f(t)\,dt = \int_{-T/2}^{x} \underbrace{f(s)}_{f(s+T) = f(s)}\,ds = F(x)$$

$s = t - T$

Teorema

Sia $f \in L^2([-T/2, T/2])$ T-periodica e a media nulla $\left(c_0(f) = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx = 0 \right)$

$F(x) = \int_{-T/2}^{x} f(t)\,dt$. Allora

se $n \neq 0$ $\quad c_n(F) = \frac{1}{in\omega} c_n(f)$

se $n = 0$ $\quad c_0(F) = \frac{1}{T}\int_{-T/2}^{T/2} F(x)\,dx = -\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(-1)^n}{in\omega} c_n(f)$

$$\hat{F}(x) = \hat{F}(0) + \sum_{n\neq 0} \frac{c_n(f)}{in\omega} e^{in\omega x}$$ (convergenza uniforme)

$$\int_{-T/2}^{x} \Big(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} c_n(f)\, e^{in\omega t} \Big)\, dt = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} c_n(f) \int_{-T/2}^{x} e^{in\omega t}\, dt \qquad \omega = \frac{2\pi}{T} \cdot \left(-\frac{T}{2}\right)$$

$$\int_{-T/2}^{x} e^{in\omega t}\, dt = \frac{1}{in\omega} e^{in\omega x} - \frac{1}{in\omega} e^{-in\pi}, \qquad e^{-in\pi} = (-1)^n$$

$$= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \underbrace{c_n(f) \frac{1}{in\omega}}_{= c_n(F)} e^{in\omega x} + \underbrace{\left(- \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(-1)^n}{in\omega} c_n(f) \right)}_{= c_0(F)}$$

$F'(x) = f(x) \qquad f \in L^2 \qquad$ se $F' \in L^2 \Rightarrow$ la serie di F converge uniformemente

$$F(x) = c_0(F) + \sum_{n \neq 0} c_n(F)\, e^{in\omega x} \qquad c_0(F) = \frac{1}{T}\int_{-T/2}^{T/2} F(x)\,dx$$

$$c_n(F) = \frac{1}{T}\int_{-T/2}^{T/2} F(x)\, e^{-in\omega x}\,dx = \frac{1}{T}\left(\underbrace{\left[\frac{1}{-in\omega}\, e^{-in\omega x} F(x)\right]_{-T/2}^{T/2}}_{=0} + \frac{1}{in\omega}\int_{-T/2}^{T/2} f(x)\, e^{-in\omega x}\,dx \right)$$

$$\int_{-T/2}^{T/2} f(x)\,dx = 0$$

$$= \frac{1}{in\omega}\, c_n(f)$$

$$\underbrace{F(T/2)}_{=0} = c_0(F) + \sum_{n \neq 0} \frac{1}{in\omega}\, c_n(f)\, \underbrace{e^{in\pi}}_{(-1)^n}$$

$$0 = c_0(F) + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{(-1)^n}{in\omega}\, c_n(f)$$

Es: $f(x) = x$ in $]-\pi, \pi]$ $\qquad \hat{f}(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \operatorname{sen}(nx)$

$$F(x) = \frac{1}{2}x^2 = \int_{-\pi}^{x} f(t)\,dt + \frac{1}{2}\pi^2$$

$$F(x) = \frac{1}{2}\pi^2 + \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2}{n} \int_{-\pi}^{x} \operatorname{sen}(nt)\,dt = \frac{1}{2}\pi^2 + \sum_{n=1}^{+\infty} \frac{2(-1)^n}{n^2} \cos(nx) - \sum_{n=1}^{+\infty} \frac{2}{n^2}$$

$$\int_{-\pi}^{x} \operatorname{sen}(nt)\,dt = -\frac{1}{n}\cos(nx) + \frac{1}{n}\cos(n\pi), \qquad \cos(n\pi) = (-1)^n$$

$$\sum_{n=1}^{+\infty} \frac{2}{n^2} = \frac{2\pi^2}{6} = \frac{\pi^2}{3}$$

$$c_0(F) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}x^2\,dx = \frac{1}{2\pi} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\pi^3 = \frac{\pi^2}{6}$$

$$\left(\frac{1}{2} - \frac{1}{3}\right)\pi^2 = \frac{3-2}{6}\pi^2 = \frac{\pi^2}{6}$$

Regolarità della funzione e ordine di infinitesimo della successione $(c_n(f))_n$

$f \in L^2([-T/2, T/2])$ Lemma di Riemann-Lebesgue $\Rightarrow \lim_{|n|\to+\infty} c_n(f) = 0$

Sia $\boxed{f \in C^k(\mathbb{R})}$ allora $c_n(f^{(k)}) = (in\omega)^k c_n(f)$

poichè la serie di $f^{(k)}$ converge si ha $\lim_{|n|\to+\infty} (n\omega)^k c_n(f) = 0$

$\Rightarrow \operatorname{ord}_{+\infty}(c_n(f)) > k$

$$\frac{c_n(f)}{\frac{1}{n^k}} = n^k c_n(f) \to 0$$

$$\boxed{\operatorname{ord}_{+\infty} c_n(f) > k}$$

⚠ Non vale il viceversa!

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x)$$

$\operatorname{ord} \frac{1}{(2n-1)^2} = 2 > 1$ ma $f \notin C^1$

$$f(x) = \frac{\pi}{4}\left(\frac{\pi}{2} - |x|\right)$$

Comunque, se ord $|c_n(f)| \geq k+1+\varepsilon$ $\quad \varepsilon > 0$ allora $f \in C^k$

se $\sum_{n\in\mathbb{Z}} |n^k c_n(f)| < +\infty$ allora $f \in C^k$

$\sum \left|(in\omega)^k c_n(f)\, e^{in\omega x}\right|$ converge per il M-test di Weierstrass

Equazione del calore

$u(0,t) = 0 \quad \forall t$

$u(x,0) = \varphi(x)$

$$\begin{cases} u_t = u_{xx} &]0,\pi[\times]0,+\infty[\\ u(0,t) = u(\pi,t) = 0 &]0,+\infty[\quad \text{(condizioni al bordo)} \\ u(x,0) = \varphi(x) & [0,\pi] \quad \text{(condizione iniziale)} \end{cases}$$

$\varphi(0) = u(0,0) = 0$

$\boxed{\varphi : [0,\pi] \to \mathbb{R}}$

sono soluzioni dell'equazione $u_t = u_{xx}$ che soddisfano le condizioni al bordo le funzioni

$$u_N(x,t) = \sum_{n=1}^{N} c_n e^{-n^2 t} \operatorname{sen}(nx) \qquad c_n \in \mathbb{R}$$

cerchiamo $$u(x,t) = \sum_{n=1}^{+\infty} c_n e^{-n^2 t} \operatorname{sen}(nx)$$

supponiamo $\varphi \in L^2([0,\pi])$ estendiamo φ a $\hat{\varphi} : [-\pi,\pi] \to \mathbb{R}$ per simmetria ponendo

$\hat{\varphi}(-x) = -\varphi(x)$ se $x \in [0,\pi]$ $\quad (\varphi(0) = 0)$

$$\hat{\varphi}(x) \simeq \sum_{n=1}^{+\infty} c_n \operatorname{sen}(nx) dx \qquad c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{\varphi}(x) \operatorname{sen}(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \operatorname{sen}(nx)\, dx$$

$$c_n = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin(nx)\, dx$$

$$u(x,t) = \sum_{n=1}^{+\infty} c_n e^{-n^2 t} \sin(nx)$$

$\forall t$

$\mathrm{ord}_{+\infty} \left| c_n e^{-n^2 t} \right|$ sopraroalle

$u(\cdot, t)$ è C^{∞}!

u è una funzione di classe C^{∞} anche se φ è discontinua!

"Effetto regolarizzante dell'equazione del calore"

Logaritmo in $\mathbb{C}$

e^z è periodico di periodo $2\pi i$

$e^z = w$ $\quad w \in \mathbb{C}$, esiste $z \in \mathbb{C}$ tale che $e^z = w$?

se $w = 0$ NO $\quad e^z = 0$ non ha soluzioni

$z = x + iy \qquad |e^z| = e^x > 0 \quad \forall x!$

se $w \neq 0 \qquad w = \rho_0 e^{i\vartheta_0}$

$$e^{x+iy} = w = \rho_0 e^{i\vartheta_0} \Leftrightarrow \boxed{e^x \cdot e^{iy} = \rho_0 e^{i\vartheta_0}}$$

$$\Downarrow$$

$$e^x = \rho_0 \qquad y = \underline{\vartheta_0 + 2k\pi} \quad k \in \mathbb{Z}$$

$x = \log(\rho_0)$

Si dice determinazione principale del logaritmo la funzione

$$\operatorname{Log} w = \log|w| + i \operatorname{Arg} w \qquad \operatorname{Log}: \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

è l'inverso della funzione e^z ristretta all'insieme $E = \{ \underset{x+iy}{\overset{\shortparallel}{z}} \in \mathbb{C} : -\pi < y \leq \pi \}$

$$\operatorname{Log}(-e) = \log \overset{1}{\overset{\shortparallel}{|e|}} + i\pi = 1 + i\pi \qquad \operatorname{Log}(-1) \overset{?}{=} \cancel{1 + e^{i\frac{\pi}{2}}} = i\pi$$

$$-e = e \cdot \underset{-1}{\underset{\shortparallel}{e^{i\pi}}} \qquad |-1| = 1 \quad \operatorname{Arg}(-1) = \pi$$

$$\log 1 = 0$$

1) $\boxed{\mathrm{Log}(e^{z}) = z}$ $\forall z \in \mathbb{C}$ NO

$\Sigma = \{ z = x + iy : x \in \mathbb{R}$

$-\pi < y \leq \pi \} \setminus \{0\}$

$\mathrm{Log}(e^{2\pi i}) = \mathrm{Log}(1) = 0 \neq 2\pi i$

$e^{2\pi i} = 1$

2) $\boxed{e^{\mathrm{Log}\, z} = z \quad \forall z \neq 0}$

$e^{\mathrm{Log}\, z} = e^{\ln|z| + i \,\mathrm{Arg}\, z} = e^{\ln(|z|)} \cdot e^{i \,\mathrm{Arg}\, z}$

$= |z| \cdot e^{i \,\mathrm{Arg}\, z} = z$

3) $\mathrm{Log}(z+w) = \mathrm{Log}(z) \cdot \mathrm{Log}(w)$ NO!!

$\log(2) \neq \ln(1) \cdot \ln(1) = 0$

4) $\mathrm{Log}(z\cdot w) \overset{?}{=} \mathrm{Log}(z) + \mathrm{Log}(w)$

$$0 = \mathrm{Log}\big(\underbrace{(-1)\cdot(-1)}_{=\,1}\big) \neq \underbrace{\mathrm{Log}(-1)}_{=\,\pi i} + \mathrm{Log}(-1) = 2\pi i$$

5) $\mathrm{Log}(z^n) \overset{?}{=} n\,\mathrm{Log}(z)$

$$0 = \mathrm{Log}\big((-1)^2\big) \neq 2\,\mathrm{Log}(-1) = 2\pi i$$

6) $D\,\mathrm{Log}\,z = \dfrac{1}{z}$ se $z \in \mathbb{C} \setminus \,]-\infty, 0[$

oss) : $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ non è continua su $]-\infty, 0[$

se $z \in \mathbb{C} \setminus]-\infty, 0]$

$$D \operatorname{Log} z \Big|_{z=z_0} = \lim_{z \to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} = \frac{1}{e^w \Big|_{w = \operatorname{Log} z_0}} = \frac{1}{z_0}$$

se $\underline{z \to z_0}$ allora posto $z = e^w$ $z_0 = e^{w_0}$ $w = \operatorname{Log} z$

$\underline{w \to w_0}$ $w_0 = \operatorname{Log} z_0$