Antitrasformata

$\mathcal{F} : L^1(\mathbb{R}) \to C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $\cancel{\mathcal{F}\{f\}(\omega) = 1}$

$\mathcal{F}$ non è suriettivo (Lemma di Riemann-Lebesgue)

$\mathcal{F}$ iniettivo?

$$f(x) \simeq \int_{-\infty}^{+\infty} \frac{1}{2\pi} \underbrace{\left(\int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt \right)}_{= \mathcal{F}\{f\}(\omega)} e^{i\omega x}\, d\omega \quad ?$$

È vero che $f(x) = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x)$?

Def: $\mathcal{F}^{-1}\{g\}(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} g(\omega)\, e^{i\omega x}\, d\omega$

Tentativo

$$\mathcal{F}\{\mathcal{F}^{-1}\{g\}\}(\lambda) = \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega)\, e^{i\omega x}\, d\omega \right) e^{-i\lambda x}\, dx \overset{?}{=}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{i(\omega-\lambda)x}\, dx \right) g(\omega)\, d\omega$$

non è integrabile

Teorema

$f \in L^1(\mathbb{R})$, sia $\hat{f} \in L^1(\mathbb{R})$, definiamo $\mathcal{F}^{-1}\{\hat{f}\}(x) = \frac{1}{2\pi}\int^{+\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$

$\frac{1}{2\pi}\mathcal{F}\{\hat{f}\}(-\omega)$

Allora $\boxed{\mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x) = f(x)}$ e $\mathcal{F}\{\mathcal{F}^{-1}\{\hat{f}\}\}(\omega) = \hat{f}(\omega)$

Dim $k \in \mathbb{N}^+$

poniamo $\varphi_k(\omega) = e^{-\frac{\omega^2}{2k^2}}$ $\Rightarrow$ $\mathcal{F}\{\varphi_k\}(t) = \frac{1}{\sqrt{2\pi}}\, k\, e^{-\frac{k^2}{2}t^2} = k\,\phi(kt)$

dove $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ Nucleo di Gauss $(G(x))$

$k\,\phi(kt) = \frac{k}{\sqrt{2\pi}}\, e^{-\frac{k^2 t^2}{2}}$

Poniamo $F_k(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\mathcal{F}\{f\}(\omega)} \cdot \varphi_k(\omega)\, e^{i\omega x}\, d\omega$ $\left[\varphi_k(\omega) = e^{-\frac{\omega^2}{2k^2}}\right]$

Vorremo vedere che

① $\lim_{k\to+\infty} F_k(x) = f(x)$

② $\lim_{k\to+\infty} F_k(x) = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x)$

①
$$F_k(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(s)\, e^{-i\omega s} \cdot \varphi_k(\omega)\, e^{i\omega x}\, d\omega\right) ds = \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\varphi_k(\omega)\, e^{-i\omega(s-x)}\, d\omega\right) f(s)\, ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{k\,\Phi(k(s-x))}_{\Phi_k(s-x)} f(s)\, ds$$

$$\| \quad \mathcal{F}\{\varphi_k\}(s-x) = k\,\Phi(k(s-x))$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\mathcal{F}\{\varphi_k\}(s-x)}_{\cancel{\varphi}} f(s)\,ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sqrt{2\pi}\, k\, e^{-\frac{k^2}{2}(s-x)^2} f(s)\,ds =$$

$$\frac{1}{\sqrt{2\pi}}\, k\, e^{-\frac{k^2}{2}t^2}$$

$$\varphi_k = e^{-\frac{\omega^2}{2k^2}} \rightsquigarrow \sqrt{2\pi}\, k\, e^{-\frac{k^2}{2}}$$

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$\hat{\varphi}_k =$$

$$\Phi_k(t) = k\,\Phi(kt) = \frac{1}{\sqrt{2\pi}}\, k\, e^{-\frac{k^2t^2}{2}}$$

$$e^{-ax^2} \rightsquigarrow \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

$$a = \frac{1}{2k^2}$$

(2) $\lim_{h \to +\infty} F_h(x) = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x)$

$\mathcal{F}\{f\}(\omega)\, \varphi_h(\omega)\, e^{i\omega x}$

$\lim_{h \to +\infty}$ " $= \hat{f}\, e^{i\omega x}$

$\varphi_h(\omega) = e^{-\frac{\omega^2}{2h^2}} \longrightarrow 1$

$\lim_{h \to +\infty} F_h(x) = \lim_{h \to +\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\omega)\, \varphi_h(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\omega)\, e^{i\omega x}\, d\omega$

(lebesgue)

$= \mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x)$

Quindi se $\boxed{f \in L^1(\mathbb{R}),\quad \hat{f} \in L^1(\mathbb{R})}$ allora

$$\boxed{\mathcal{F}^{-1}\{f\}(\omega) = \frac{1}{2\pi}\,\mathcal{F}\{f\}(-\omega)}$$ [Formula di dualità]

Per $\mathcal{F}^{-1}$ valgono tutte le proprietà che valgono per $\mathcal{F}$

Inoltre $\mathcal{F}$ è iniettiva $\quad \mathcal{F}\{f_1\}(\omega) = \mathcal{F}\{f_2\}(\omega)$ q.o. in $\mathbb{R}$

$\Rightarrow \quad f_1(x) = f_2(x)$ q.o. in $\mathbb{R}$

Teorema (Trasformata del prodotto)

$f_1, f_2, f_1 \cdot f_2, \hat{f}_1, \hat{f}_2, \hat{f}_1 \cdot \hat{f}_2, \widehat{f_1 f_2} \in L^1(\mathbb{R})$

Allora $\boxed{\mathcal{F}\{f_1 \cdot f_2\} = \frac{1}{2\pi} \hat{f}_1 * \hat{f}_2}$

$$\mathcal{F}^{-1}\{\hat{f}_1 * \hat{f}_2\}(x) = \frac{1}{2\pi} \mathcal{F}\{\hat{f}_1 * \hat{f}_2\}(-x) =$$

$$= \frac{1}{2\pi}\left(\mathcal{F}\{\hat{f}_1\}(-x) \cdot \mathcal{F}\{\hat{f}_2\}(-x) \right) = 2\pi \left(\frac{1}{2\pi} \mathcal{F}\{\hat{f}_1\}(-x) \cdot \frac{1}{2\pi} \mathcal{F}\{\hat{f}_2\}(-x) \right) =$$

$$\frac{1}{2\pi} \mathcal{F}\{\hat{f}_1\}(-x) = \mathcal{F}^{-1}\{\hat{f}_1\}(x) \cdot \mathcal{F}^{-1}\{\hat{f}_2\}(x)$$

$$= \boxed{2\pi\, f_1(x) \cdot f_2(x)}$$

Applico $\mathcal{F}$ $\Rightarrow$ $\hat{f}_1 * \hat{f}_2 = 2\pi\, \mathcal{F}\{f_1 \cdot f_2\}$

$L^2(\mathbb{R})$

es: $\operatorname{sinc} x = \begin{cases} \dfrac{\operatorname{sen} \pi x}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ è L^2 ma non L^1 !

Idea $f \cdot \chi_{[-n,n]}$ $\qquad \lim\limits_{n \to +\infty} f \cdot \chi_{[-n,n]}(x) = f(x)$

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} f(x)\, \chi_{[-n,n]}(x)\, dx = \lim_{n \to +\infty} \int_{-n}^{n} f(x)\, dx = PV \int_{-\infty}^{+\infty} f(x)\, dx$$

$f \in L^2(\mathbb{R}) \Rightarrow f_n = f \cdot \chi_{[-n,n]} \in L^1(\mathbb{R})$ $\qquad$ idea definire $\tilde{\mathcal{F}}\{f\} = \lim\limits_{n \to +\infty} \mathcal{F}\{f_n\}$

Teorema di Plancherel

È possibile definire un operatore lineare continuo e biettivo $\tilde{\mathcal{F}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ che verifica:

1) $\tilde{\mathcal{F}}\{f\} = \mathcal{F}\{f\} \quad \forall f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

2) $\tilde{\mathcal{F}}\{f\} = \lim_{n \to +\infty} \mathcal{F}\{f \cdot \chi_{[-n,n]}\}$ (limite in L^2)

3) $\langle \tilde{\mathcal{F}}\{f\}, \tilde{\mathcal{F}}\{g\} \rangle_{L^2} = 2\pi \langle f, g \rangle_{L^2} \quad \forall f, g \in L^2(\mathbb{R})$

$\| \tilde{\mathcal{F}}\{f\} \|_{L^2} = \sqrt{2\pi} \, \| f \|_{L^2}$

Identità di Parseval

$\| f \|_{L^2} = \sqrt{2\pi} \, \| \hat{f}(n) \|_{\ell^2}$

Dim

Lemma Siano $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$, allora $\langle \hat{\varphi}_1, \hat{\varphi}_2 \rangle_{L^2} = 2\pi \langle \varphi_1, \varphi_2 \rangle_{L^2}$

$\|\hat{\varphi}\|_{L^2}^2 = 2\pi \|\varphi\|_{L^2}^2$

Dim $\langle \varphi_1, \varphi_2 \rangle_{L^2} = \int_{-\infty}^{+\infty} \varphi_1(x) \cdot \overline{\varphi_2(x)}\, dx =$

$= \int_{-\infty}^{+\infty} \mathcal{F}^{-1}\{\mathcal{F}\{\varphi_1\}\}(x) \cdot \overline{\varphi_2(x)}\, dx = \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}_1(\omega)\, e^{i\omega x}\, d\omega \right) \cdot \overline{\varphi_2(x)}\, dx =$

Fubini Tonelli

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}_1(\omega) \cdot \overline{\left(\int_{-\infty}^{+\infty} e^{-i\omega x} \varphi_2(x)\, dx \right)}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\varphi}_1(\omega) \cdot \overline{\hat{\varphi}_2(\omega)}\, d\omega = \frac{1}{2\pi} \langle \hat{\varphi}_1, \hat{\varphi}_2 \rangle$

$\int_{-\infty}^{+\infty} e^{-i\omega x} \varphi_2(x)\, dx = \hat{\varphi}_2(\omega)$

Sia ora $f \in L^2(\mathbb{R})$ e prendiamo una successione $(f_n)_n$ $f_n \in C_0^\infty(\mathbb{R})$ tale che $f_n \to f$ in L^2.

Allora la successione $(\hat{f}_n)_n$ è di Cauchy:

$$\| \hat{f}_n - \hat{f}_m \|_{L^2}^2 = \| \mathcal{F}\{f_n - f_m\} \|_{L^2}^2 \underset{\text{Lemma}}{=} 2\pi \| f_n - f_m \|_{L^2}^2 \to 0$$

$L^2(\mathbb{R})$ è completo, esiste $\lim_{n \to +\infty} \hat{f}_n = \hat{\mathcal{F}}\{f\}$

Poiché $f \cdot \chi_{[-n,n]} \to f$ in $L^2(\mathbb{R})$ si ottiene la formula 2

Valgono anche le 3 per continuità.

Stesso discorso per $\tilde{\mathcal{F}}^{-1}\{g\} = \lim_{n\to+\infty} \mathcal{F}^{-1}\{g \cdot \chi_{[-n,n]}\}$

Es: $\hat{\mathcal{F}}\{\operatorname{sinc}(x)\}(\omega) = ?$ $\quad \operatorname{sinc}(x) \in L^2 - L^1$

$\widehat{P_{2a}} = 2a \operatorname{sinc}\left(\frac{a\omega}{\pi}\right)$ applico $\tilde{\mathcal{F}}^{-1} : L^2 \to L^2$

$\tilde{\mathcal{F}}^{-1}\{\hat{P}_{2a}\}(x) = 2a\, \tilde{\mathcal{F}}^{-1}\left\{\underline{\operatorname{sinc}\left(\frac{a\omega}{\pi}\right)}\right\}(x)$ prendiamo $a=\pi$

$\parallel$

$P_{2a}(x)$ $\quad P_{2\pi}(x) = 2\pi\, \hat{\mathcal{F}}^{-1}\{\operatorname{sinc}(\omega)\}(x) \overset{\text{quindi}}{=} \cancel{2\pi} \cdot \frac{1}{\cancel{2\pi}}\, \tilde{\mathcal{F}}\{\operatorname{sinc}(\omega)\}(-x)$

$\Rightarrow \hat{\mathcal{F}}\{\operatorname{sinc}(\omega)\}(x) = P_{2\pi}(-x) = P_{2\pi}(x)$ $\quad \boxed{\hat{\mathcal{F}}\{\operatorname{sinc}(x)\}(\omega) = P_{2\pi}(\omega)}$

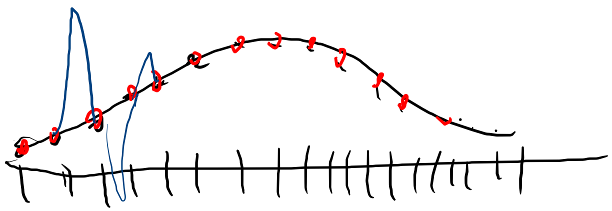
$$\mathcal{F}\left\{ \operatorname{senc}\left(\frac{a}{\pi}x\right)\right\}(\omega) = \frac{\pi}{a} P_{2a}(\omega)$$

$$\mathcal{F}\left\{ \frac{\operatorname{sen} x}{x}\right\}(\omega) = \pi\, P_2(\omega)$$

$$\operatorname{senc}\left(\frac{x}{\pi}\right) =$$

Applicazioni

$$f(\omega \cdot n)$$



Teoria dei campionamenti

$f \in L^1(\mathbb{R})$; f è a banda limitata se $\mathcal{F}\{f\}$ è a supporto compatto

$\exists K$:

$$\hat{f}(\omega) = 0 \quad \forall\, |\omega| > K$$

Es: $$\operatorname{sinc}\left(\frac{a}{\pi}x\right) = \begin{cases} \dfrac{\operatorname{sen} ax}{ax} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

è a banda limitata

$$\operatorname{sinc}_a(x) = \operatorname{sinc}\left(\frac{a}{\pi}x\right)$$

Sia $f \in L^2(\mathbb{R})$ $a>0$ la funzione $f * \mathrm{senc}_a$ è a banda limitata

perché $\mathcal{F}\{f * \mathrm{senc}_a\} = \hat{f} \cdot \frac{\pi}{a} p_{2a}$ ha supporto compatto contenuto in $[-a,a]$

Sia f a banda limitata; consideriamo

$\sigma_0 = \min\{\sigma : \hat{f}(\omega)=0 \quad \forall \omega \quad |\omega|>\sigma\}$ $\qquad \nu = \frac{\sigma_0}{2\pi}$ $\qquad \nu$ si

2ν si dice la frequenza di campionamento di Nyquist e rappresenta la minima frequenza di campionamento che permette la ricostruzione esatta del segnale a partire dalla sequenza dei campionamenti $\left(f\left(\frac{n}{2\nu}\right)\right)_{n \in \mathbb{Z}}$

Teorema di Shannon-Nyquist

Sia $f \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ e a banda limitata; $\sigma_0 = \min\{\sigma > 0 : \hat{f}(\omega) = 0 \text{ se } |\omega| > \sigma\}$

$\boxed{\nu = \frac{\sigma_0}{2\pi}}$. Allora $\forall x \in \mathbb{R}$

$$f(x) = \sum_{n=-\infty}^{+\infty} f\left(\frac{n}{2\nu}\right) \operatorname{sinc}\left(2\nu x - n\right)$$

Dim $\hat{f} \in L^\infty(\mathbb{R}) \cap L^2(\overbrace{[-\sigma_0, \sigma_0]}^{T = 2\sigma_0}) \cap L^1(\mathbb{R})$; $\hat{f}$ sviluppabile in serie di Fourier

$$\hat{f}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{2\pi}{2\sigma_0}t} \qquad c_n = \frac{1}{2\sigma_0}\int_{-\sigma_0}^{\sigma_0} \hat{f}(t)\, e^{-in\frac{2\pi}{2\sigma_0}t}\, dt = * \qquad \frac{\pi}{\sigma_0} = \frac{1}{2\nu}$$

$$* = \frac{2\pi}{2\sigma_0} \frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0} \hat{f}(t)\, e^{it\left(-n\frac{1}{2\nu}\right)}\, dt \quad \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\right) = \frac{1}{2\nu} f\left(-n\frac{1}{2\nu}\right)$$

$$f(x) = \frac{1}{2\pi}\int_{\cancel{-\infty}\;-\sigma_0}^{\cancel{+\infty}\;\sigma_0} \underline{\hat{f}(t)}\, e^{itx}\,dt = \frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0} \sum_{n=-\infty}^{+\infty} \frac{1}{2w} f\left(-\frac{n}{2w}\right) e^{in\frac{1}{2w}t}\, e^{itx}\,dt$$

$$it\left(\frac{n}{2w}+x\right)$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{4\pi w} f\left(-\frac{n}{2w}\right) \underbrace{\int_{-\sigma_0}^{\sigma_0} e^{i\frac{t(2wx+n)}{2w}}\,dt}$$