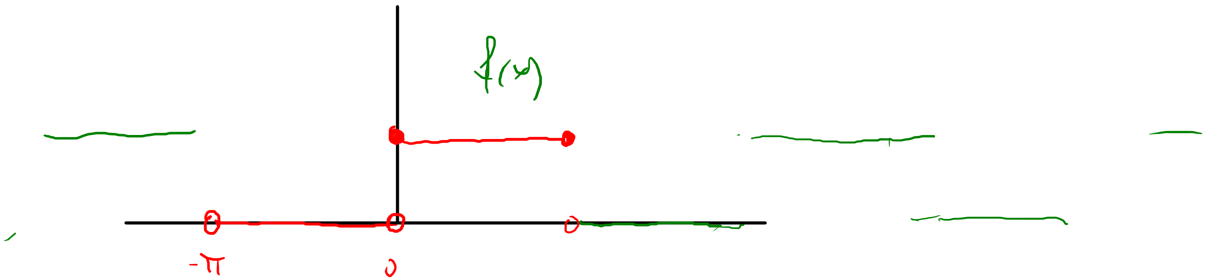
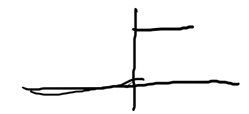
$f \in L^1([-\pi,\pi])$ $\qquad P_N[f](x) = \sum_{n=-N}^{N} c_n e^{inx}$ $\quad$ o $\quad P_N[f](x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \,\mathrm{sen}(nx)$

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(nx)\, dx$

Es: $f(x) = \chi_{[0,\pi]}(x)$ $\qquad f: \,]-\pi,\pi] \to \mathbb{R}$ esteso per periodicità a $\mathbb{R}$

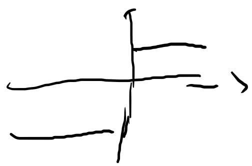


$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 1 \qquad a_n = \frac{1}{\pi}\int^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int \cos(nx)\,dx = 0$$



$$c_n = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\,\frac{1}{-in}\Big(\underbrace{e^{-in\pi}}_{(-1)^n} - 1\Big)$$

$$\frac{a_0}{2}$$



$$b_n = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen}(nx)\,dx = \frac{1}{n\pi}\Big[-\underbrace{\cos(n\pi)}_{(-1)^n} + 1\Big] = \frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$\boxed{1-(-1)^n = \begin{cases} 0 & \text{se } n \text{ è pari} \\ 2 & \text{se } n \text{ è dispari} \end{cases}}$$

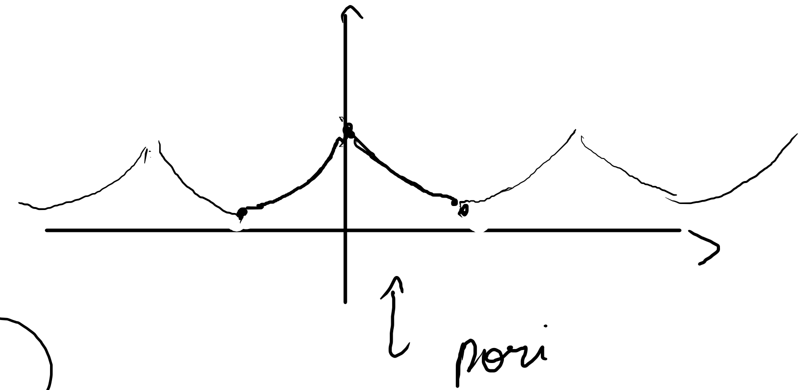
$$P_N[f](x) = \frac{1}{2} + \sum_{n=1}^{N} \frac{1-(-1)^n}{n\pi}\operatorname{sen}(nx)$$

n dispari $\quad n = 2k+1 \quad k \in \mathbb{N}$

$$P[f](x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\frac{N-1}{2}} \frac{1}{2k+1}\operatorname{sen}\left((2k+1)x\right)$$

$$2k+1 = N \qquad k = \frac{N-1}{2}$$

$$f(x) = e^{-|x|} \quad \text{su} \quad]-\pi, \pi]$$



$$f(\pi) = e^{-\pi}$$

$$f(-\pi) = e^{-\pi}$$

$$\frac{e^x}{n^2+1}\left(n \operatorname{sen}(nx) + \cos(nx)\right)$$

$$a_n = \frac{2}{\pi} \int_{-\pi}^{0} e^x \cos(nx)\, dx = \frac{2}{\pi(n^2+1)}\left(1 - (-1)^n e^{-\pi}\right)$$

$$a_0 = \frac{2}{\pi}\left(1 - e^{-\pi}\right)$$

$$P_N[f](x) = \frac{1 - e^{-\pi}}{\pi} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{\left(1 - (-1)^n e^{-\pi}\right)}{n^2+1} \cos(nx)$$

Se il periodo T non è 2π? $\quad f:]-T/2, T/2] \quad \omega = \frac{2\pi}{T} \quad T\omega = \underline{2\pi}$

funzioni base: $\cos(n\omega x) \quad \operatorname{sen}(n\omega x) \quad e^{in\omega x}$

Basi $\quad \mathcal{F} = \left\{ \frac{e^{in\omega x}}{\sqrt{T}} \quad n \in \mathbb{Z} \right\} \quad \mathcal{F} = \left\{ \frac{1}{\sqrt{T}}, \frac{\cos(n\omega x)}{\sqrt{T/2}}, \frac{\operatorname{sen}(n\omega x)}{\sqrt{T/2}} \; n \in \mathbb{N}^+ \right\}$

$$\| e^{in\omega x} \|_2^2 = \int_{-T/2}^{T/2} | e^{in\omega x} | \, dx = T$$

$$P_N[f](x) = \sum_{-N}^{N} \left(\int_{-T/2}^{T/2} f(x) \cdot \frac{1}{\sqrt{T}} e^{-in\omega x} dx \right) \cdot \frac{1}{\sqrt{T}} e^{in\omega x} = \sum_{-N}^{N} \underbrace{\boxed{\frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-in\omega x} dx}}_{c_n} \cdot e^{in\omega x}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\,dx \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(n\omega x)\,dx \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(n\omega x)\,dx$$

Oss: sia $f(x)$ T-periodica

allora $g(x) = f\left(\frac{T}{2\pi}x\right)$ è 2π-periodica

$$\left[\; g(x+2\pi) = f\left(\frac{T}{2\pi}(x+2\pi)\right) = f\left(\frac{T}{2\pi}x + T\right) = f\left(\frac{T}{2\pi}x\right) = g(x) \;\right]$$

$$P_N[g](x) = \sum_{-N}^{N} C(g)_n\, e^{inx}$$

$$P_N[f](x) = ?$$

$$C(f)_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\, e^{-in\frac{2\pi}{T}x}\,dx \qquad C(g)_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right) e^{-inx}\,dx = \frac{1}{\cancel{2\pi}}\int_{-T/2}^{T/2} f(y)\cdot\frac{\cancel{2\pi}}{T}\, e^{-in\frac{2\pi}{T}y} = \frac{1}{T}\int_{-T/2}^{T/2} f(y)\, e^{-in\omega y}\,dy$$

$$y = \frac{T}{2\pi}x \qquad dx = \frac{2\pi}{T}dy \qquad x = \pi \,(-\pi) \rightsquigarrow y = \frac{T}{2} \,\left(-\frac{T}{2}\right)$$

$$C(g)_n = C(f)_n \quad !!$$

$$P_N[f](x) = \sum_{-N}^{N} C(g)_n \, e^{i n \overbrace{\frac{2\pi}{T}}^{=\omega} x}$$

a_n b_n stesso caso! $a(f)_n = a(g)_n$ $b(f)_n = b(g)_n$

Es: $f(x) = \chi_{[0,1]}(x)$ su $]-1,1]$ $g(x) = \chi_{[0,\pi]}(x)$ su $]-\pi,\pi]$

$f(x) = g(\pi x)$ f è periodica di periodo 2 $g(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\cdots} \frac{1}{2n+1} \operatorname{sen}((2n+1)x)$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\frac{N-1}{2}} \frac{1}{2n+1} \operatorname{sen}((2n+1)\pi x)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \qquad \underline{P_N[f](x)} = \sum_{n=-N}^{N} c_n e^{inx}$$

$f \in L^2([-\pi,\pi])$ diremo energia di f il numero $\|f\|_2^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$

Energia del polinomio di Fourier

$$\|P_N[f]\|_2^2 = \Big\|\sum_{n=-N}^{N} c_n e^{inx}\Big\|_2^2 = \sum_{n=-N}^{N} |c_n|^2 \|e^{inx}\|_2^2 = 2\pi \sum_{n=-N}^{N} |c_n|^2$$

In forma reale

$$\|P_N[f]\|_2^2 = \Big\|\frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx)\Big\|_2^2 = \frac{|a_0|^2}{4}\cdot 2\pi + \sum |a_n|^2 \cdot \pi + \sum |b_n|^2 \cdot \pi =$$

$$\|a_0\| = |a_0|\sqrt{2\pi} \qquad = \pi\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{N}\left(|a_n|^2 + |b_n|^2\right)\right]$$

Ricordo che se $v \in H$ $\quad P_N[v] = \sum_{k=1}^{N} \langle v, e_k \rangle e_k \qquad v - P_N[v] \perp P_N[v]$

$$\|v\|^2 = \| \underbrace{v - P_N[v] + P_N[v]}_{\text{ortogonali}} \|^2 = \underbrace{\|v - P_N[v]\|^2}_{\geq 0} + \|P_N[v]\|^2 \geq \|P_N[v]\|^2$$

L'energia di v è sempre maggiore uguale all'energia della proiezione.

$\forall N \in \mathbb{N}$ si ha che $\left\| \sum_{n=1}^{N} \langle v, e_n \rangle e_n \right\| = \sum_{n=1}^{N} |\langle v, e_n \rangle|^2 \leq \|v\|^2$

questa è la somma parziale della serie $\sum_{n=1}^{+\infty} |\langle v, e_n \rangle|^2$

questa serie numerica a termini positivi è limitata, quindi è convergente!

Inoltre

$$\boxed{\sum_{n=1}^{+\infty} |\langle v, e_n \rangle|^2 \leq \|v\|^2}$$

Disuguaglianza di Bessel

$\|P_N[f]\|^2$

Nel nostro caso avremo

$$\boxed{2\pi \sum_{n=-\infty}^{+\infty} |c_n|^2 \leq \|f\|^2}$$

Ora, osserviamo che $\|f\|^2 = \|f - P_N[f] + P_N[f]\|^2 = \|f - P_N[f]\|^2 + \|P_N[f]\|^2$

Si avrà $\lim_{N \to +\infty} \|P_N[f]\| = \|f\|$ se e solo se $\lim_{N \to +\infty} \|f - P_N[f]\| = 0$

cioè se $\lim_{N \to +\infty} P_N[f] = f$ in norma L^2

$\lim_{n\to+\infty} P_n[f] = f$ in norma se e solo se vale l'identità di Parseval

$$\sum_{n=1}^{+\infty} |\langle f, e_n\rangle|^2 = \|f\|^2$$

per Fourier : $\sum_{n=-\infty}^{+\infty} c_n e^{inx}$ converge a f in norma L^2 se e solo se

$$\boxed{2\pi \sum_{n=-\infty}^{+\infty} |c_n|^2 = \|f\|^2}$$

$$\left(\pi\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} \left(|a_n|^2 + |b_n|^2\right)\right] = \|f\|^2 \right)$$

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx) + \sum_{n=0}^{+\infty} b_n \sin(nx)$$

Teorema di convergenza in media quadratica (o in energia)

$f \in L^2(]-\pi,\pi[)$. Allora la serie di Fourier di f $\sum_{n=-\infty}^{+\infty} c_n e^{inx}$ converge a f nella norma di L^2. In particolare vale l'identità di Parseval

$$\|f\|^2 = 2\pi \sum_{n=-\infty}^{+\infty} |c_n|^2 \qquad \left[\, c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx \,\right]$$

[oppure, in forma reale, $\|f\|^2 = \pi\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)\right]$]

$$\mathcal{F}(f)$$

$$\|f\| = \sqrt{2\pi}\cdot\|(c_n)_n\|$$

Dim. omessa

$$f \in L^2(]-\pi,\pi[) \rightsquigarrow \sum_{-\infty}^{+\infty} c_n e^{inx}$$

$$\downarrow$$

$$(c_n)_n$$

$$\mathcal{F} : \underbrace{L^2(]-\pi,\pi[)}\longrightarrow \underline{\ell^2} = \left\{ (c_n)_n \quad n \in \mathbb{Z} : \sum_{n=-\infty}^{+\infty} |c_n|^2 < +\infty \right\}$$

$$\langle (c_n)_n, (d_n)_n \rangle = \sum_{n=-\infty}^{+\infty} c_n \overline{d_n}$$ è un prodotto Hermitiano

ℓ^2 è uno sp. di Hilbert

$$\mathcal{F}(f) = (c_n)_n$$ $\mathcal{F}$ è biiettiva!

$\|f\|_{L^2} = \sqrt{2\pi}\ \|\mathcal{F}(f)\|_{\ell^2}$

$\mathcal{F}$ è un isomorfismo tra spazi di Banach

$\mathcal{F} : f \rightsquigarrow (c_n)_n$

è una trasformata

$f \qquad \hat{f}(n)$

$f : \mathbb{R} \to \mathbb{C} \quad \rightsquigarrow \quad \hat{f} : \mathbb{Z} \to \mathbb{C}$

OSS: Disuguaglianza di Bessel $\underbrace{2\pi \sum_{n=-\infty}^{+\infty} |c_n|^2}_{\uparrow} \leq \|f\|^2$

la serie è convergente, quindi la successione $|c_n|^2$ deve essere infinitesima

$$\lim_{\substack{n\to-\infty \\ +\infty}} c_n = 0 \qquad \lim_{n\to\pm\infty} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = 0$$

Analogamente $\displaystyle \lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \lim_{n\to+\infty} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = 0$

Lemma di Riemann-Lebesgue

Sia $f \in L^1(]-\pi,\pi])$, allora

$$\lim_{|\lambda|\to+\infty}\int_{-\pi}^{\pi} f(x)\, e^{-i\lambda x}\, dx = 0$$

$$\lim_{\lambda\to+\infty}\int_{-\pi}^{\pi} f(x)\cos(\lambda x)\, dx = 0 \qquad \lim_{\lambda\to+\infty}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(\lambda x)\, dx = 0$$

Il problema della convergenza puntuale