THE GOVERNING EQUATIONS

- **<u>Approx based on orders of magnitude (scale analysis)</u>**
  - **Considerations on continuity equation…**
  - **… […]**
  - **Considerations on x,y - momentum equations…**
  - **… […]**
  - **Considerations on z - momentum equations…**
  - **… […]**
  - **Considerations on energy equation…**
  - **… […]**

THE GOVERNING EQUATIONS

- **Approx based on orders of magnitude (scale analysis)**
  - Considerations on continuity equation…
  - … […]
  - Considerations on x,y - momentum equations…
  - … […]
  - Considerations on z - momentum equations…
  - … […]
  - Considerations on energy equation…
  - … […]
  - Large-scale geophysical flows tend to be fully hydrostatic even in presence of substantial motions

Primitive Equations of GFD

x − momentum:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial u}{\partial z}\right) \quad (4.21a)$$

y − momentum:
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial v}{\partial z}\right) \quad (4.21b)$$

z − momentum:
$$0 = -\frac{\partial p}{\partial z} - \rho g \quad (4.21c)$$

continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4.21d)$$

energy:
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z}\left(\kappa_E\frac{\partial \rho}{\partial z}\right) \quad (4.21e)$$

5 equations for 5 variables u, v, w, ρ, p

$f = 2\Omega \sin\varphi$ and ρ_0, g constant, $\nu_E(z)$, $\kappa_E(z)$

THE PRIMITIVE EQUATIONS

- **Scale analysis on x,y – momentum equations:**

$$\frac{U}{T}, \frac{U^2}{L}, \frac{U^2}{L}, \frac{WU}{H}, \Omega U, \frac{P}{\rho_0 L}, \frac{\nu U}{H^2}$$

$\Rightarrow$ **Rotation (Coriolis) term ΩU is fundamental to measure the importance of the terms relative to it:**

$$\frac{1}{\Omega T}, \frac{U}{\Omega L}, \frac{U}{\Omega L}, \frac{WU}{H} \cdot \frac{1}{\Omega U} \cdot \frac{L}{L}, 1, \frac{P}{\rho_0 L \Omega U}, \frac{\nu}{\Omega H^2}$$

- def. Temporal Rossby number: $Ro_T = \frac{1}{\Omega T} = \frac{local\ velocity\ variation}{Coriolis\ term}$ ... (ω)
- def. Rossby number: $Ro = \frac{U}{\Omega L} = \frac{advection}{Coriolis\ term}$ ... (ε)
- $\frac{WU}{H} \cdot \frac{1}{\Omega U} \cdot \frac{L}{L} = \frac{WL}{UH} \cdot \frac{U}{\Omega L} = \frac{WL}{UH} \cdot Ro = \frac{vertical\ convergence/divergence}{horizontal\ convergence/divergence} \cdot Ro$
- def. Ekman number: $Ek = \frac{\nu}{\Omega H^2} = \frac{friction\ (z)}{Coriolis\ term}$

$\Rightarrow$ **GFD has** $Ro_T \lesssim 1 \ldots Ro \lesssim 1 \ldots Ek \ll 1\ far\ from\ B.L.$

- def. Reynolds number: $Re = \frac{UL}{\nu} = \frac{U}{\Omega L} \cdot \frac{\Omega H^2}{\nu} \cdot \frac{L^2}{H^2} = \frac{Ro}{Ek} \cdot \frac{L^2}{H^2} = \frac{advection}{friction\ (xy)} \gg 1$

THE PRIMITIVE EQUATIONS

- **Fluid turbulence at subgeophysical scales (small eddies) can act as dissipative mechanism: molecular viscosity ν can be substituted by a much larger EDDY VISCOSITY ν_T or ν_E**
  - For water: $\nu \sim 10^{-6} m^2/s$ and $\nu_T \sim 10^{-2} m^2/s$
- **Even with eddy viscosity, Ekman number remains small ($Ek \sim 10^{-2}$) but friction is essential near boundary layers ($Ek \sim 1$)**
- $\frac{P}{\rho_0 L\Omega U} = \frac{pressure\ gradient}{Coriolis\ term} \Rightarrow if\ terms\ are\ comparable: P \sim \rho_0 L\Omega U$
- **from z-momentum:** $0 = -\frac{\partial p}{\partial z} - \rho g \Rightarrow P \sim \Delta\rho g H$

$$\frac{\Delta\rho g H}{\rho_0 L\Omega U} \cdot \frac{U}{U} = \frac{\Delta\rho g H}{\rho_0 U^2} \cdot \frac{U}{L\Omega} = Ri \cdot Ro$$

- def. Richardson number: $Ri = \frac{\Delta\rho g H}{\rho_0 U^2} = \frac{potential\ energy}{kinetic\ energy}$ ... $(1/\sigma)$

✓ *Exercise: find scales for $\rho_0 L\Omega U$, $\Delta\rho g H$, $\rho_0 g H$ in Ocean and Atmosphere using Table 4.1*

GEOSTROPHIC FLOWS

- **After inertial oscillations, homogeneous geostrophic flows are the second simple case where NSEq. can be solved, and can describe natural GFD phenomena**
- **Hypoteses:**
  - **Coriolis term dominates others (= rapidly rotating flows):** $Ro_T \ll 1 \; and \; Ro \ll 1$
  - **Homogeneous fluids:** $\rho_0 = cost \; and \; \rho' = 0$
  - **Ignore frictional effect (= far from B.L.):** $Ek \ll 1$
- **Primitive equations:**

$$
\begin{aligned}
-fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
+fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- ∂_z **of x,y-momentum eq. [...] :** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$

=> Taylor-Proudman theorem: *"horizontal velocity field has no vertical shear and all the particles on the same vertical move in concert"*

GEOSTROPHIC FLOWS

- **Solving the x,y-momentum eqs.:**

$$u = \frac{-1}{\rho_0 f}\frac{\partial p}{\partial y}, \qquad v = \frac{+1}{\rho_0 f}\frac{\partial p}{\partial x}$$

- $(\boldsymbol{u}, \boldsymbol{v}) = \overline{\boldsymbol{u}} \perp \nabla \boldsymbol{p}$ **the flow is across-gradient (or isobaric):**
  - **NO pressure work is performed either on the fluid or by the fluid: Once initiated the flow can persist without a continuous energy source** $\overline{\boldsymbol{u}} \cdot \nabla \boldsymbol{p} = 0$

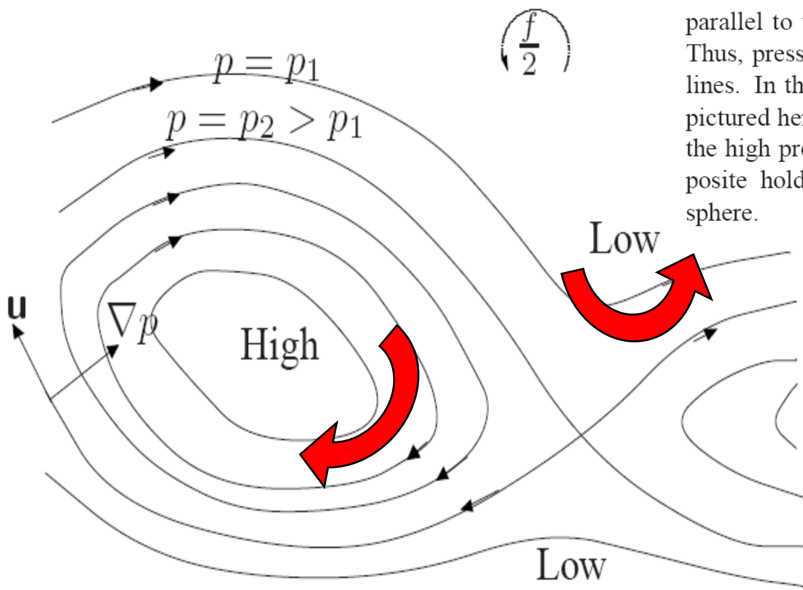
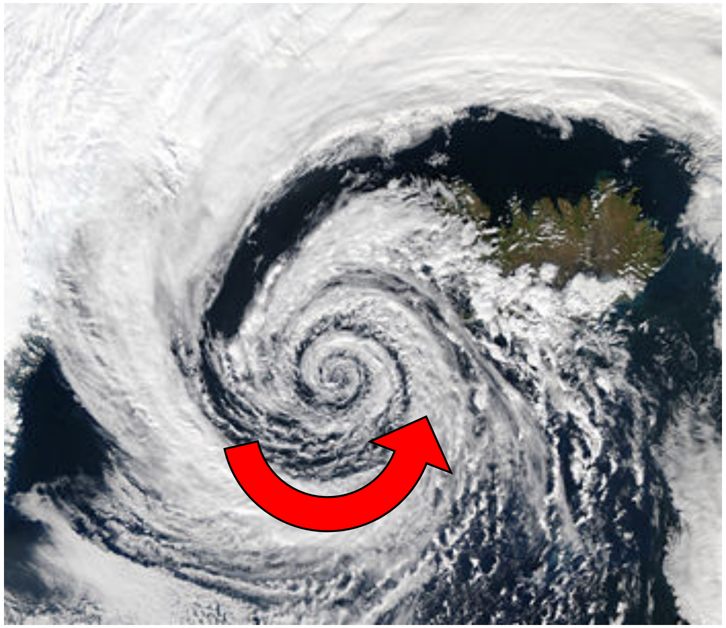


Figure 7-1 Example of geostrophic flow. The velocity vector is everywhere parallel to the lines of equal pressure. Thus, pressure contours act as streamlines. In the Northern Hemisphere (as pictured here), the fluid circulates with the high pressure on its right. The opposite holds for the Southern Hemisphere.

***f* > 0**



http://it.wikipedia.org/wiki/Immagine:Low_pressure_system_over_Iceland.jpg

GEOSTROPHIC FLOWS

- GEOSTROPHY comes from $\gamma\eta = Earth$ and $\sigma\tau\rho o\varphi\eta = turning$
- Balance between Coriolis force and pressure gradient
- All geostrophic flows are isobaric:
  - Northern hemisphere $f > 0$: currents flows with H on their right
  - Southern hemisphere $f < 0$: currents flows with H on their left
- IF the flow extends over a meridional span not too wide $L_y \ll L_x$: $\frac{\partial f}{\partial y} \rightarrow 0 \Rightarrow f = cost \Rightarrow f - PLANE$ and the horizontal divergence is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right) = 0$$

- Geostrophic flows are naturally non-divergent on the f-plane

$\frac{\partial w}{\partial z} = 0 \Rightarrow w = cost$ and, if the fluid is bounded in the vertical by a flat surface: $w = 0$

- Geostrophic flows are 2-dimensional

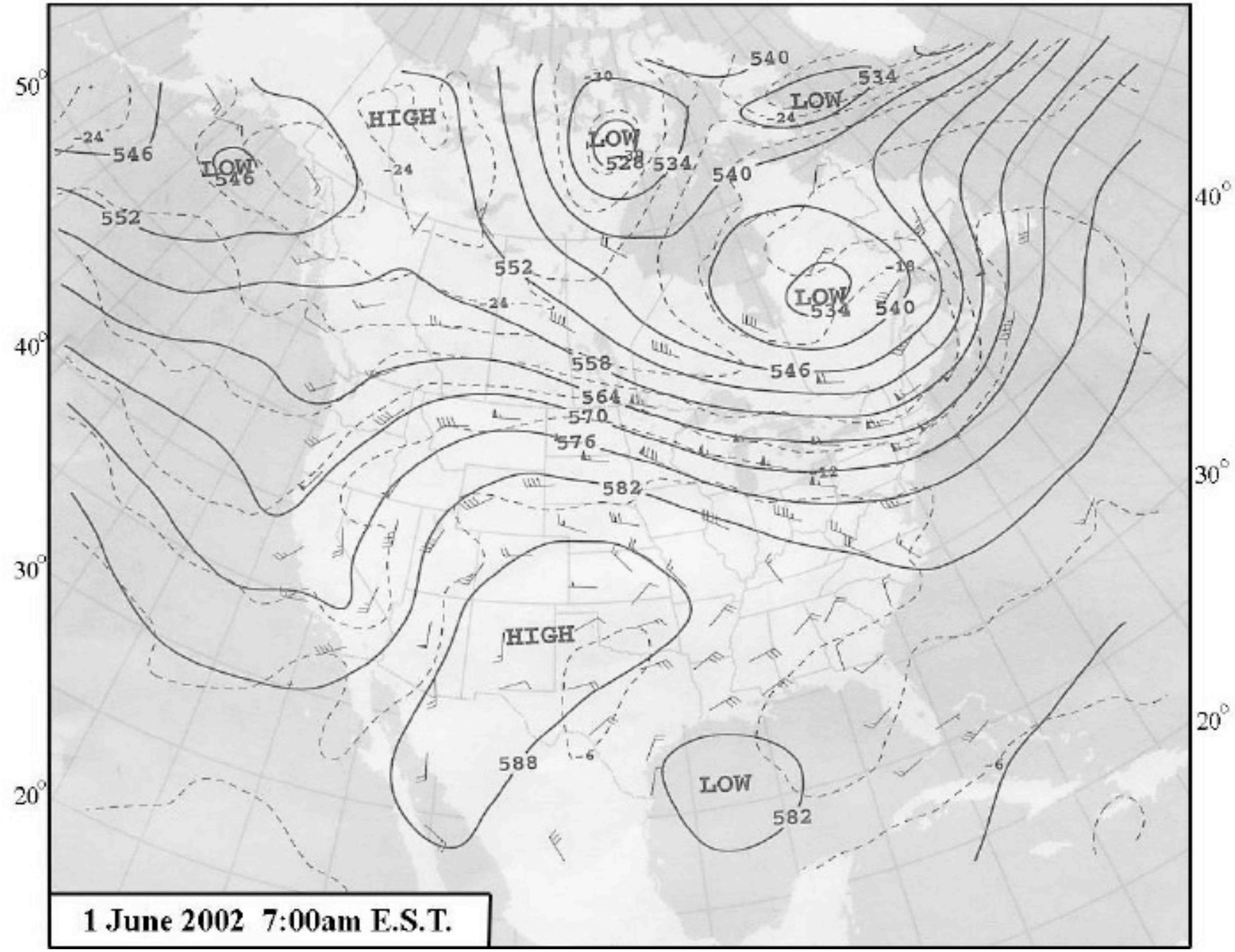
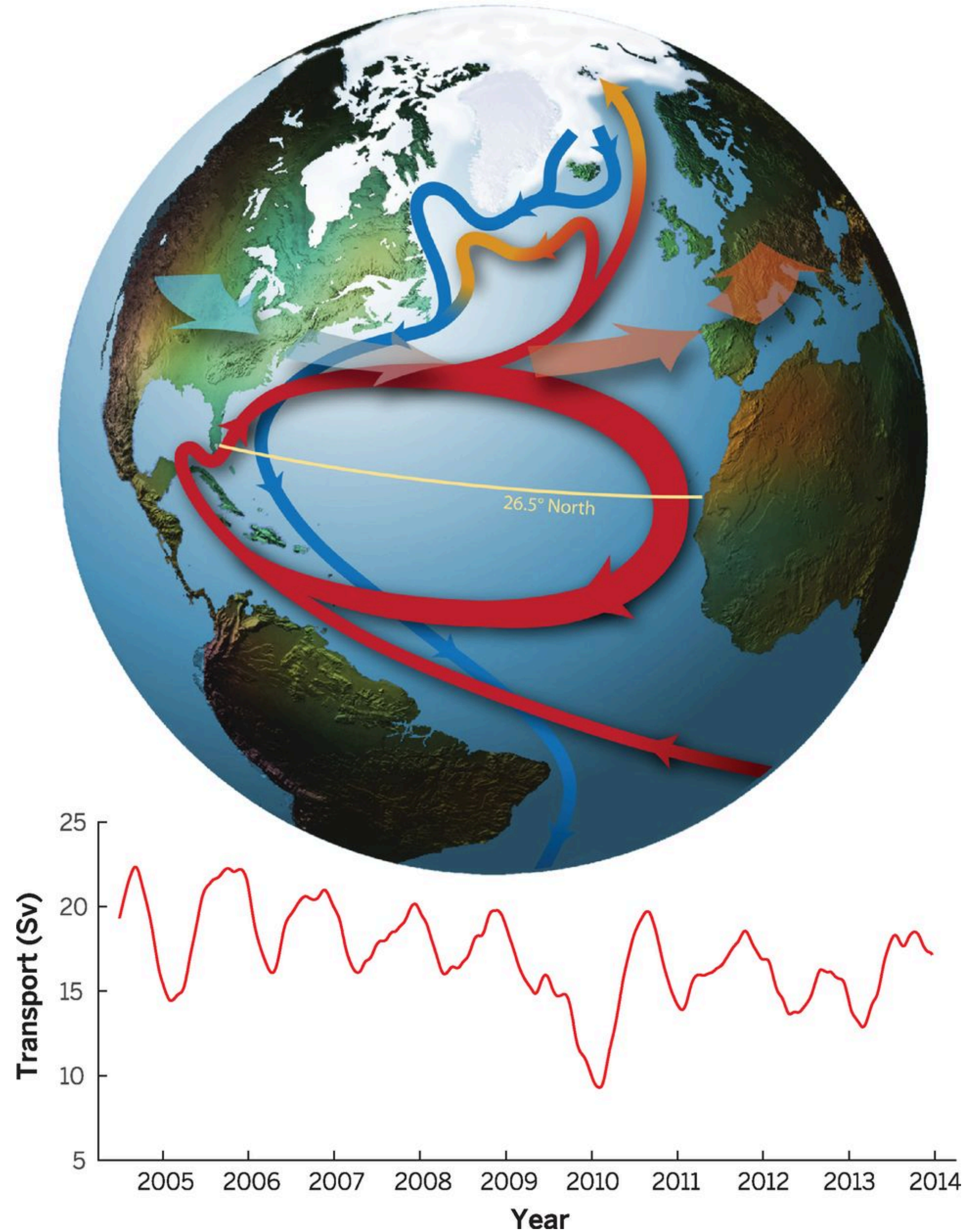


Figure 7-2 A meteorological example showing the high degree of parallelism between wind velocities and pressure contours (isobars), indicative of geostrophic balance. The solid lines are actually height contours of a given pressure (500 mb in this case) and not pressure at a given height. However, because atmospheric pressure variations are large in the vertical and weak in the horizontal, the two sets of contours are nearly identical by virtue of the hydrostatic balance. According to meteorological convention, wind vectors are depicted by arrows with flags and barbs: on each tail, a flag indicates a speed of 50 knots, a barb 10 knots and a half-barb 5 knots (1 knot = 1 nautical mile per hour = 0.5144 m/s). The wind is directed toward the bare end of the arrow, because meteorologists emphasize where the wind comes from, not where it is blowing. The dashed lines are isotherms. (*Chart by the National Weather Service, Department of Commerce, Washington, D.C.*)

A simplified schematic (top) of the AMOC. Warm water flows north in the upper ocean (red), gives up heat to the atmosphere (atmospheric flow gaining heat represented by changing color of broad arrows), sinks, and returns as a deep cold flow (blue).

Latitude of the 26.5° N AMOC observations is indicated. The actual flow is considerably more complex. (**Bottom**) The 10-year (April 2004 to March 2014) time series of the AMOC strength at 26.5° N in Sverdrups (1 Sv = 10^6 m^3 s^{-1}). This is the 180-day filtered version of the time series. Visible are the low AMOC event in 2009–2010 and the overall decline in AMOC strength over the 10-year period.

http://www.sciencemag.org/content/348/6241/1255575



Published by AAAS

M. A. Srokosz, and H. L. Bryden Science 2015;348:1255575

Science AAAS

GEOSTROPHIC FLOWS OVER IRREGULAR BOTTOM

- **Same framework, but with no-flat bottom:**

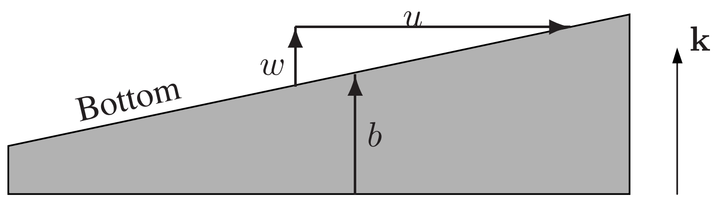


Figure 7-3 Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.

- Bottom elevation (bathymetry / topography): $b = b(x, y)$
- $w = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \bar{u} \cdot \nabla z = 0 + u\frac{\partial b}{\partial x} + \mathrm{v}\frac{\partial b}{\partial y} + w\frac{\partial b}{\partial z} = u\frac{\partial b}{\partial x} + \mathrm{v}\frac{\partial b}{\partial y}$
- But on the f-plane $w = \mathrm{cost}$ and since $w(z = H) = 0 \Rightarrow w = 0\ \forall z$
- $\Rightarrow \boldsymbol{\bar{u}} \cdot \boldsymbol{\nabla b} = \mathbf{0}$ **flow is directed to zones of equal depth: FREE geostrophic flows can occur only along closed isobaths**
- **ISOBARS = ISOBATHS**
- **If bumps or dips exist, the fluid can only go around them: due to vertical rigidity, fluid particles at all levels must likewise go around: TAYLOR COLUMNS are permanent tubes of fluid above bumps or dips**

GEOSTROPHIC FLOWS OVER IRREGULAR BOTTOM

- **Same framework, but with no-flat bottom:**

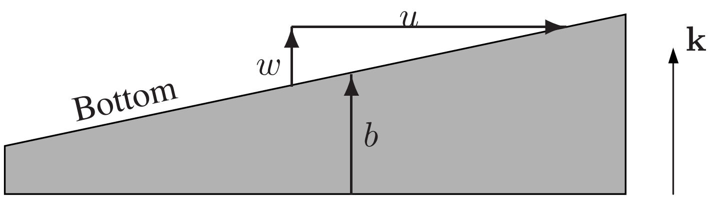


Figure 7-3 Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.

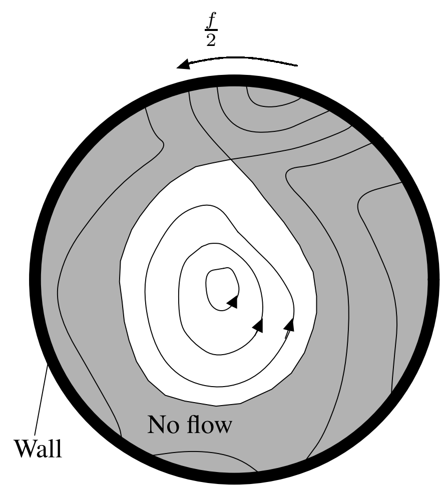


Figure 7-4 Geostrophic flow in a closed domain and over irregular topography. Solid lines are isobaths (contours of equal depth). Flow is permitted only along closed isobaths.

- **ISOBARS = ISOBATHS**

- **If bumps or dips exist, the fluid can only go around them: due to vertical rigidity, fluid particles at all levels must likewise go around: TAYLOR COLUMNS are permanent tubes of fluid above bumps or dips**

BAROTROPIC FLOWS

- **Generalization to non-geostrophic flows: “second level” of flows**
- **Hypoteses:**
  - **Coriolis term DOES NOT dominate others:** $Ro_T \sim 1 \; and \; Ro \sim 1$
  - **Homogeneous fluids:** $\rho_0 = cost \; and \; \rho' = 0$
  - **Ignore frictional effect (SLIP is allowed [...]):** $Ek \ll 1$
- **Primitive equations:**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- **IF T.-P. theorem still holds** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ **initially, it will hold also at all future time**
- **Advection, Coriolis and Pressure terms remain z-independent**

BAROTROPIC FLOWS

- Generalization to non-geostrophic flows: “second level” of flows
- Hypoteses:
  - Coriolis term DOES NOT dominate others: $Ro_T \sim 1$ *and* $Ro \sim 1$
  - Homogeneous fluids: $\rho_0 = cost$ *and* $\rho' = 0$
  - Ignore frictional effect (SLIP is allowed [...]): $Ek \ll 1$
- Primitive equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
$$

- IF T.-P. theorem still holds $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ initially, it will hold also at all future time
- Advection, Coriolis and Pressure terms remain z-independent

BAROTROPIC FLOWS

- Generalization to non-geostrophic flows: “second level” of flows
- Hypoteses:
  - Coriolis term DOES NOT dominate others: $Ro_T \sim 1 \; and \; Ro \sim 1$
  - Homogeneous fluids: $\rho_0 = cost \; and \; \rho' = 0$
  - Ignore frictional effect (SLIP is allowed [...]): $Ek \ll 1$
- Primitive equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
0 &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}$$

- IF T.-P. theorem still holds $\boxed{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0}$ initially, it will hold also at all future time
- Advection, Coriolis and Pressure terms remain z-independent

BAROTROPIC FLOWS

- **Although the flow has NO VERTICAL SHEAR, this remains the only similarity with geostrophic flows: Barotropic flows are not required to be aligned with isobars, neither be non-divergent on the horizontal plane, so they can develop a vertical velocity $w \neq 0$**
- **Integrating continuity eq. over the entire fluid depth […]:**

$$\int_{b}^{b+h} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} dz = 0$$

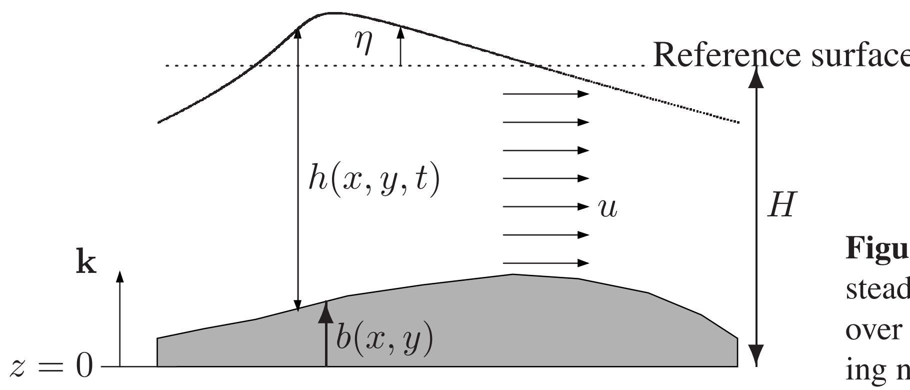


Figure 7-5 Schematic diagram of unsteady flow of a homogeneous fluid over an irregular bottom and the attending notation.

➢ **NEW CONTINUITY EQUATION:**

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

with $\eta = b + h - H$ and $\partial_t \eta = \partial_t h$

Shallow-water model

in case of flat bottom *b(x,y)=0*

3 unknowns in 3 equations

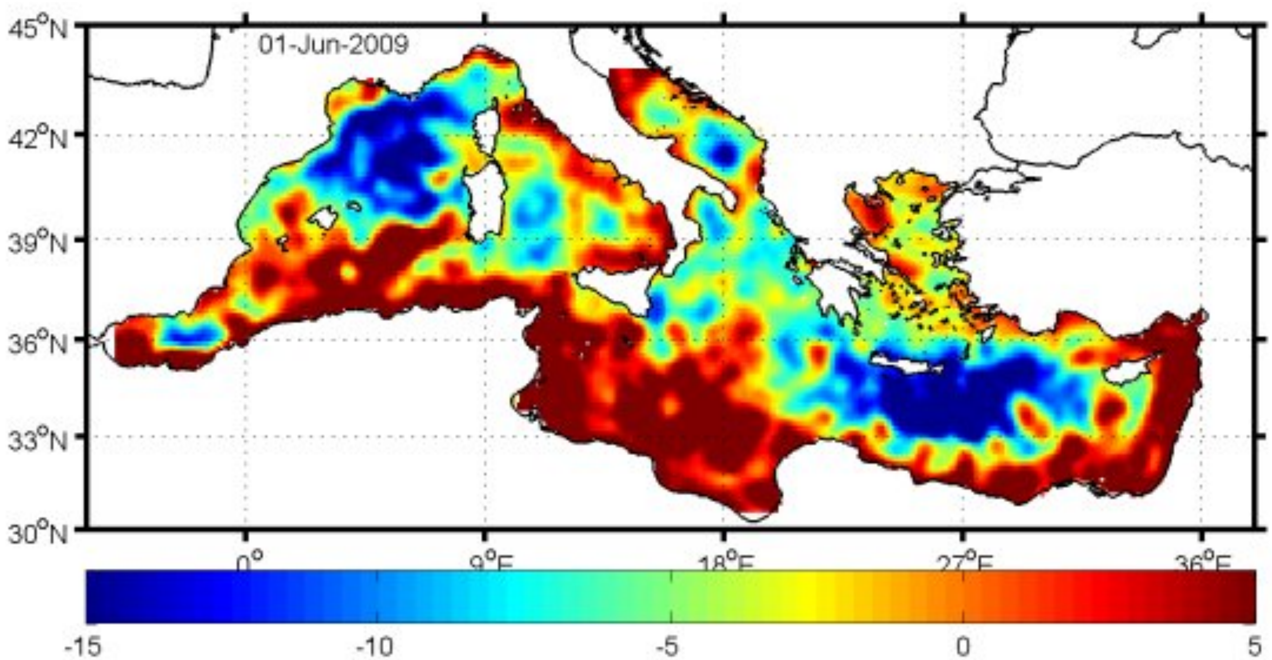
$$\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - f v &= -g\,\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + f u &= -g\,\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}$$

Geostrophy and altimetry

Variation of η with *x,y* measured from satellite gives info on geostrophic currents

$$-\ fv\ =\ -g\,\frac{\partial \eta}{\partial x}$$

$$+\ fu\ =\ -g\,\frac{\partial \eta}{\partial y}$$



Example of absolute dynamic topography (in cm) of the Mediterranean Sea on 1 June 2009 using the Rio et al. (2007) synthetic mean dynamic topography. http://www.goceitaly.asi.it/GoceIT/index.php?Itemid=94

VORTICITY DYNAMICS

- **Geostrophic flows are non-divergent on the f-plane, with 2d-divergence equal to zero: let's investigate the role of the horizontal divergence in barotropic flows**
- **Subtract y-derivative of x-mom.eq from x-derivative of y-mom.eq of barotropic flow system (or the shallow-water model) [...]**

- def. <u>ambient vorticity</u> f
- def. <u>relative vorticity</u> $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (the vertical component of $\nabla \times \bar{u}$)
- def. <u>total vorticity</u> $f + \zeta$

- $\frac{d}{dt}(f+\zeta) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(f+\zeta) = 0 \Rightarrow total\ vorticity\ ruled\ by\ horiz.div.$
- $\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \Rightarrow fluid\ column\ heigth\ ruled\ by\ horiz.div.$
- $\frac{d}{dt}(h \cdot dS) = 0 \Rightarrow parcel's\ volume\ is\ conserved\ in\ time$
- Combining the above equations... [...]

VORTICITY DYNAMICS

- **Kelvin's theorem for 2-d rotating flows**: *"in barotropic flows without friction the circulation is conserved"*
- **This conservation principle has the same meaning of that of the angular momentum for an isolated system**

$$\frac{d}{dt}\left[(f + \zeta)\, ds\right] = 0$$

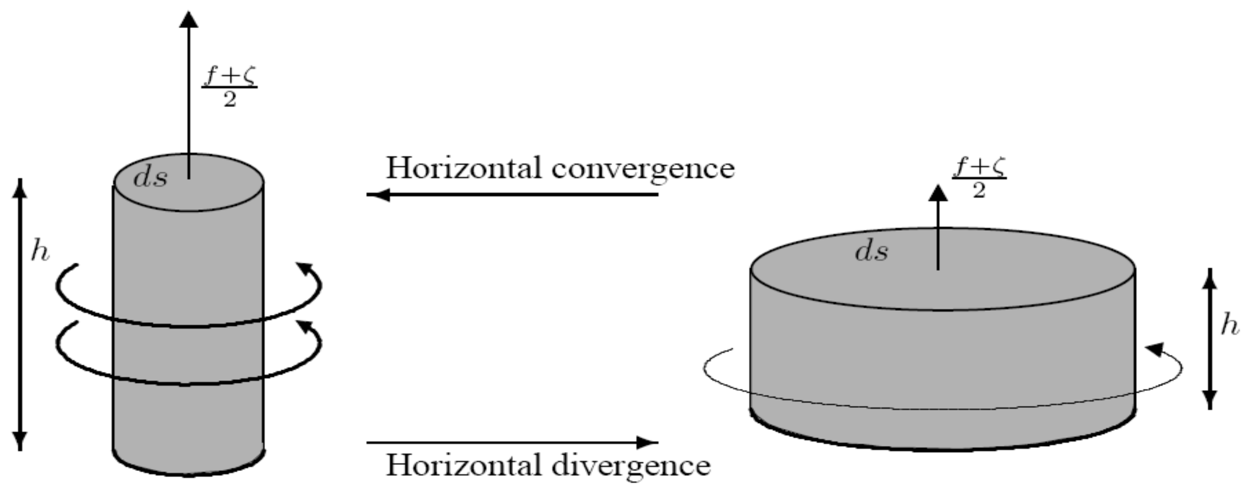
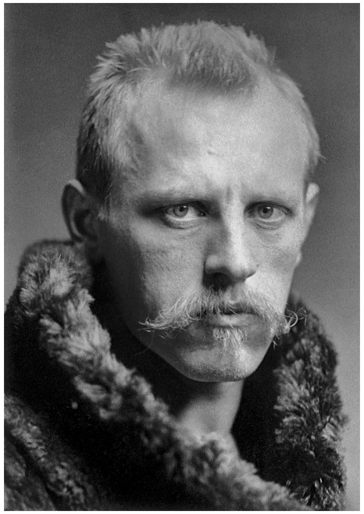


Figure 7-6 Conservation of volume and circulation of a fluid parcel undergoing vertical squeezing or stretching. The products $h\,ds$ and $(f + \zeta)\,ds$ are conserved during the transformation. As a corollary, the ratio $(f + \zeta)/h$, called the potential vorticity, is also conserved.

VORTICITY DYNAMICS

- **Kelvin's theorem for 2-d rotating flows**: *"in barotropic flows without friction the circulation is conserved"*
- **This conservation principle has the same meaning of that of the angular momentum for an isolated system**
- **IF both circulation and volume are conserved, so is their ratio, allowing to eliminate dependency on cross-section**
- $\frac{d}{dt}\left(\frac{f+\zeta}{h}\right)=0$ **where** $q=\frac{f+\zeta}{h}=\frac{f+\partial v/\partial x-\partial u/\partial y}{h}$
- **and q is called POTENTIAL VORTICITY, or "circulation per volume", thus obtaining the conservation of potential vorticity**
- **<u>For rapidly rotating flows</u>:** $Ro=\frac{U}{\Omega L}\ll 1 \Rightarrow f+\zeta\sim\Omega+\frac{U}{L}\sim\Omega\Rightarrow q=\frac{f}{h}$
- **and, IF $f=cost$ each fluid column must conserve its height f: and in particular, if the upper boundary is flat, fluid parcels must follow the isobaths => <u>barotropic flows become geostrophic</u>**

The Ekman models



Fridtjof Nansen (1861 –1930)

Norwegian scientist, explorer, diplomat.

Nobel Peace Prize 1922

https://en.wikipedia.org/wiki/Fridtjof_Nansen

Vagn Walfrid Ekman (1874 –1954)

Swedish oceanographer.

http://en.wikipedia.org/wiki/Vagn_Walfrid_Ekman

- Prandtl hypothesis on Boundary Layers
- $\boldsymbol{Ek} \rightarrow \mathbf{1}$ close to the wall - $\boldsymbol{Ek} \ll \mathbf{1}$ far from the wall
- study of iceberg's motion (Nansen/Fram → Bjerknes → Ekman)

The Ekman models

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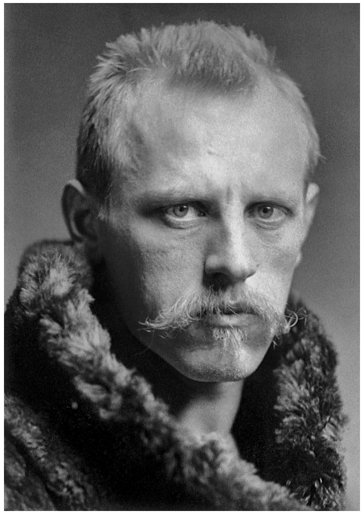

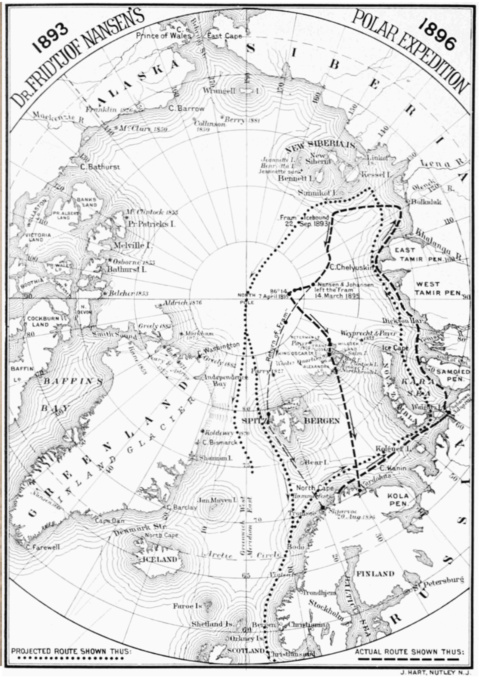

https://frammuseum.no/polar-history/vessels/the-polar-ship-fram/

https://en.wikipedia.org/wiki/Nansen%27s_Fram_expedition

EKMAN LAYER

- **As seen from the scale analysis of the primitive eqs. vertical friction has a very minor role in the balance of forces ($\boldsymbol{Ek \ll 1}$) and may be omitted**
- **But we lost something, since the frictional terms have the highest derivative order => when $\boldsymbol{Ek \ll 1}$ not all the BCs can be applied, the result is that SLIPPING ON THE BOUNDARY is allowed**
- **Prandtl hypothesis**: *the fluid has 2 distinct behaviors:*
  - *far from the boundary (INTERIOR, vertical scale **H**), friction can be neglected ($\boldsymbol{Ek \ll 1}$): $\boldsymbol{Ek = \frac{\upsilon_T}{\Omega H^2} \sim \frac{10^{-2} m^2/s}{10^{-4} s^{-1} \cdot (10^3 m)^2} \sim 10^{-4}}$*
  - *across a short distance near the boundary (BOUNDARY LAYER, vertical scale **d**), friction acts to bring the finite interior velocity to zero at the wall ($\boldsymbol{Ek \sim 1}$): $\boldsymbol{Ek = \frac{\upsilon_T}{\Omega d^2} \sim 1 \Rightarrow d = \sqrt{\frac{\upsilon_T}{\Omega}} \sim 10\, m \Rightarrow d \ll H}$*
- **Because of the Coriolis effect, the frictional layer of the geophysical flows, called EKMAN LAYER, greatly differs from the BL in non-rotating flows ($\boldsymbol{\delta}$), which does not have a thickness and grows downstream ($\boldsymbol{\delta \propto \sqrt{x}}$)**

THE BOTTOM EKMAN LAYER

- **The bottom exerts a frictional stress against the flow bringing its interior velocity gradually to zero within a thin layer above the wall $d \ll H$**
- **Hypotheses:**
  - **Interior flow is uniform and geostrophic:** $Ro_T \ll 1$ *and* $Ro \ll 1$
  - **Homogeneous fluid:** $\rho_0 = cost$ *and* $\rho' = 0$
  - **Flat bottom**
- **Primitive equations:**

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_E \frac{\partial^2 u}{\partial z^2}$$

$$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_E \frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z},$$

- **Boundary conditions:**

$$\text{Bottom } (z=0): \quad u = 0, \quad v = 0,$$

$$\text{Toward the interior } (z \gg d): \quad u = \bar{u}, \quad v = 0, \quad p = \bar{p}(x,y)$$

- **Interior flow is uniform, no horizontal gradient**
- **[…]**

THE BOTTOM EKMAN LAYER

- **Solutions:**

$$u = \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right)$$
$$v = \bar{u}\, e^{-z/d}\sin\frac{z}{d}\,.$$

$$d = \sqrt{\frac{2\nu_E}{f}}$$ **Ekman depth**

- **Considerations:**
  - **As expected, the Ekman depth corresponds to $Ek \sim 1$**
  - **Although the driving interior flow is along x, we have a transversal velocity (along y) which is not negligible**
  - **Close to wall $z \to 0$ or $\frac{z}{d} \ll 1 \Rightarrow u \sim v \sim \bar{u}z/d$ …the velocity near the bottom is at 45 degree to the left of the interior velocity (with f>0) […]**
  - **Where u reaches its maximum at $z = \frac{3\pi}{4}d$ the velocity is $u = 1.07\bar{u}$ that is, larger than its interior value**
  - **The net transport of fluid transverse to the main flow is $V = \int_0^\infty v dz = \bar{u}d/2$ while $U = -\bar{u}d/2$**

THE BOTTOM EKMAN LAYER

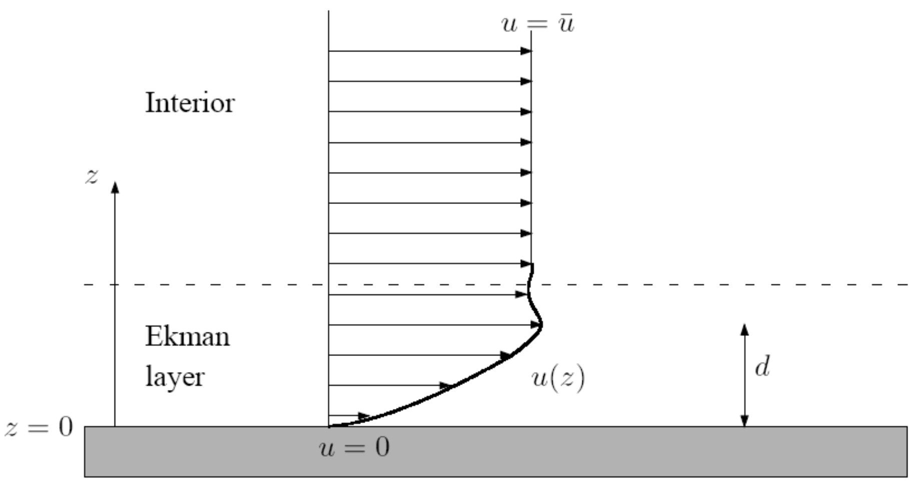

Figure 8-3 Frictional influence of a flat bottom on a uniform flow in a rotating framework.

$$u = \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right)$$
$$v = \bar{u}\, e^{-z/d}\sin\frac{z}{d}\,.$$

$$d = \sqrt{\frac{2\nu_E}{f}}$$

$$f > 0$$

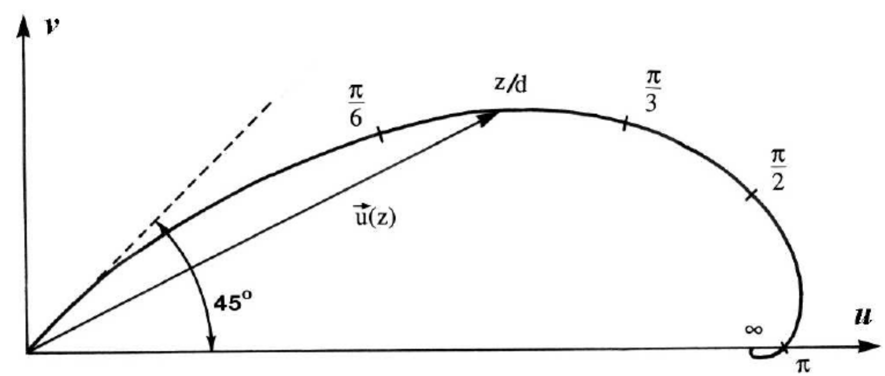

Figure 8-4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.