THE BOTTOM EKMAN LAYER - GENERALIZED

- **Interior geostrophic flow varying on a scale sufficiently large to be in geostrophic equilibrium:**

$$-f\bar{v} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \qquad f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$

- **for a constant Coriolis parameter (on f-plane) the flow is non-divergent:** $\partial\bar{u}/\partial x + \partial\bar{v}/\partial y = 0$

- **The BL equations:**

$$\begin{aligned} -f(v-\bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\ f(u-\bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

- **…with BCs** $\boldsymbol{u \to \bar{u}}$ ***and*** $\boldsymbol{v \to \bar{v}}$ ***for*** $\boldsymbol{z \to \infty}$ ***and*** $\boldsymbol{u(z=0) = v(z=0) = 0}$

- **…and solutions:**

$$\begin{aligned} u &= \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right) - \bar{v}\,e^{-z/d}\sin\frac{z}{d} \\ v &= \bar{u}\,e^{-z/d}\sin\frac{z}{d} + \bar{v}\left(1 - e^{-z/d}\cos\frac{z}{d}\right). \end{aligned}$$

THE BOTTOM EKMAN LAYER - GENERALIZED

- **We can compute the transport related to the Ekman bottom layer:**

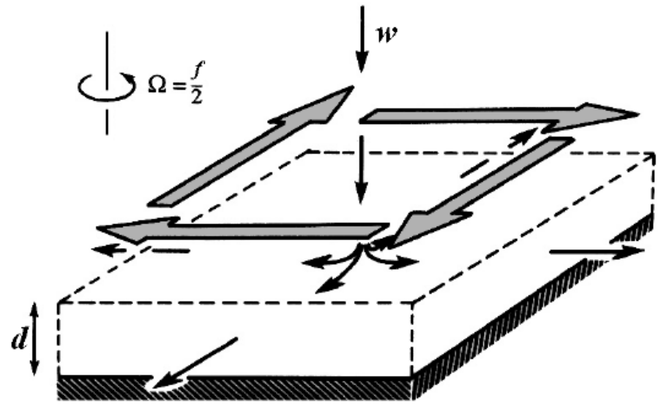
$$\mathrm{U} = \int_0^\infty (u - \bar{u})\, dz = -\frac{d}{2}(\bar{u} + \bar{v})$$
$$\mathrm{V} = \int_0^\infty (v - \bar{v})\, dz = \frac{d}{2}(\bar{u} - \bar{v}).$$

- **this transport is not necessarily parallel to the interior geostrophic flow and may be divergent [...]:**

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = -\frac{d}{2}\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) = -\frac{d}{2\rho_0 f}\nabla^2 \bar{p}.$$

- **The flow in the BL converges/diverges if interior has a relative vorticity $\bar{\zeta} = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \neq 0$ (pos/neg):**
  - **Divergence in BEL and compensating downwelling in the interior + ACyc gyre**
  - **Convergence in BEL and compensating upwelling in the interior + Cyc gyre**
- **Due to the solid bottom, the only possibility to provide convergence / divergence which supports upwelling / downwelling is a vertical velocity $\bar{w}$ from the interior**

THE BOTTOM EKMAN LAYER - GENERALIZED



$f > 0$

Figure 8-5 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such a situation arises in the presence of an anticyclonic gyre in the interior, as depicted by the large horizontal arrows. Similarly, interior cyclonic motion causes convergence in the Ekman layer and upwelling in the interior.

- **Interior is geostrophic and on f-plane $\partial_z \bar{w} = 0 \Rightarrow \bar{w} = cost \Rightarrow$ the vertical velocity must occur throughout the depth of the fluid**
- **Since divergence is $\propto d \ll H \Rightarrow$ the vertical velocity is very weak […]**
- **def. EKMAN PUMPING: $\bar{w} = \frac{d}{2}\bar{\zeta} = \frac{d}{2\rho_0 f}\nabla^2\bar{p} = -\nabla \cdot (U, V) = -\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)$**
- **The larger the vorticity from the interior, the greater the upwelling / downwelling, with an effect increasing toward the equator ($f \rightarrow 0$)**

THE BOTTOM EKMAN LAYER OVER UNEVEN TERRAIN

- **Irregular topography has an effect over the structure of BEL**
- **Terrain with elevation $\mathbf{z = b}(x, y)$ above a horizontal reference level**
- **Since GFD flows are almost 2D: $\nabla b(x, y) = (\partial_x b, \partial_y b) \ll 1$**
- **Interior geostrophic flow not uniform**
- **The BL equations:**

$$
\begin{aligned}
-\ f(v - \bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\
f(u - \bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2}
\end{aligned}
$$

- **…with BCs $u \to \bar{u}$ and $v \to \bar{v}$ for $z \to \infty$ and $u(z = b) = v(z = b) = 0$**
- **…and solutions are the same as previous case with $z \to z - b$:**

$$
\begin{aligned}
u &= \bar{u} - e^{(b-z)/d} \left(\bar{u} \cos \frac{z-b}{d} + \bar{v} \sin \frac{z-b}{d} \right) \\
v &= \bar{v} + e^{(b-z)/d} \left(\bar{u} \sin \frac{z-b}{d} - \bar{v} \cos \frac{z-b}{d} \right)
\end{aligned}
$$

THE BOTTOM EKMAN LAYER OVER UNEVEN TERRAIN

- **Computing vertical velocity from continuity eq.:**

$$
\begin{aligned}
\frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\
&= e^{(b-z)/d} \left\{ \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right) \sin \frac{z-b}{d} \right. \\
&\quad + \frac{1}{d} \frac{\partial b}{\partial x} \left[(\bar{u} - \bar{v}) \cos \frac{z-b}{d} + (\bar{u} + \bar{v}) \sin \frac{z-b}{d} \right] \\
&\quad \left. + \frac{1}{d} \frac{\partial b}{\partial y} \left[(\bar{u} + \bar{v}) \cos \frac{z-b}{d} - (\bar{u} - \bar{v}) \sin \frac{z-b}{d} \right] \right\}
\end{aligned}
$$

- **…then we can integrate from $z = b, w = 0$ to $z \to \infty, w = \bar{w}$:**

$$
\bar{w} = \left(\bar{u} \frac{\partial b}{\partial x} + \bar{v} \frac{\partial b}{\partial y} \right) + \frac{d}{2} \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right)
$$

- **The first component was found during the analysis of geostrophic flow over irregular bottom, and it ensures no normal flow to the bottom; the second is the Ekman pumping as in the flat bottom case, which is not affected by the bottom slope**

THE SURFACE EKMAN LAYER

- **The frictional stress against the flow is exerted by the WIND STRESS (historically, the 1st problem investigated by Ekman)**

- **Hypotheses:**
  - **Interior flow is uniform and geostrophic:** $Ro_T \ll 1 \; and \; Ro \ll 1$
  - **Homogeneous fluid:** $\rho_0 = cost \; and \; \rho' = 0$
  - **Presence of wind stress:** $\vec{\tau} = (\tau_x, \tau_y)$

- **Primitive equations + BCs:**

$$
\begin{aligned}
-f\,(v-\bar{v}) &= \nu_E \frac{\partial^2 u}{\partial z^2} \\
+f\,(u-\bar{u}) &= \nu_E \frac{\partial^2 v}{\partial z^2} \\
\text{Surface } (z=0): &\quad \rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x, \quad \rho_0 \nu_E \frac{\partial v}{\partial z} = \tau^y \\
\text{Toward interior } (z \to -\infty): &\quad u = \bar{u}, \quad v = \bar{v}.
\end{aligned}
$$

- **Solutions:**

$$
\begin{aligned}
u &= \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) \right] \\
v &= \bar{v} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]
\end{aligned}
$$

THE SURFACE EKMAN LAYER

- The solution has a wind-driven component fully related to the wind stress $\vec{\tau}$, independent by the interior flow but dependent on $1/d \Rightarrow$ the wind-driven component can be very large if d is very small (for example with almost inviscid flow with ν_E very small), and even a moderate wind stress may generate a large wind-driven component

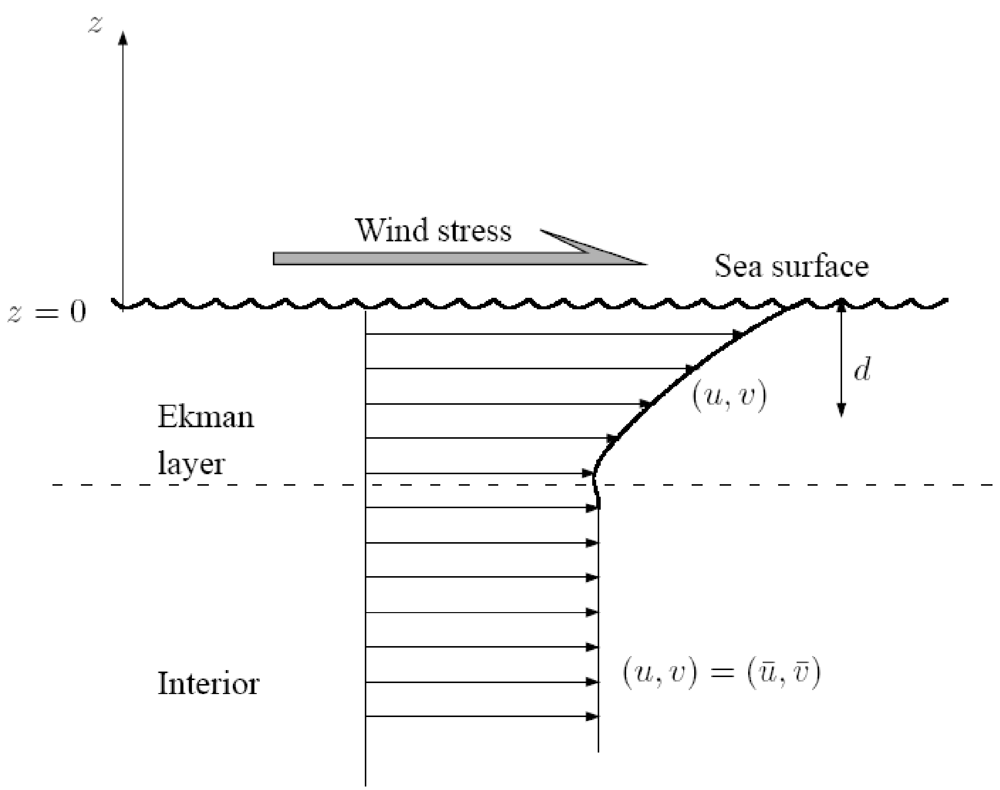


Figure 8-6 The surface Ekman layer generated by a wind stress on the ocean.

THE SURFACE EKMAN LAYER

- The **wind-driven (Ekman) transport** in the SEL has components […]:

$$\mathrm{U} = \int_{-\infty}^{0} (u - \bar{u})\, dz = \frac{1}{\rho_0 f}\, \tau^y$$

$$\mathrm{V} = \int_{-\infty}^{0} (v - \bar{v})\, dz = \frac{-1}{\rho_0 f}\, \tau^x.$$

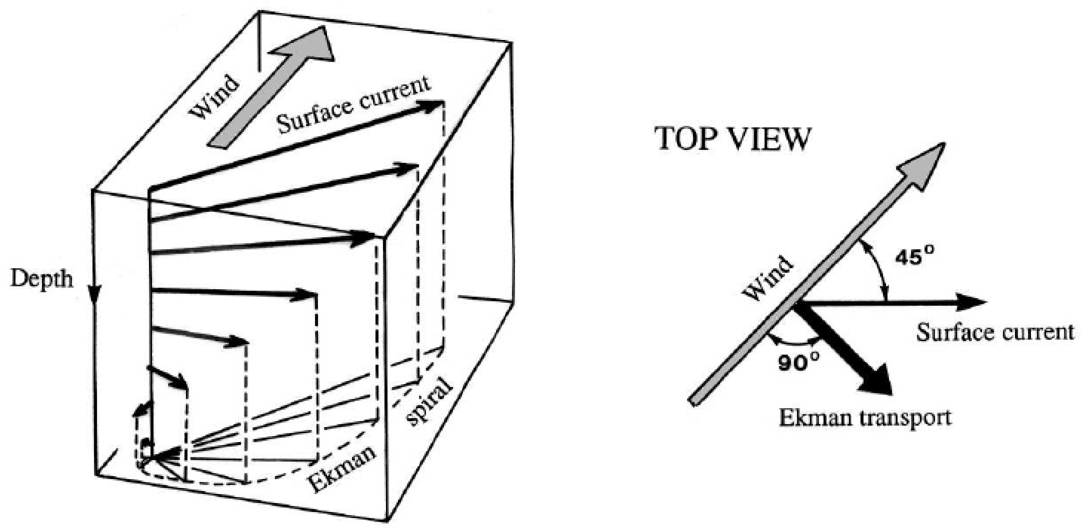


Figure 8-7 Structure of the surface Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the right of the surface stress. The reverse holds for the Southern Hemisphere.

$$f > 0$$

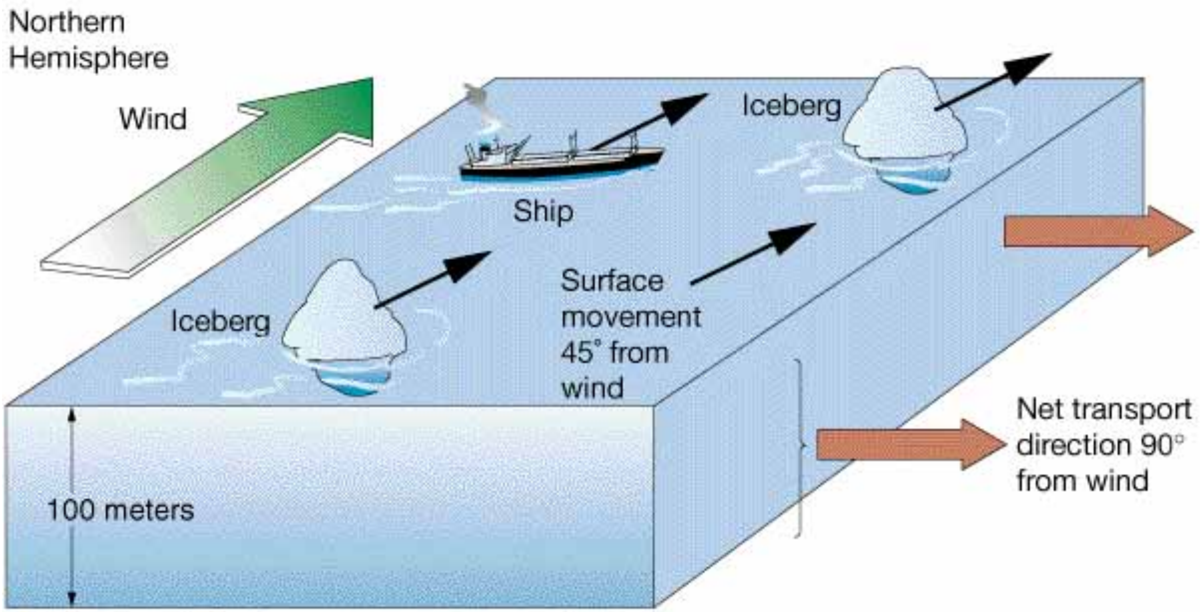
THE SURFACE EKMAN LAYER

- The **Ekman transport** is perpendicular to the wind stress, to the right in the N. Hemisphere, to the left in the S. Hemisphere, explaining why icebergs, mostly floating underwater, drift to the right of the wind as observed by Fridtjof Nansen
- The surface velocity $\overrightarrow{u_0} = \vec{u}(z = 0)$ has an angle of $45°$ with $\vec{\tau}$ [...]

https://people.ucsc.edu/~mdmccar/migrated/ocea1/01_Public/lectures/lect_notes_2/14_SURF_Ocean_Circ_12Fall.pdf

$$f > 0$$

Nansen noticed icebergs moving
20° - 40° to right of wind direction



THE SURFACE EKMAN LAYER

- **Compute the divergence of the Ekman transport (as done for BEL):**

$$\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y} = \int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = \frac{1}{\rho_0}\left[\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right)\right]$$

- **The divergence is totally independent by $\boldsymbol{v_E}$ and is entirely dependent on $\vec{\boldsymbol{\tau}}$: $\nabla \cdot \vec{\boldsymbol{U}} \propto \nabla \times \vec{\boldsymbol{\tau}}|_z \rightarrow$ *wind − stress curl***

- **On f-plane: $\nabla \cdot \vec{\boldsymbol{U}} = \dfrac{1}{\rho_0 f} \nabla \times \vec{\boldsymbol{\tau}}|_z$**

- **IF $\nabla \times \vec{\boldsymbol{\tau}}|_z \neq 0$ the divergence of the Ekman transport must be provided by a vertical velocity throughout the interior (as in BEL) [...] :**

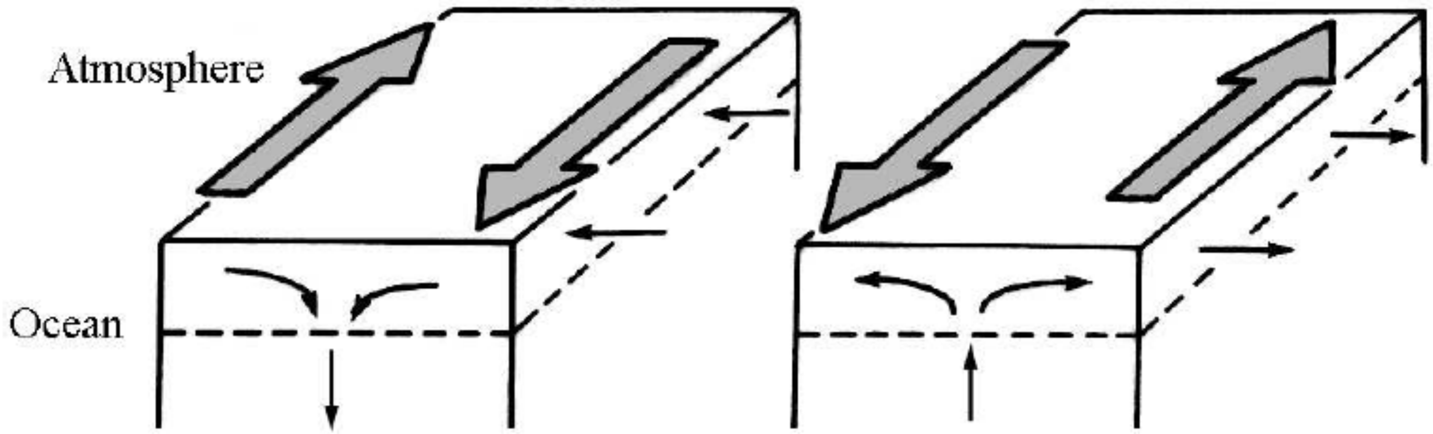
$$\bar{w} = + \int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = \frac{1}{\rho_0}\left[\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right)\right] = w_{\mathrm{Ek}}$$

- **def. EKMAN PUMPING: $\bar{\boldsymbol{w}} = \boldsymbol{w_{Ek}} = \dfrac{1}{\rho_0} \nabla \times \dfrac{\vec{\boldsymbol{\tau}}}{\boldsymbol{f}}|_z$**

- **Ekman pumping on f-plane: $\boldsymbol{w_{Ek}} = \dfrac{1}{\rho_0 f} \nabla \times \vec{\boldsymbol{\tau}}|_z$**

THE SURFACE EKMAN LAYER

- **Ekman pumping on f-plane:** $w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z \lesseqgtr 0$
- **Ekman pumping: a very effective mechanism to drive subsurface ocean currents through the action of winds**



$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z < 0$$

Clockwise wind pattern

Convergence in SEL

Downwelling

$$w_{Ek} = \nabla \cdot \vec{U} = \frac{1}{\rho_0 f} \nabla \times \vec{\tau}|_z > 0$$

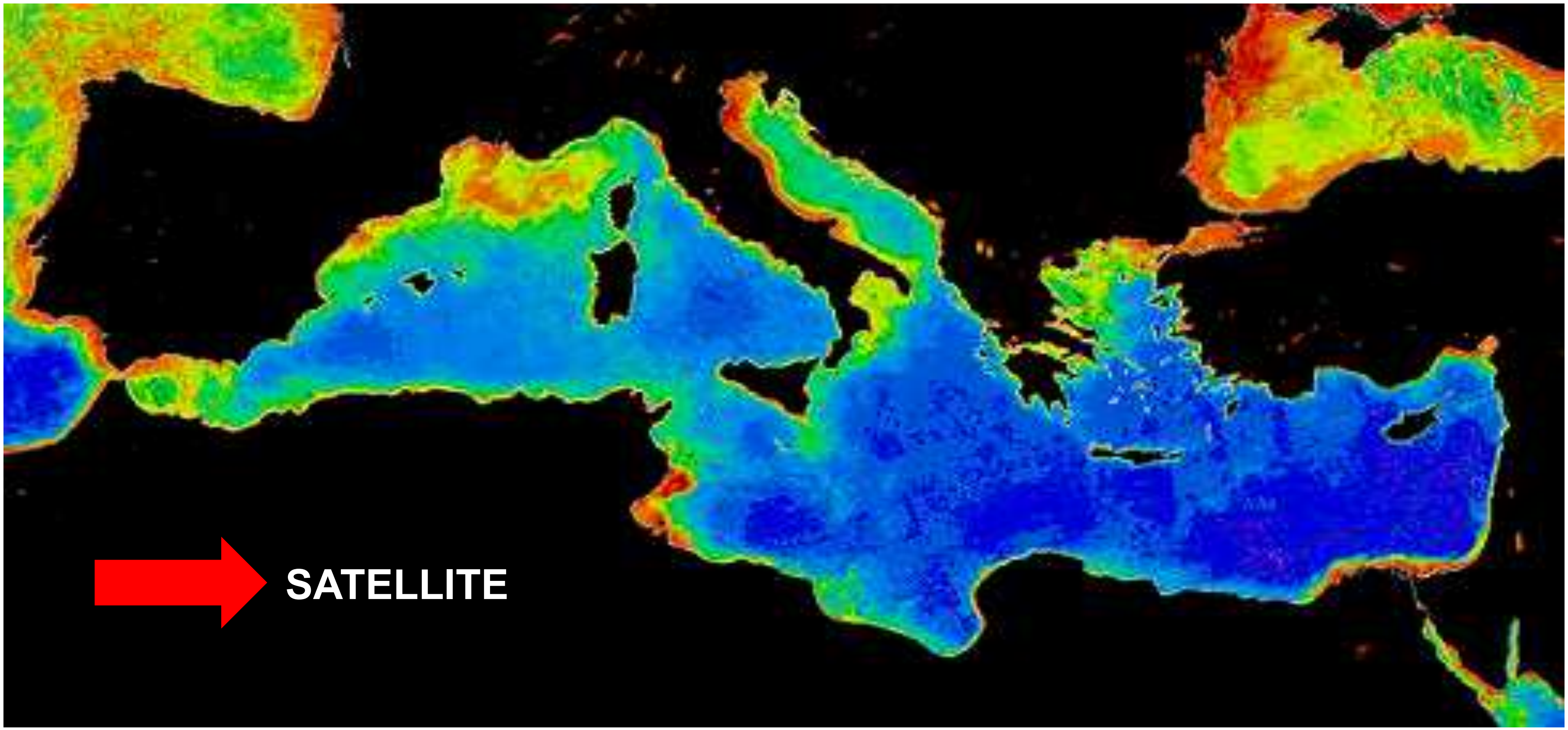
Anticlockwise wind pattern

Divergence in SEL

Upwelling

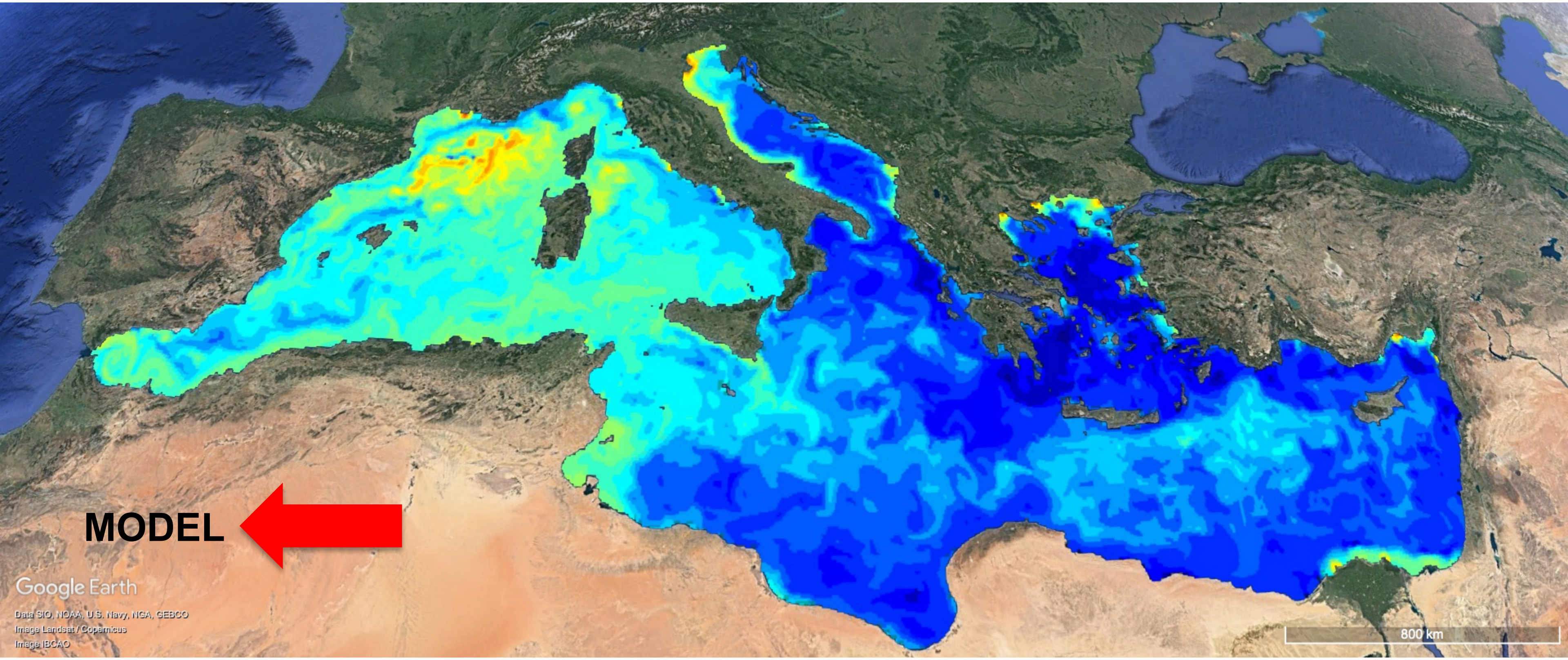
$$f > 0$$

Effect of upwelling on biogochemistry in the Mediterranean Sea



North-Western Med Sea is an area of upwelling and high productivity due to Ekman pumping

Effect of upwelling on biogochemistry in the Mediterranean Sea



North-Western Med Sea is an area of upwelling and high productivity due to Ekman pumping

THE EKMAN LAYER IN REAL GEOPHYSICAL FLOWS

- Real geophysical flows are characterized by turbulence and stratification $\Rightarrow$ observations cannot match the highly idealized models of BEL and SEL
- GFD flows have $Re \gg 1 \Rightarrow$ we can replace υ with υ_E to account for the enhanced momentum transfer in a turbulent flow
- Ekman layers are SHEAR FLOWS and turbulence is not homogenous, increasing with the shear and suppressed close to the boundary where size of turbulent eddies is limited $\Rightarrow$ a general theory of turbulence does not exist, as a minimum $\upsilon_E = \upsilon_E(z)$ but observations do not agree with simple models
- The angle between near-boundary velocity / surface velocity and interior in BEL / SEL is $< 45°$ ranging $5° \div 20°$
- Eddy viscosity scales with friction velocity $u^* = \sqrt{|\vec{\tau}|/\rho_0}$ and d as mixing length (the size of the most turbulent eddies): $\upsilon_E \sim u^* d$
- **Ekman depth scales with $d \sim \sqrt{\upsilon_E/f} \sim \sqrt{u^* d\ /f} \Rightarrow d \sim u^*/f$**
- Empirically: $d = 0.4\, u^*/f$

THE EKMAN LAYER IN REAL GEOPHYSICAL FLOWS

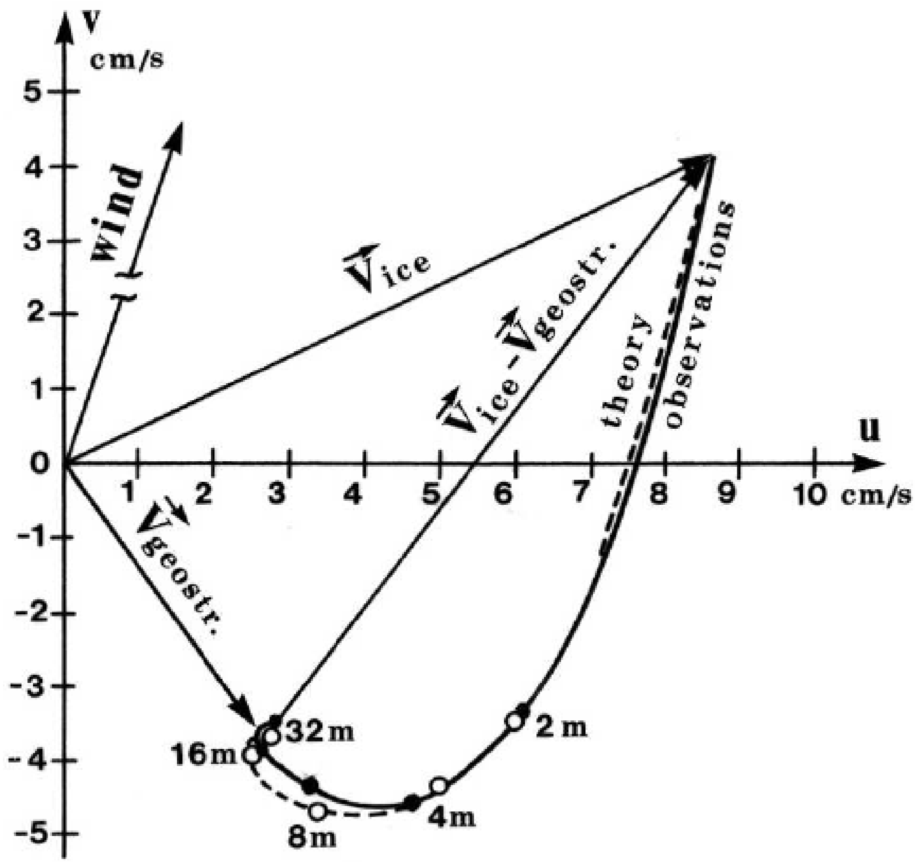
- **Real geophysical flows are characterized by vertical density stratification $\rho = \rho(z)$: the gradual change of density with z hinders vertical movements ⇒ reduction of vertical mixing of momentum by turbulence and decoupling motions at separate levels**

- **Stratification reduces the Ekman depth and increases the veering of the velocity vector with z**

- **Surface atmospheric layer during daytime over land and above warm currents at sea is frequently in a state of CONVECTION due to the heating from below: the Ekman dynamics is related to convective motions, driven both by the geostrophic flow aloft and by the intensity of the surface heat flux ⇒ Atmospheric Boundary Layer (ABL)**

- **Ekman depth scales with $d = \dfrac{1.3\, u^*}{f\left(1+\frac{N^2}{f^2}\right)^{1/4}}$**

One of the few cases when obs → theory



=> *surface current*

Figure 8-9 Comparison between observed currents below a drifting ice floe at 84.3°N and theoretical predictions based on an eddy viscosity $\nu_E = 2.4 \times 10^{-3}$ m^2/s. (Reprinted from *Deep-Sea Research*, 13, Kenneth Hunkins, Ekman drift currents in the Arctic Ocean, p. 614, ©1966, with kind permission from Pergamon Press Ltd, Headington Hill Hall, Oxford 0X3 0BW, UK)

observations: angle u_0 and u_{INT} < 45º

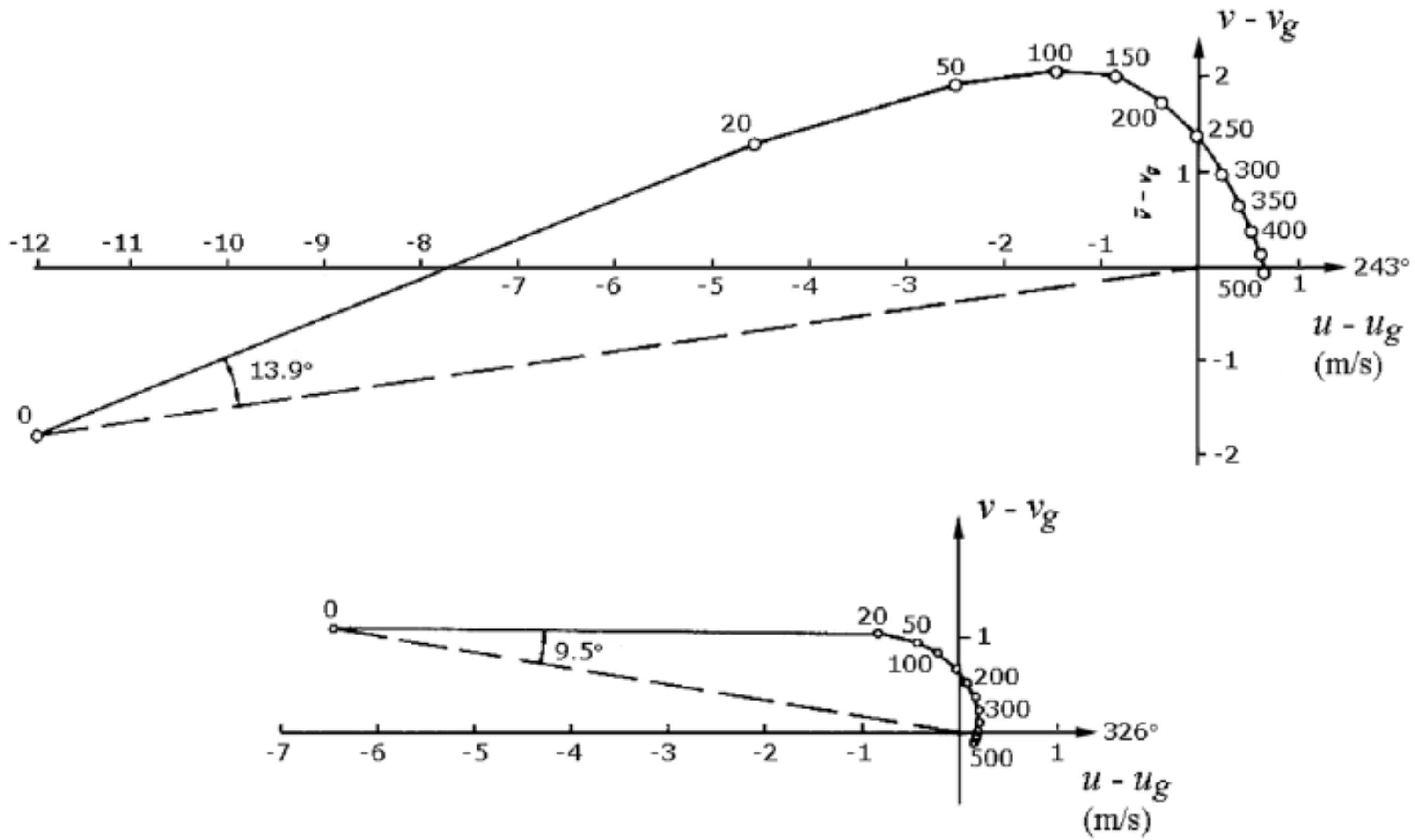


Figure 8-10 Wind vectors minus geostrophic wind as a function of height (in meters) in the maritime friction layer near the Scilly Isles. *Top diagram*: Case of warm air over cold water. *Bottom diagram*: Case of cold air over warm water. (Adapted from Roll, 1965)

BAROTROPIC WAVES

- Another possible simplification of the governing eqs. of GFD is to LINEARIZE the eqs. → restrictions must be imposed on the flows
- Coriolis terms are linear → no need to simplify
- Advection terms are non-linear → need to be simplified (Ro $= \frac{U}{\Omega L} \ll \mathbf{1}$)
  - → relatively weak flows (small U)
  - → relatively large scales (large L)
  - → in LAB: fast rotation rates (large Ω)
- Local time rate of velocity change is linear → no need to simplify ($\mathrm{Ro}_T = \frac{1}{\Omega T} \sim \mathbf{1}$)
- → consider slow flow fields under rotation that evolve relatively fast = rapidly moving disturbances do not require large velocities = information (or energy) may travel faster then material particles → the flow is a WAVE FIELD !
- → WAVES supported by <u>inviscid</u>, <u>homogeneous</u> fluid <u>in rotation</u>
- Velocity scale: "celerity" $\mathrm{C} = \frac{\textit{distance covered by the signal}}{\textit{nominal evolution time}} = \frac{L}{T} \sim L\Omega \gg U$

BAROTROPIC WAVES

- slow flow fields under rotation that evolve relatively fast = rapidly moving disturbances do not require large velocities = information (or energy) may travel faster then material particles

- *NOTE: look Appendix B of Cushman to review wave dynamics (already done in the first part of the course)*

BAROTROPIC WAVES

System governing the linear wave dynamics of an inviscid, homogeneous, shallow-water fluid in rotation (for f > 0) => **start from the shallow-water model excluding advection**

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x}\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y}\\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

$$h(x, y, t) = \eta(x, y, t) + H(x, y)$$

BAROTROPIC WAVES

System governing the linear wave dynamics of an inviscid, homogeneous, shallow-water fluid in rotation (for f > 0) => **if $H(x,y) = cost$ [flat bottom] and through the scale analysis we obtain a linearized form of the continuity equation which brings to small amplitude waves [...]**

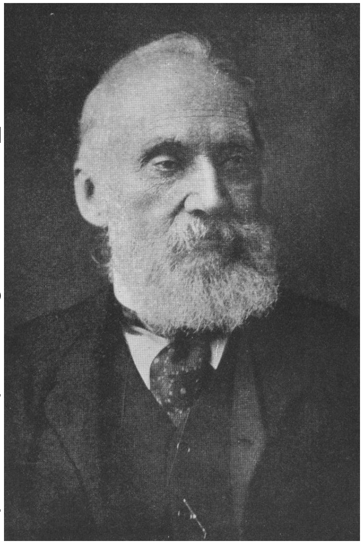
$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

KELVIN WAVE 1

https://it.wikipedia.org/wiki/William_Thomson



A traveling disturbance requiring a lateral boundary layer as a support: u = v = 0 at x = 0 (coastline)

Lord Kelvin's hypothesis was that u = 0 in the whole domain

From the previous equations **[…]** :

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial y^2}$$

$$c = \sqrt{gH}$$

Wave equation => propagation of 1-d non-dispersive waves => speed of surface gravity waves in non-rotating shallow waters

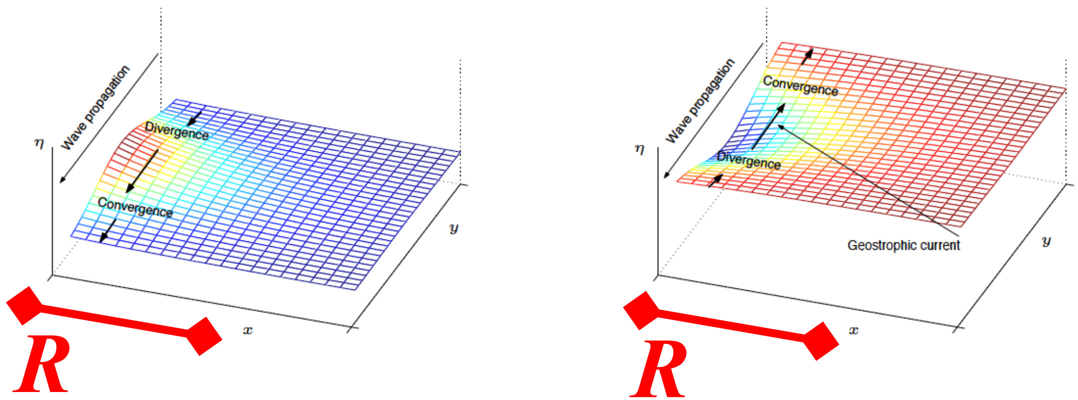
Solution: **[…]**

$$\begin{aligned} u &= 0 \\ v &= \sqrt{gH}\, F(y+ct)\, e^{-x/R} \\ \eta &= -\,H\ F(y+ct)\, e^{-x/R}, \end{aligned}$$

$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}$$

Rossby radius of deformation

KELVIN WAVE 2



$$- fv = -g \frac{\partial \eta}{\partial x}$$

Figure 9-1 Upwelling and downwelling Kelvin waves. In the Northern Hemisphere, both waves travel with the coast on their right, but the accompanying currents differ. Geostrophic equilibrium in the x– momentum equation leads to a velocity v that is maximum at the bulge and directed as the geostrophic equilibrium requires. Because of the different geostrophic velocities at the bulge and further away, convergence and divergence patterns create a lifting or lowering of the surface. The lifting and lowering is such that the wave propagates towards negative y in either case (positive or negative bulge).

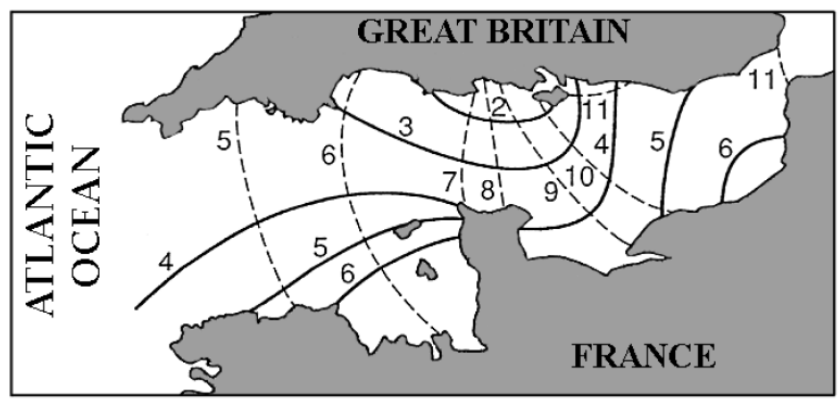


Figure 9-2 Cotidal lines (dashed) with time in lunar hours for the M2 tide in the English Channel showing the eastward progression of the tide from the North Atlantic Ocean. Lines of equal tidal range (solid, with value in meters) reveal larger amplitudes along the French coast, namely to the right of the wave progression in accordance with Kelvin waves. (From Proudman, 1953, as adapted by Gill, 1982.)

POINCARE’ WAVES 1

https://it.wikipedia.org/wiki/Henri_Poincar%C3%A9#/media/File:JH_Poincare.jpg

Keep u $\neq$ 0 in the whole domain

The system has to be solved entirely → all coeffs. are constant and a Fourier-like solution can be set:

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re \begin{pmatrix} A \\ U \\ V \end{pmatrix} e^{\mathrm{i}\,(k_x x + k_y y - \omega t)}$$

Dispersion relation **[…]** : $\omega\,[\omega^2 - f^2 - gH\,(k_x^2 + k_y^2)] = 0$

solution:

$\omega = 0$ → steady geostrophic flow

$\omega = \sqrt{f^2 + gH\,k^2}$ → **superinertial travelling waves** (PW)

and cases **[…]**: f = 0, HF, LF with $\frac{\omega}{f} = \sqrt{1 + R^2 k^2}$

POINCARE' WAVES 2

$$1 \leq \frac{\omega}{f} \leq Rk$$

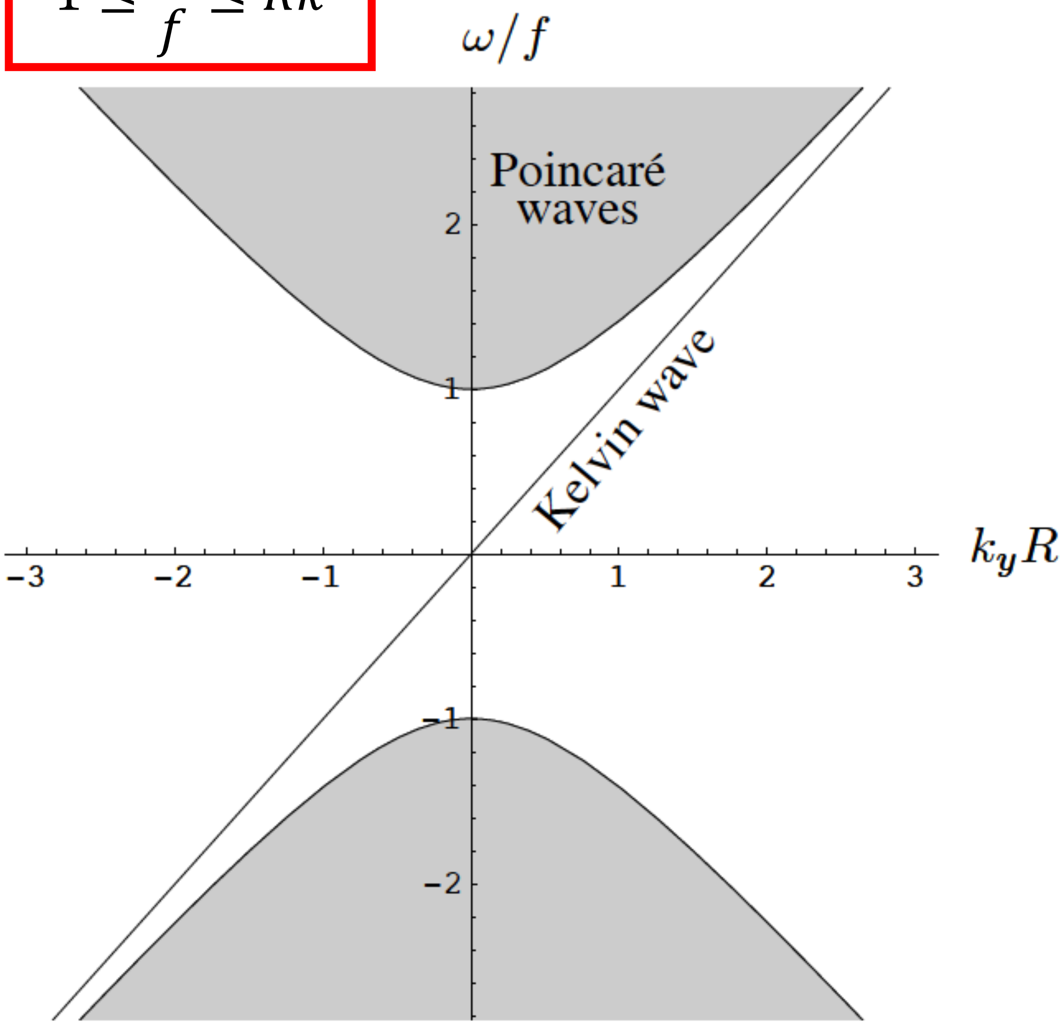


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

The solution of KW as $\omega/f = k_y R$ can be found with Fourier-like solution in the eqs. for KW with $e^{i(ky+\omega t)}$

POINCARE' WAVES 2

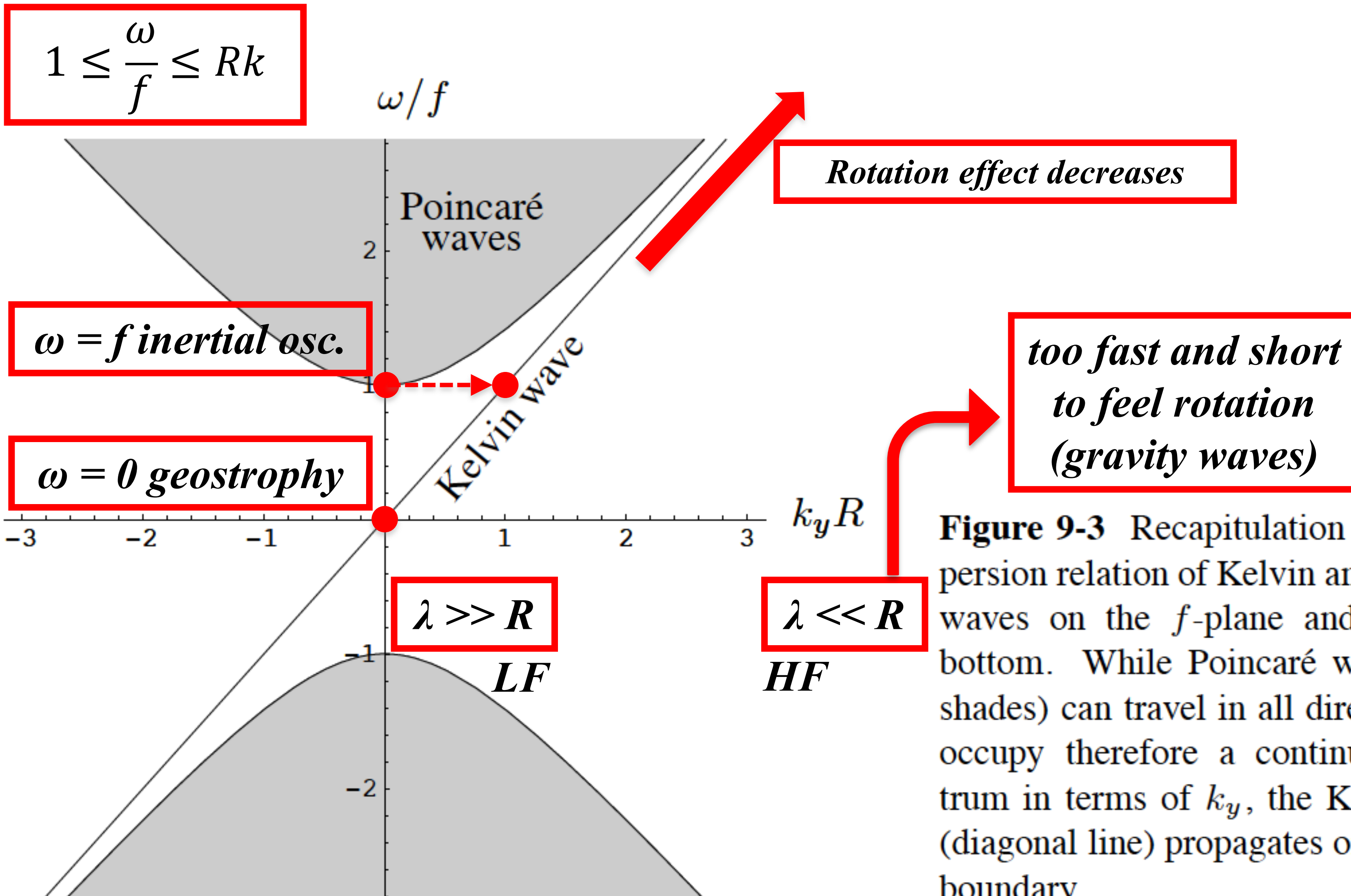


Figure 9-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom. While Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of k_y, the Kelvin wave (diagonal line) propagates only along a boundary.

The solution of KW as $\omega/f = k_y R$ can be found with Fourier-like solution in the eqs. for KW with $e^{i(ky+\omega t)}$

PLANETARY or ROSSBY WAVES 1

http://paoc.mit.edu/paoc/education/rossby.htm

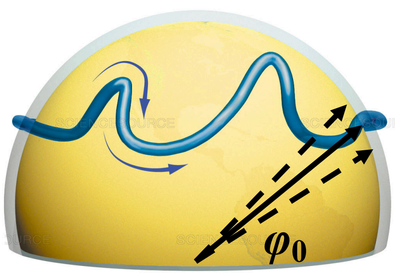
KW and PW are relatively fast waves ($\omega \geq f$): do exist other slower waves ($\omega \ll f$), **associated with evolution of disturbances in the geostrophic flow**?

Coriolis parameter: $f = 2\Omega \sin \varphi$

Taylor expansion around a reference latitude φ_0:

$$f = 2\Omega \sin\varphi_0 + 2\Omega \frac{y}{a} \cos\varphi_0 + \ldots$$

$$f = f_0 + \beta_0 y$$



$$\beta_0 = 2(\Omega/a)\cos\varphi_0$$ BETA PARAMETER

$$\beta = \frac{\beta_0 L}{f_0} \ll 1$$ PLANETARY NUMBER

The system for the β-plane has "large" and "small" terms → **[…]** retaining the large ones we obtain the geostrophic flow (u_g,v_g 1st-approx solution)

PLANETARY or ROSSBY WAVES 2

solving the system with (u_g, v_g) we obtain: velocity = geostr + ageostr

[…] and then including (u,v) in the continuity equation:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} = 0$$

Using a Fourier-like solution for η we have the dispersion relation:

$$\omega = -\beta_0 R^2 \frac{k_x}{1 + R^2 (k_x^2 + k_y^2)}$$

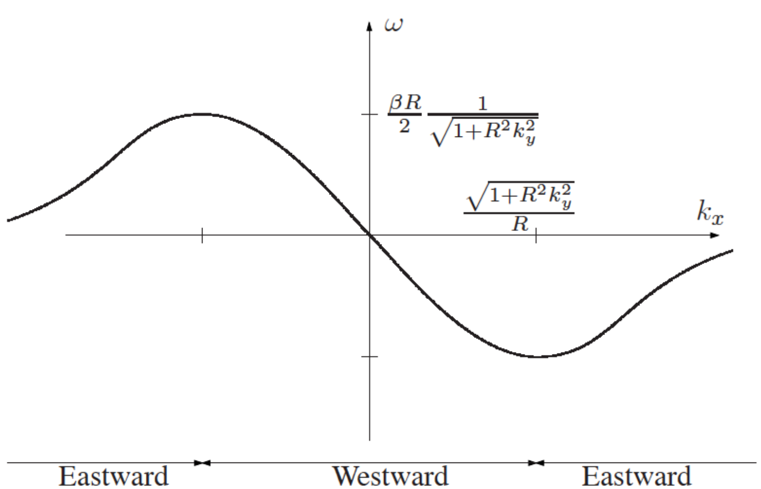


Figure 9-4 Dispersion relation of planetary (Rossby) waves. The frequency ω is plotted against the zonal wavenumber k_x at constant meridional wavenumber k_y. As the slope of the curve reverses, so does the direction of zonal propagation of energy.

- For both cases of LW and SW: $\omega \ll f_0$ subinertial wave
- $c_x = c_x(k)$ dispersive wave
- $c_x = \frac{\omega}{k_x} < 0$ westward propagation
- $c_g = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}\right)$ is westward for LW and eastward for SW

PLANETARY or ROSSBY WAVES 3

- Rewriting the dispersion relation we obtain eq. for circles in (k_x, k_y) at constant ω: $\omega_1 < \omega_2 < \omega_3 < \omega_{max}$ **[…]**

- $\omega_{max} = \frac{\beta_0 R}{2}$ max frequency

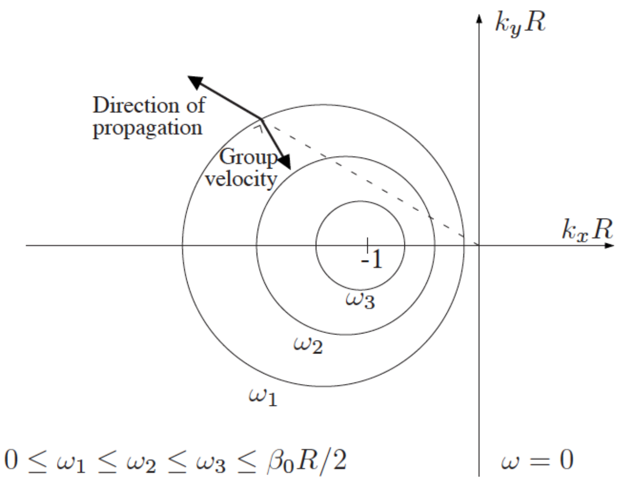
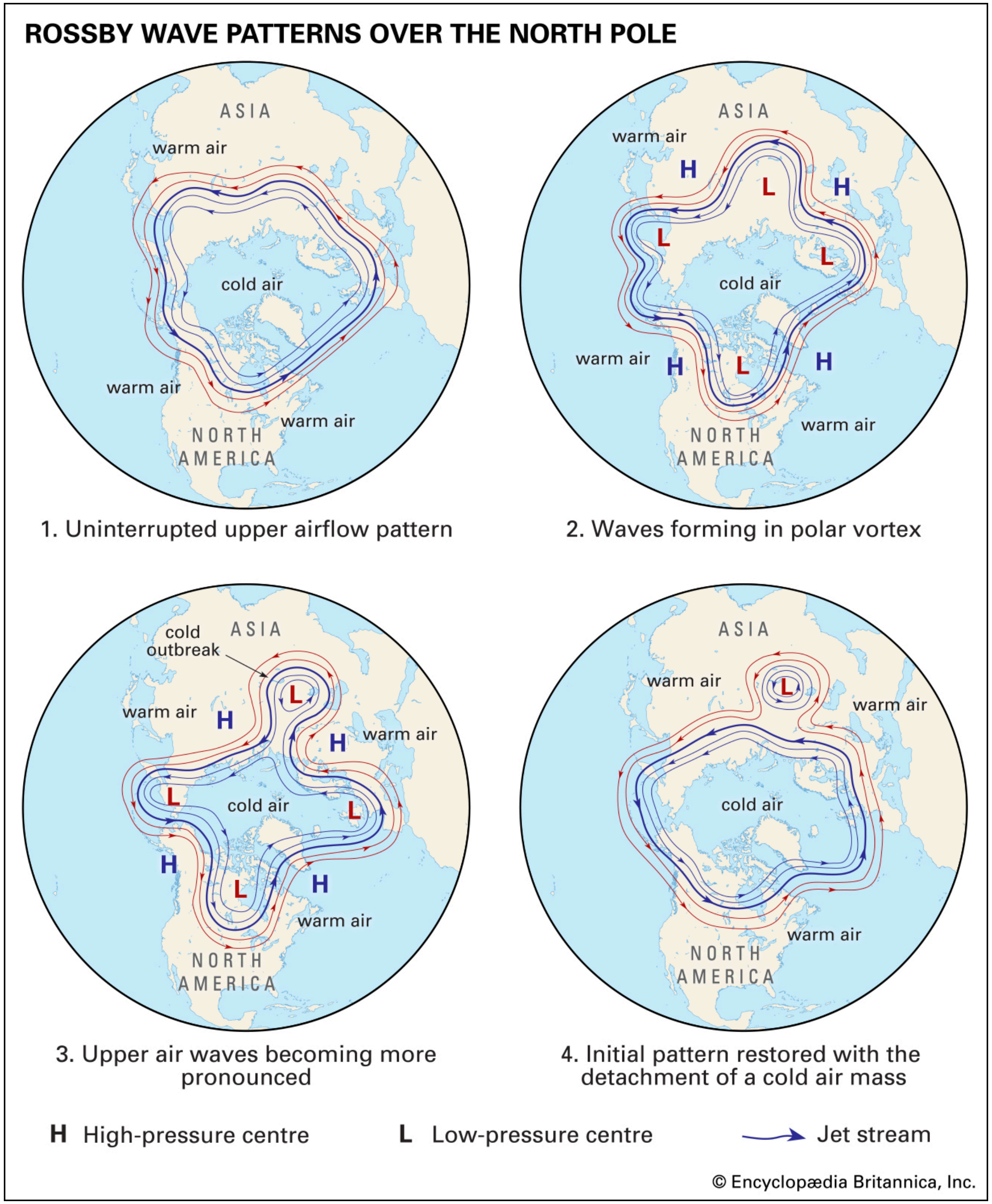


Figure 9-5 Geometric representation of the planetary-wave dispersion relation. Each circle corresponds to a fixed frequency, with frequency increasing with decreasing radius. The group velocity of the (k_x, k_y) wave is a vector perpendicular to the circle at point (k_x, k_y) and directed toward its center.

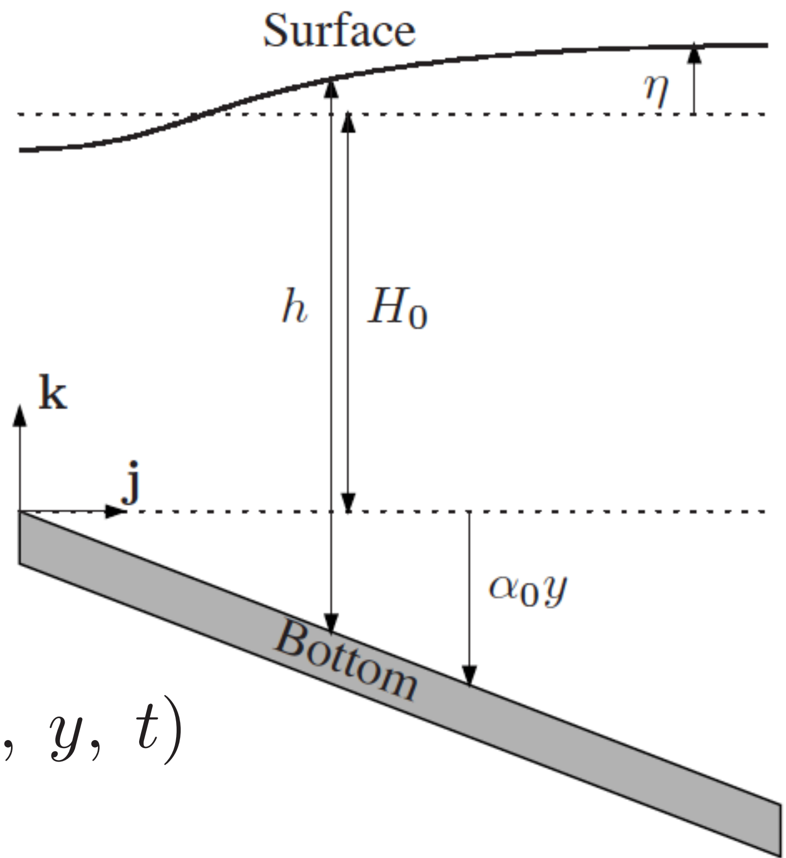


TOPOGRAPHIC WAVES 1

Perturbing effect is small and associated with weak bottom irregularity (not uncommon in the GFD phenomena…)

$$H = H_0 + \alpha_0 y$$

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1$$



$$h(x,\ y,\ t) = H_0 + \alpha_0 y + \eta(x,\ y,\ t)$$

The system has "large" and "small" terms which scale as Ro_T: retaining the large ones we obtain the geostrophic flow (u_g,v_g 1st-approx solution)

TOPOGRAPHIC WAVES 2

solving the system with (u_g, v_g) we obtain: velocity = geostr + ageostr

[…] and then including (u,v) in the continuity equation:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2\eta + \frac{\alpha_0 g}{f}\frac{\partial \eta}{\partial x} = 0$$

Using a Fourier-like solution for η we have the dispersion relation:

$$\omega = \frac{\alpha_0 g}{f}\frac{k_x}{1 + R^2\left(k_x^2 + k_y^2\right)}$$

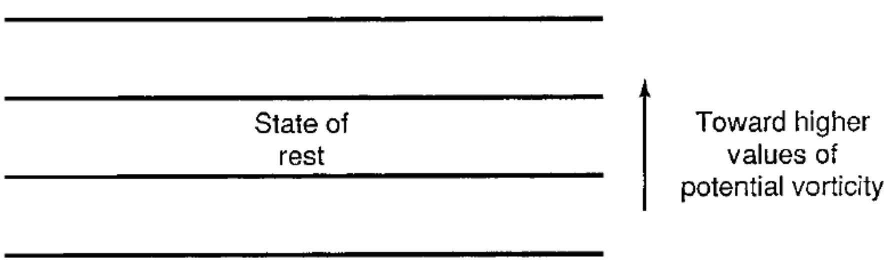
- Phase speed c_x has the same sign as α_0 → TW propagate in the Northern Hemisphere with the shallower side on their right
- For both cases of LW and SW: $\omega \ll f$ subinertial wave
- Since RW always propagate westward = with north on their right, analogy with RW is: “shallow-north” and “deep-south”
- Similar considerations as RW to obtain $\omega_{max} = \frac{\alpha_0 g}{2fR}$ max frequency

ANALOGY BETWEEN PLANETARY AND TOPOGRAPHIC WAVES

Potential vorticity on β-plane and sloping bottom: $$q = \frac{f_0 + \beta_0 y + \partial v/\partial x - \partial u/\partial y}{H_0 + \alpha_0 y + \eta}$$

Taylor expansion: $$q = \frac{1}{H_0}\left(f_0 + \boxed{\beta_0 y} - \boxed{\frac{\alpha_0 f_0}{H_0}\, y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f_0}{H_0}\eta\right)$$

=> gradient of potential vorticity



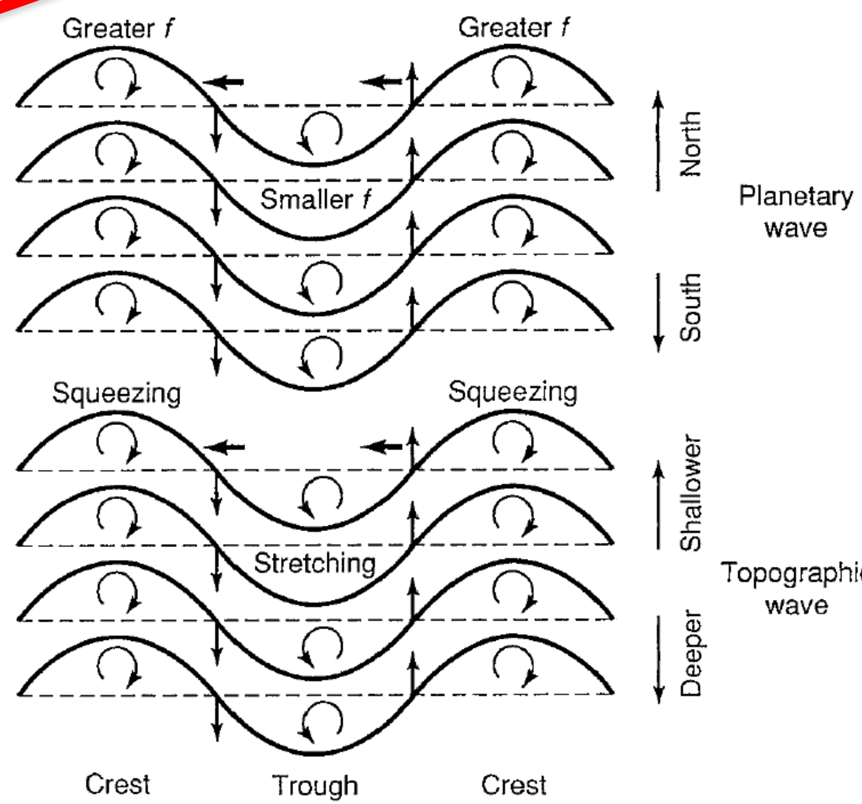


Figure 9-7 Comparison of the physical mechanisms that propel planetary and topographic waves. Displaced fluid parcels react to their new location by developing either clockwise or counterclockwise vorticity. Intermediate parcels are entrained by neighboring vortices, and the wave progresses forward.