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Let us first consider a fluid in static equilibrium. Lack of motion can occur only in the absence of horizontal forces and thus in the presence of horizontal homogeneity. Stratification is then purely vertical (Figure 11-1).

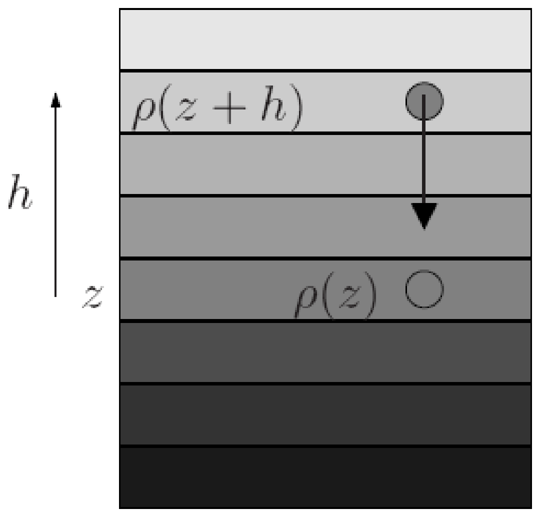

Figure 11-1 When an incompressible fluid parcel of density $\rho(z)$ is vertically displaced from level z to level $z+h$ in a stratified environment, a buoyancy force appears because of the density difference $\rho(z)-\rho(z+h)$ between the particle and the ambient fluid.

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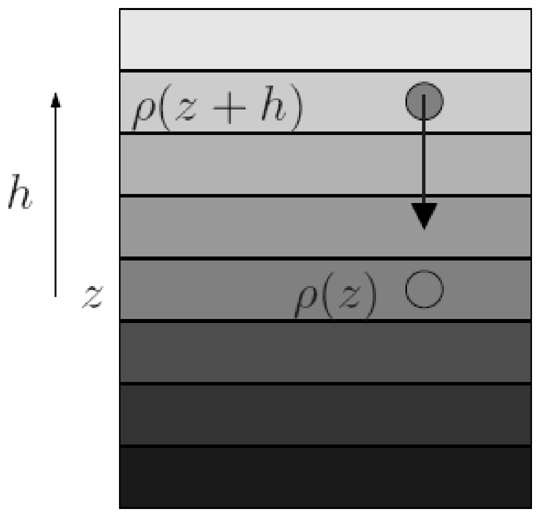

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With incompressible fluids, the displaced parcel retains its former density and at the new level is subject to a *net downward force equal to its own weight minus the weight of the displaced fluid* (Archimede's principle):

$$\rho(z)\, V\, \frac{d^2 h}{dt^2} = g\,[\rho(z+h) - \rho(z)]\, V$$

[…] using the **Boussinesq approx**

$$\rho(z+h) - \rho(z) \simeq \frac{d\rho}{dz}\, h$$

$$\frac{d^2 h}{dt^2} - \frac{g}{\rho_0}\frac{d\rho}{dz}\, h = 0$$

=> 2 cases: STABLE and UNSTABLE

=> $\frac{d\rho}{dz} < 0$ and $\frac{d\rho}{dz} > 0$ […]

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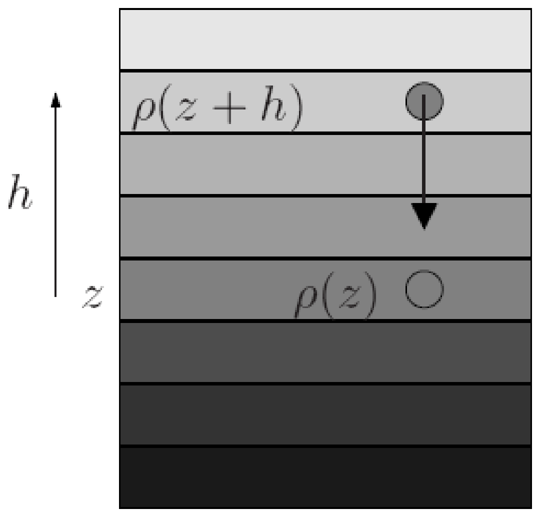

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$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz}$$

=> 2 cases: STABLE and UNSTABLE

=> $\frac{d\rho}{dz} < 0$ and $\frac{d\rho}{dz} > 0$ […]

BUOYANCY FORCE

We will restrict to STABLY STRATIFIED FLUIDS => N^2

Can we found an adimensional number with similar role as Ro ?

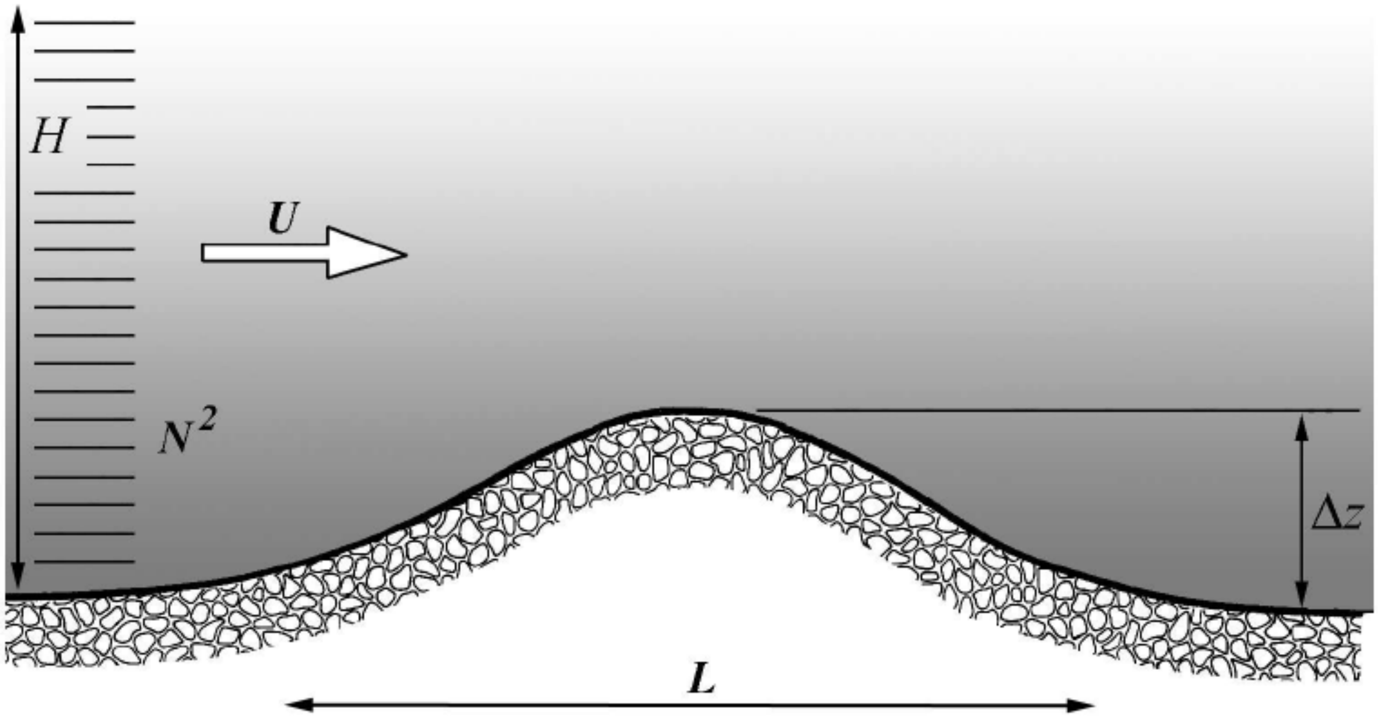

Figure 11-5 Situation in which a stratified flow encounters an obstacle, forcing some fluid parcels to move vertically against a buoyancy force.

Stratification will act to restrict or minimize the vertical displacement, forcing the flow to pass around rather the over the obstacle: the greater the restriction, the greater the importance of stratification

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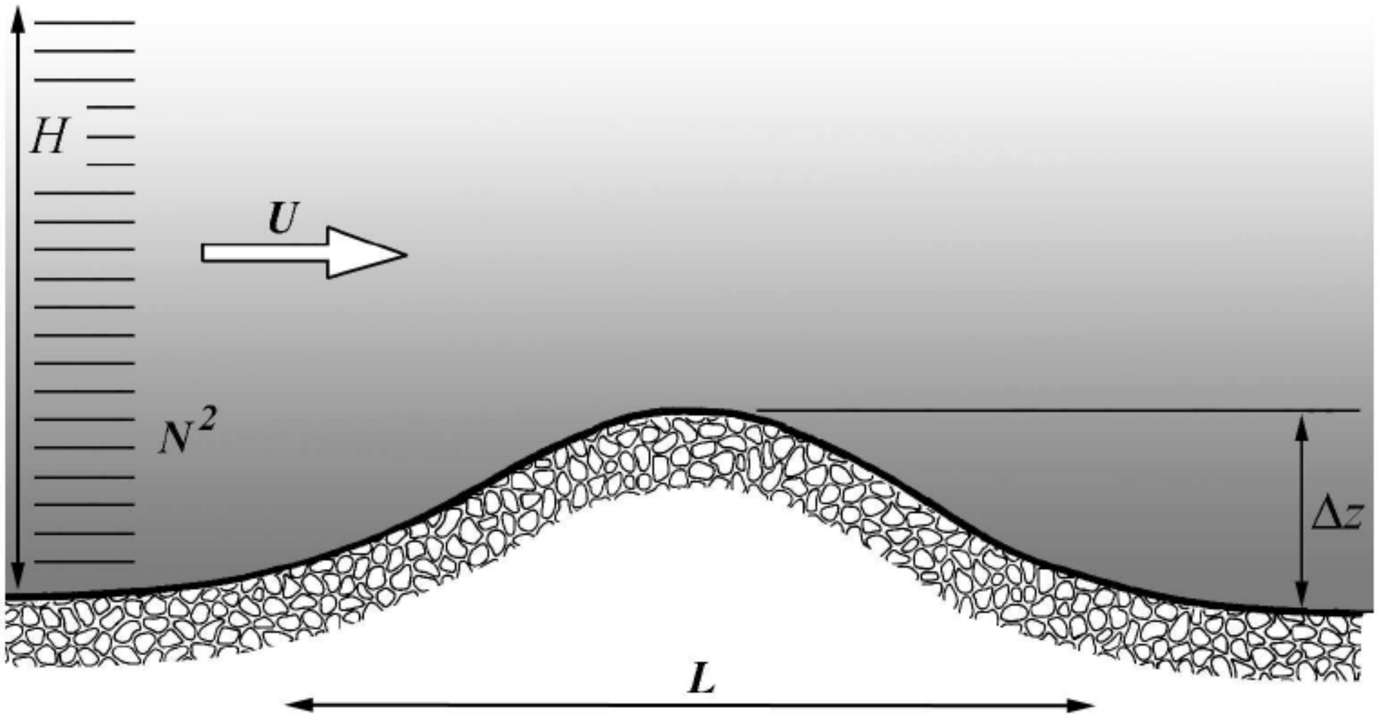

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[…] $\frac{vert.conv.}{horiz.div.} = \frac{\partial_z w}{\partial_x u + \partial_y v} = \frac{\Delta z}{H} = \frac{W/H}{U/L} = \frac{U^2}{N^2 H^2} = Fr^2$ **=> Froude n. =>** $Fr \lesssim 1$

1. $Ro \ll 1$

$Fr^2/Ro > Ro$

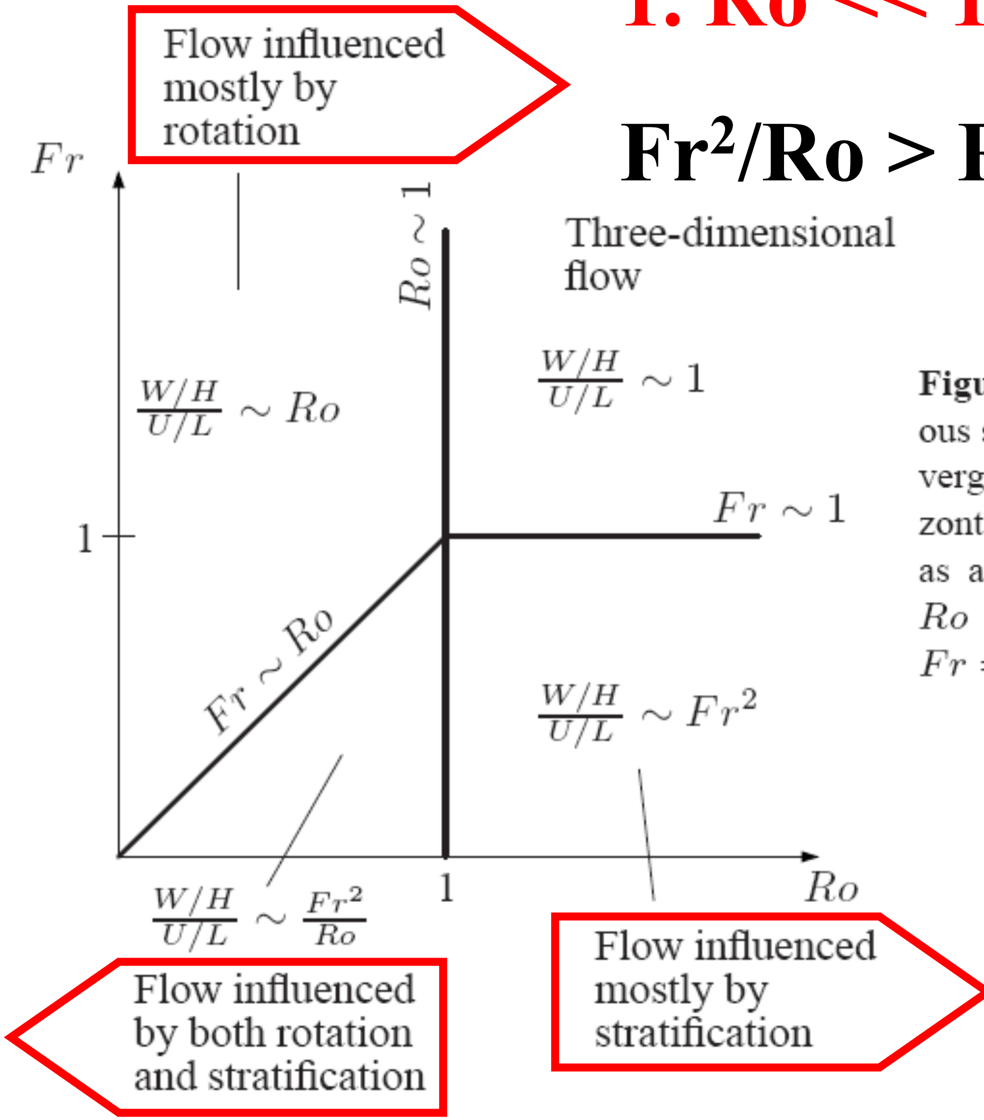

Figure 11-6 Recapitulation of the various scalings of the ratio of vertical convergence (divergence), W/H, to horizontal divergence (convergence), U/L, as a function of the Rossby number, $Ro = U/(\Omega L)$, and Froude number, $Fr = U/(NH)$.

2. $Fr \ll 1$

$Fr^2/Ro < Ro$

$$\frac{vert.\,conv.}{horiz.\,div.} = \frac{W/H}{U/L} = \begin{cases} Fr^2 \rightarrow strat.\,dominates \\ Fr^2/Ro, \rightarrow strat. + rot. \\ Ro, \rightarrow rot.\,dominates \end{cases}$$

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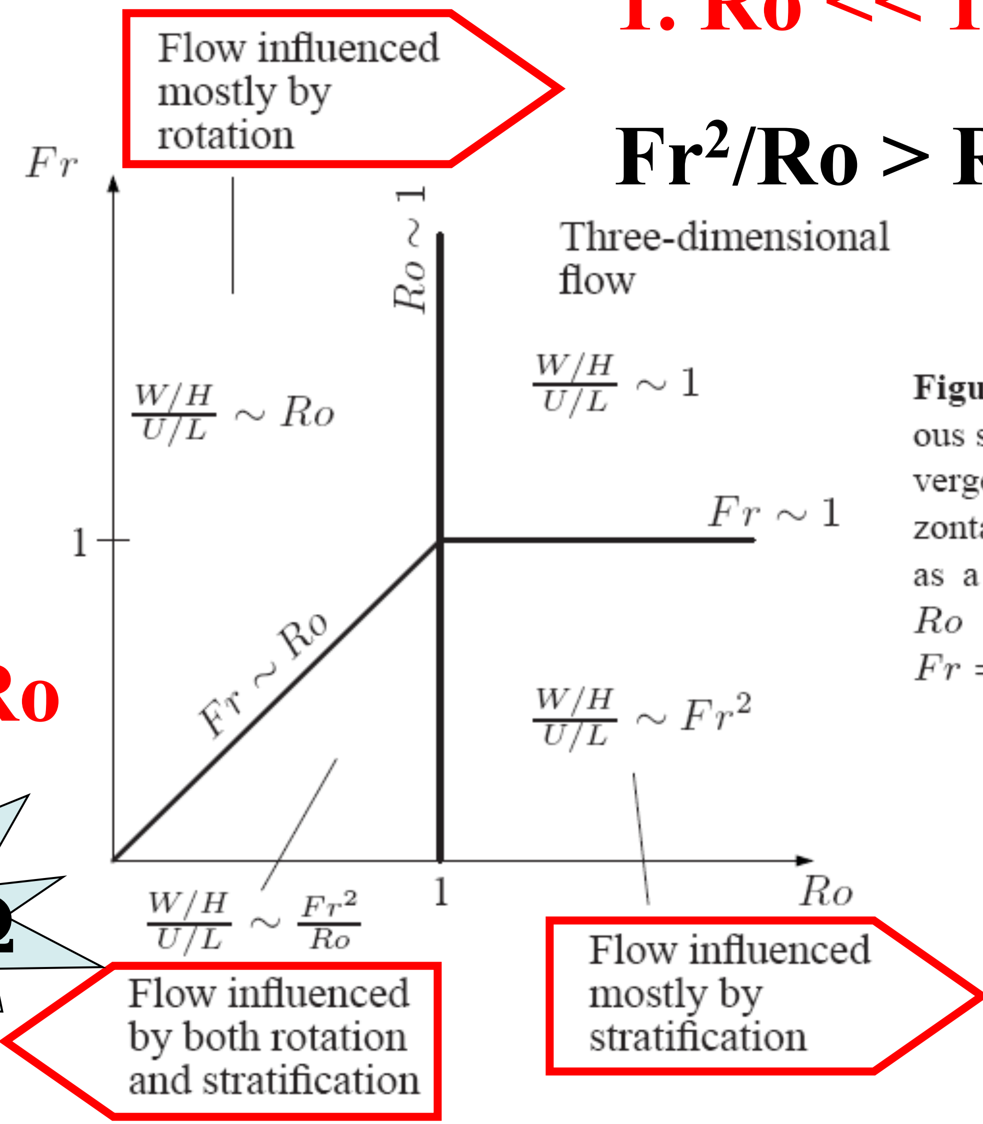

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3. $Fr^2 \lesssim Ro$

$L = NH/\Omega$

2. $Fr \ll 1$

$Fr^2/Ro < Ro$

$$Bu = \left(\frac{NH}{\Omega L}\right)^2 = \left(\frac{Ro}{Fr}\right)^2$$ **Burger n.: if Bu=1 => L =NH/Ω**

2-layer stratified fluid + shear

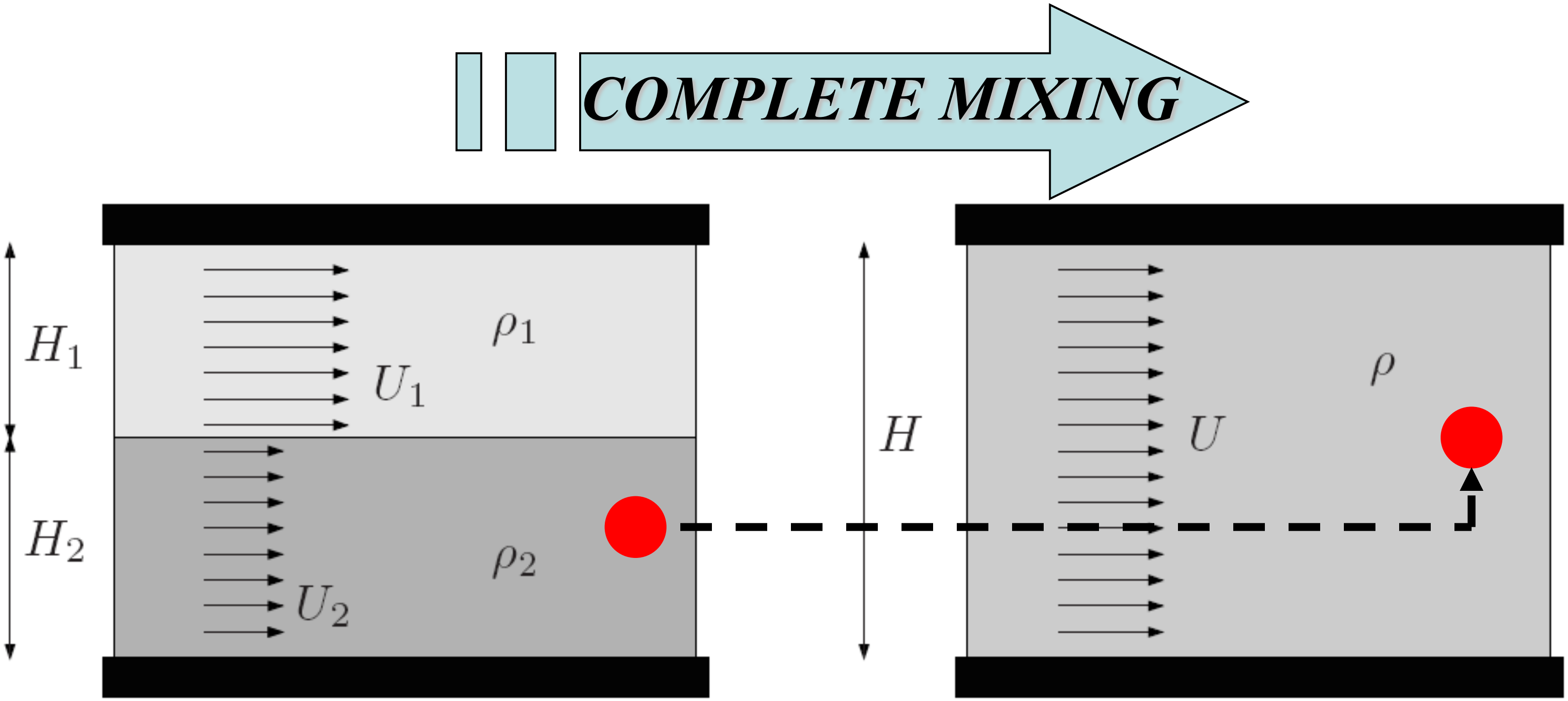

Figure 14-1 Mixing of a two-layer stratified fluid with velocity shear. Rising of dense fluid and lowering of light fluid both require work against buoyancy forces and thus lead to an increase in potential energy. Concomitantly, the kinetic energy of the system decreases during mixing. Only when the kinetic-energy drop exceeds the potential-energy rise can mixing proceed spontaneously.

● = center of gravity

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$$\begin{aligned} PE \text{ gain} &= \int_0^H \rho_{\text{final}}\, gz\, dz - \int_0^H \rho_{\text{initial}}\, gz\, dz \\ &= \frac{1}{2}\rho g H^2 - \left[\frac{1}{2}\rho_2 g \frac{H^2}{4} + \frac{1}{2}\rho_1 g \frac{3H^2}{4}\right] \\ &= \frac{1}{8}(\rho_2 - \rho_1) g H^2. \end{aligned}$$

$$\begin{aligned} KE \text{ loss} &= \int_0^H \frac{1}{2}\rho_0 u_{\text{initial}}^2\, dz - \int_0^H \frac{1}{2}\rho_0 u_{\text{final}}^2\, dz \\ &= \frac{1}{2}\rho_0 U_2^2 \frac{H}{2} + \frac{1}{2}\rho_0 U_1^2 \frac{H}{2} - \frac{1}{2}\rho_0 U^2 H \\ &= \frac{1}{8}\rho_0 (U_1 - U_2)^2 H. \end{aligned}$$

COMPLETE VERTICAL MIXING is naturally possible when $KE_{loss} > PE_{gain}$:

$$\frac{(\rho_2 - \rho_1) g H}{\rho_0 (U_1 - U_2)^2} < 1$$

…meaning that the density variation barrier is not too strong or the shear is enough large to supply the necessary amount of energy to overcome the stratification

In the opposite case ($KE_{loss} < PE_{gain}$) we have LOCALIZED MIXING occurring only near the initial interface, not extending to the entire system…

We know (Kundu, 1990) that a sinusoidal perturbation of wavenumber k is unstable if

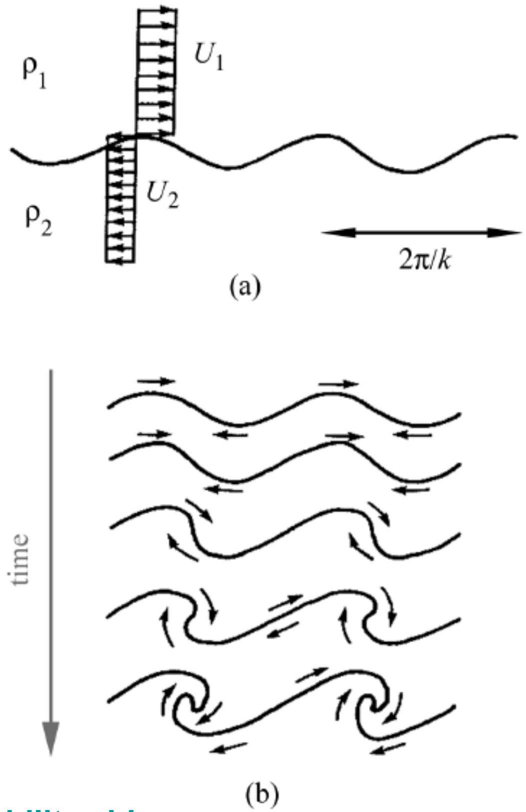
$$(\rho_2^2 - \rho_1^2)g < \rho_1\rho_2 k\,(U_1 - U_2)^2,$$

or for a Boussinesq fluid ($\rho_1 \simeq \rho_2 \simeq \rho_0$),

$$2(\rho_2 - \rho_1)g < \rho_0 k\,(U_1 - U_2)^2.$$

In a stability analysis, waves of all wavenumbers must be considered

$\Rightarrow \forall k\ \exists$ *at least a* $\tilde{k}$ *short enough to cause instability*



a 2-layer shear flow is always unstable

(localized mixing)

Figure 14-2 Kelvin–Helmholtz instability: (a) initial perturbation of wavenumber k, (b) temporal evolution of an unstable perturbation. The system is always unstable to short waves, which steepen, overturn and ultimately cause mixing. As waves overturn, their vertical and lateral dimensions are comparable.

Kelvin-Helmoltz instability video

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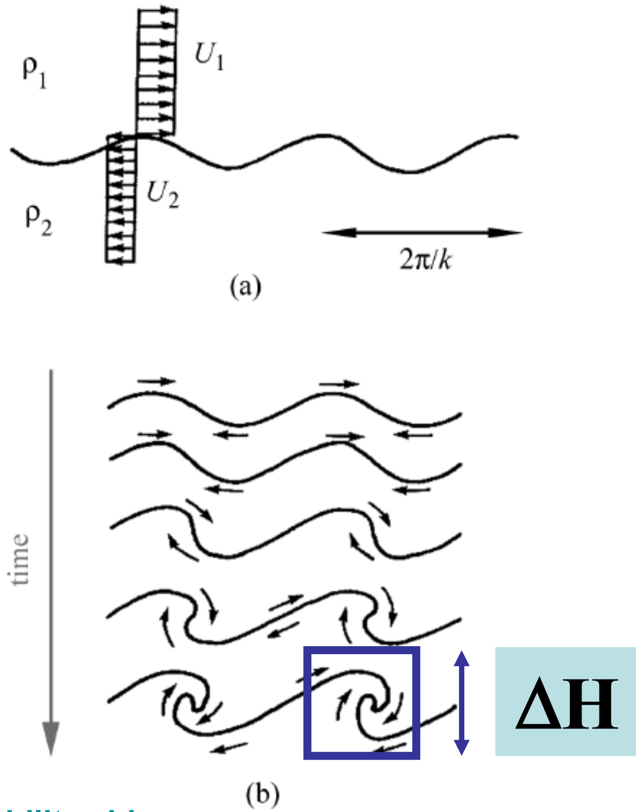

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Kelvin-Helmoltz instability video

Interfacial unstable waves grow and form ROLLS of height comparable to their width

ΔH = mixing zone

$$\Delta H \propto \lambda_{max} = \frac{2\pi}{k_{min}} \sim \frac{\rho_0 (U_1 - U_2)^2}{2(\rho_2 - \rho_1) g}$$

…link with H from $KE_{loss} < PE_{gain}$

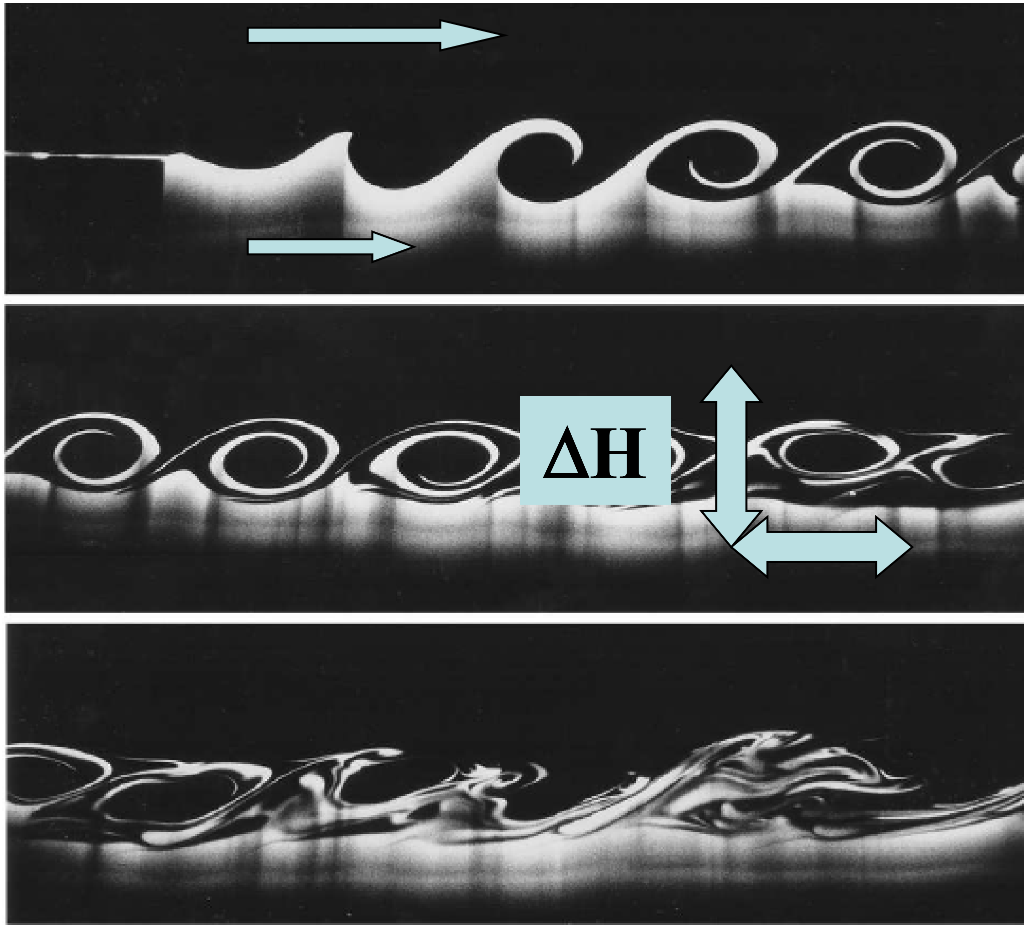

Figure 14-3 Development of a Kelvin–Helmholtz instability in the laboratory. Here, two layers flowing from left to right join downstream of a thin plate (visible on the left of the top photograph). The upper and faster moving layer is slightly less dense than the lower layer. Downstream distance (from left to right on each photograph and from top to bottom panel) plays the role of time. At first, waves form and overturn in a two-dimensional fashion (in the vertical plane of the photo) but, eventually, three-dimensional motions appear that lead to turbulence and complete the mixing. (Courtesy of Greg A. Lawrence. For more details on the laboratory experiment, see Lawrence *et al.*, 1991.)

ROLLING + BREAKING = TURBULENT MIXING

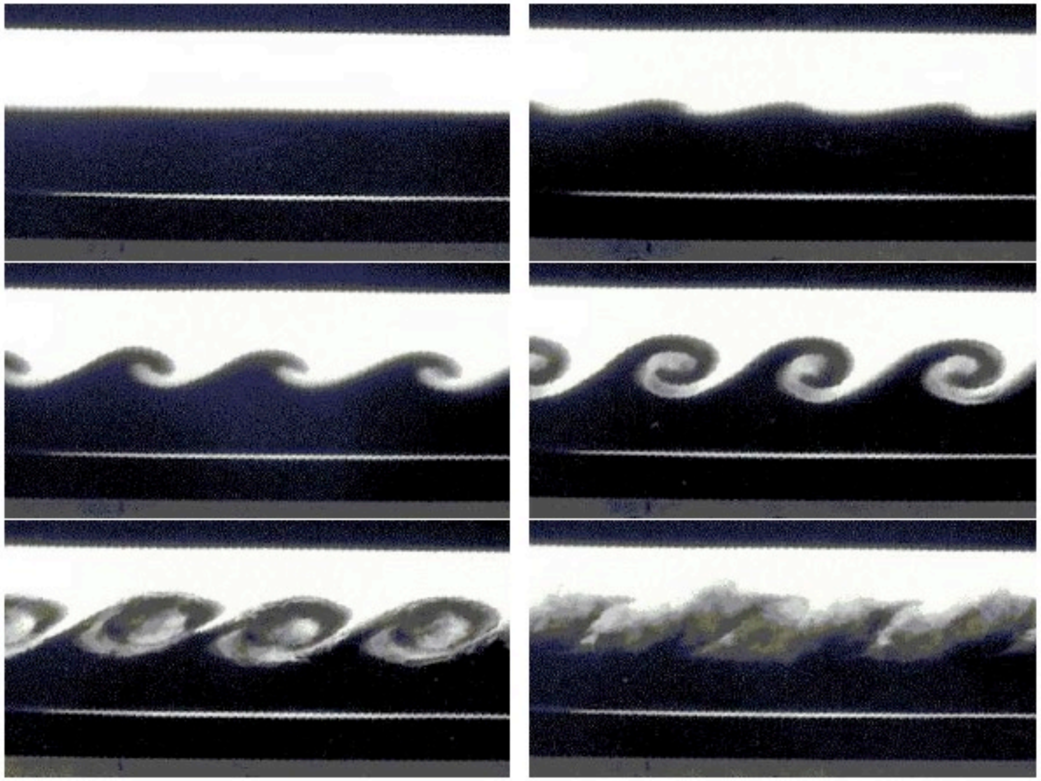

Figure 14-4 Kelvin–Helmholtz instability generated in a laboratory with fluids of two different densities and colours. (*Adapted from GFD-online, Satoshi Sakai, Isawo Iizawa, Eiji Aramaki*)

Figure 14-5 Kelvin–Helmholtz instability in the Algerian sky. (*Photo by the second author*)

Kelvin-Helmoltz instability

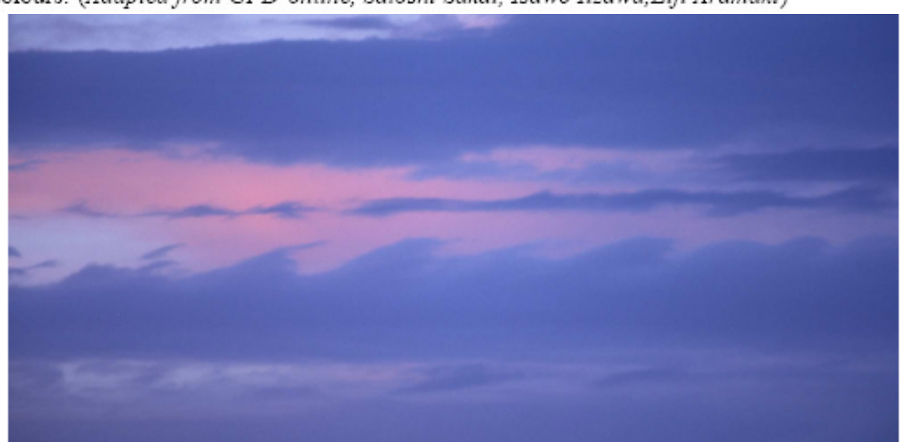

Figure 14-6 Kelvin–Helmholtz instability of the Sahara desert. (*Photo by the second author*)

Instability of a stratified shear flow

Q: For a given density stratification (N^2), what is the critical velocity shear for the instability → mixing?

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Consider a 2-d (x,z) inviscid, non-rotating, non-diffusive fluid with velocity (u,w), dynamic pressure p and density anomaly ρ

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} &= 0.
\end{aligned}
$$

Basic state + small Perturbation and Linearization

Our basic state consists of a steady, sheared horizontal flow $[u = \bar{u}(z), w = 0]$ in a vertical density stratification $[\rho = \bar{\rho}(z)]$. The accompanying pressure field $\bar{p}(z)$ obeys $d\bar{p}/dz = -g\bar{\rho}(z)$. The addition of an infinitesimally small perturbation ($u = \bar{u} + u'$, $w = w'$, $p = \bar{p} + p'$, $\rho = \bar{\rho} + \rho'$) and a subsequent linearization of the equations yield: → $O(2) \to 0$

1) $$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + w'\frac{d\bar{u}}{dz} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

2) $$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0}$$

3) $$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

4) $$\frac{\partial \rho'}{\partial t} + \bar{u}\frac{\partial \rho'}{\partial x} + w'\frac{d\bar{\rho}}{dz} = 0.$$

def. Perturbation streamfunction ψ : $u' = +\partial\psi/\partial z,\ w' = -\partial\psi/\partial x$

Basic state + small Perturbation and Linearization

Hypotheses:

- a linear density vertical profile: $\boldsymbol{N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} = cost}$
- the horizontal and temporal evolution of the perturbations $u'w'\rho'\psi'$ are formally harmonic functions in (x,t), propagating with $\lambda_x = 2\pi/l$ and having a Fourier-like wave structure $\boldsymbol{\sim e^{il(x-ct)}}$
- substituting in Eq. 4 we obtain the density perturbation $\rho' = \frac{-N^2\rho_0}{g(\bar{u}-c)\psi}$
- [*after some maths*] we obtain the Taylor-Goldstein equation governing the vertical structure of a perturbation $\boldsymbol{\psi' = \psi(z)e^{il(x-ct)}}$ in a stratified shear flow:

$$(\bar{u}-c)\left(\frac{d^2\psi}{dz^2} - k^2\psi\right) + \left(\frac{N^2}{\bar{u}-c} - \frac{d^2\bar{u}}{dz^2}\right)\psi = 0$$

- with the boundary conditions $\mathrm{w}'(0) = \mathrm{w}'(\mathrm{H}) = 0 \Rightarrow \psi(0) = \psi(H) = 0$ in a domain vertically bounded by two horizontal planes at $z = 0, H$ we obtain an eigenvalue problem which in general may have complex eigenvalues $c = c_r + ic_i$ and $c^* = c_r - ic_i$

Basic state + small Perturbation and Linearization

- from $c = c_r + ic_i$ and $c^* = c_r - ic_i \Rightarrow \psi' \sim e^{il[x-(c_r \pm ic_i)t]} \sim e^{ilx} e^{-ilc_r t} e^{\mp lc_i t}$
- real exponential: the presence of $c_i \neq 0 \Rightarrow \exists$ at least one unstable mode
- the flow is stable $\Leftrightarrow c_i = 0$
- using integral constraints we can analyze the T.-G. eq.: with $\psi = \sqrt{\bar{u} - c}\,\phi$.

$$\frac{d}{dz}\left[(\bar{u} - c)\frac{d\phi}{dz}\right] - \left[k^2(\bar{u} - c) + \frac{1}{2}\frac{d^2\bar{u}}{dz^2} + \frac{1}{\bar{u} - c}\left(\frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 - N^2\right)\right]\phi = 0$$

$$\phi(0) = \phi(H) = 0.$$

- [*after some math and using the BCs*] we obtain a complex equation:

$$\int_0^H \left[N^2 - \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2\right]\frac{|\phi|^2}{\bar{u} - c}\,dz = \int_0^H (\bar{u} - c)\left(\left|\frac{d\phi}{dz}\right|^2 + k^2|\phi|^2\right)dz + \frac{1}{2}\int_0^H \frac{d^2\bar{u}}{dz^2}|\phi|^2\,dz$$

- whose imaginary part is $c_i \int_0^H \left[N^2 - \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2\right]\frac{|\phi|^2}{|\bar{u} - c|^2}\,dz = -\,c_i \int_0^H \left(\left|\frac{d\phi}{dz}\right|^2 + k^2|\phi|^2\right)dz,$

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- IF $N^2 > \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 \Rightarrow c_i \cdot (> 0) = -c_i \cdot (> 0) \Rightarrow c_i = 0 \Rightarrow$ STABLE
- IF $N^2 < \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 \Rightarrow c_i \cdot (< 0) = -c_i \cdot (> 0) \Rightarrow \forall c_i \Rightarrow$ STABLE or UNSTABLE

Q: For a given density stratification (N^2), what is the critical velocity shear for the instability → mixing?

- [from $c = c_r + ic_i$ and $c^* = c_r - ic_i \Rightarrow \psi' \sim e^{il[x-(c_r \pm ic_i)t]} \sim e^{ilx} e^{-ilc_r t} e^{\mp lc_i t}$]
- def. Richardson number $Ri = \frac{N^2}{\left(\frac{d\bar{u}}{dz}\right)^2} \rightarrow \begin{cases} IF\ Ri > \frac{1}{4} \Rightarrow STABLE \\ IF\ Ri < \frac{1}{4} \Rightarrow STABLE\ or\ UNSTABLE \end{cases}$
- *sufficient condition* for stability is $N^2 > \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 : Ri > \frac{1}{4} \ldots (\Rightarrow STABLE)$
- *necessary condition* for instability is $N^2 < \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2 : Ri < \frac{1}{4} \ldots (\Leftarrow UNSTABLE)$
- *But measurements in atmosphere / ocean / laboratory indicate that* $Ri < 1/4$ *is a reliable condition of instability* $\Rightarrow \left(\frac{d\bar{u}}{dz}\right)^2 > 4N^2$
- if the shear flow has linear variations of velocity and density we may refer to the 2-layer flow case, finding a similarity with $KE_{loss} > PE_{gain}$ (complete vertical mixing) $\Rightarrow Ri$ is the ratio between PE and KE !!!
- $Ri = \frac{potential\ energy\ barrier\ that\ mixing - if\ occurring - must\ overcome}{kinetic\ energy\ available\ in\ the\ shear\ flow}$

Q: can we say something more about the properties of the growing perturbation?

- if we introduce the vertical displacement a caused by the small wave perturbation: $\frac{\partial a}{\partial t} + \bar{u}\frac{\partial a}{\partial x} = w$ which corresponds to $(\bar{u} - c)\, a = -\psi.$

$w' = -\partial\psi/\partial x$

- we can rewrite the T.-G. eq. obtaining an equivalent problem for a :

$$\frac{d}{dz}\left[(\bar{u} - c)^2 \frac{da}{dz}\right] + \left[N^2 - k^2(\bar{u} - c)^2\right] a = 0$$
$$a(0) = a(H) = 0.$$

- [*after some math and using the BCs*]: $\int_0^H (\bar{u} - c)^2 P\, dz = \int_0^H N^2 |a|^2\, dz$

- through the analysis of the real and the imaginary parts of the integral and requiring the instability condition $c_i \neq 0$:
  - $U_{\min} < c_r < U_{\max}$ the growing perturbation travels with the flow at some intermediate speed
  - $\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2$ in the complex plane, $c = c_r + ic_i$ must lie within the circle with the range of $\bar{u}$ as the diameter on the real axis

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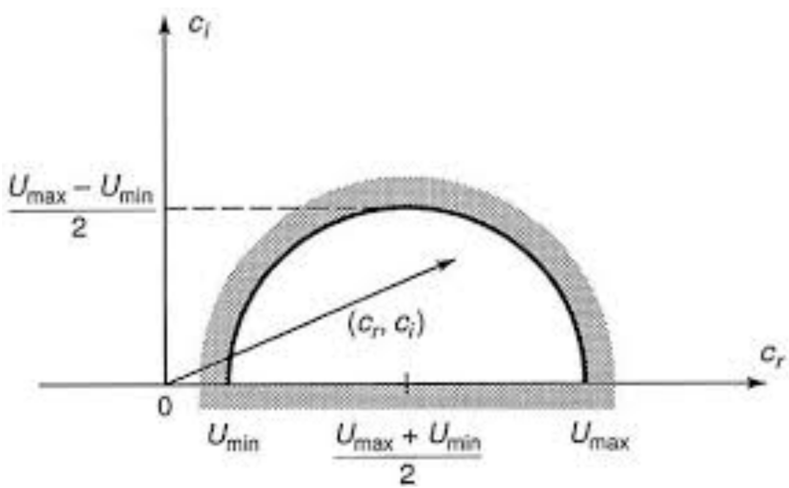
  - $\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2$ in the complex plane (c_r, c_i): $c = c_r + ic_i$ must lie within the circle with the range of $\bar{u}$ as the diameter on the real axis
- since instability requires $c_i > 0 \Rightarrow \psi' \sim e^{ilx} e^{-ilc_r t} e^{lc_i t}$ we are interested in the upper part of the circle $\Rightarrow$ Howard semicircle theorem
- $c_i \leq \frac{U_{max} - U_{min}}{2} \Rightarrow lc_i \leq \frac{l}{2}(U_{max} - U_{min}) \Rightarrow$ the perturbation does not grow to infinite



https://www.dartmouth.edu/~cushman/books/EFM/chap5.pdf