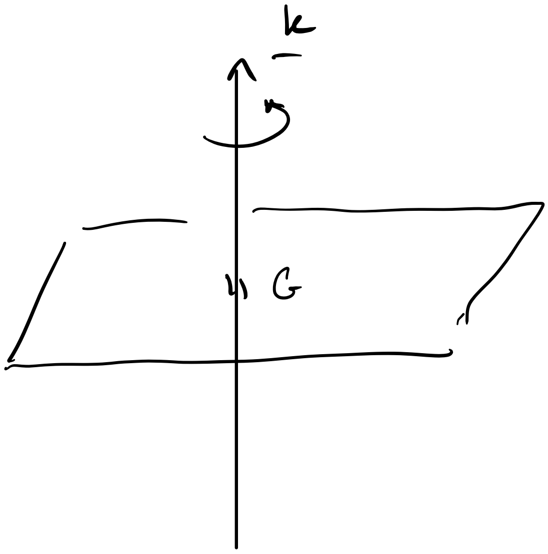
MECCANICA RAZIONALE

Eq di Lagrange → problemi di dinamica

Moti di rigidi 3D → ECD



$\downarrow g$

Lamina rettangolare omogenea

cerniera cilindrica in G

Vincoli lisci

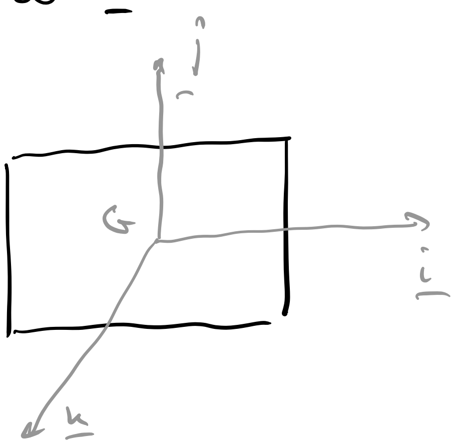
ECD :

$$\underline{R}^e = M\underline{g} + \underline{F}_G^r = \dot{\underline{P}} = 0$$

$$\underline{M}^e(G) = \underline{\mu}_G^r = \frac{d}{dt}\underline{L}(G)$$

momento di reazione

$\underline{\omega} = \omega\,\underline{k}$



$$\underline{L}(G) = I_G(\underline{\omega})$$

$$= \omega J_3\,\underline{k}$$

$$J_3 = \frac{M}{12}(a^2+b^2)$$

$$\frac{d}{dt}\underline{L}(G) = \dot\omega J_3 \underline{k}$$

ECD: momenti lungo $\underline{k}$

$$J_3\,\dot\omega = 0 \longrightarrow \omega(t) = \omega(t_0) = \omega_0$$

rotazione uniforme intorno a $\underline{k}$

Momenti lungo $\underline{i}, \underline{j} = \underline{\mu}^2 = \underline{0}$

Risultante: $\underline{F}_G^2 = -M\underline{g}$

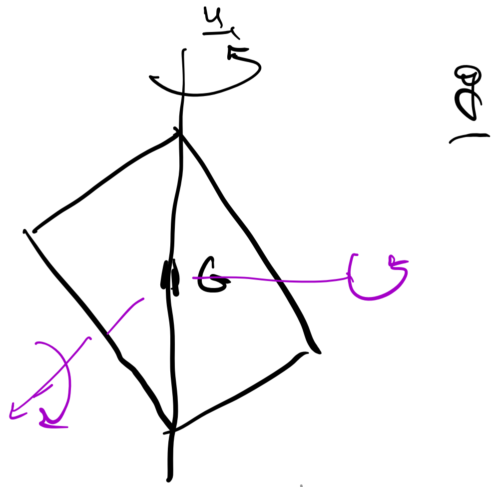
Attrito $\longrightarrow$ momento $-\nu\,\underline{\omega}$, $\nu > 0$

Allora l'eq. dei momenti in direzione $\underline{k}$ $\Rightarrow$ $\underline{J_3\,\dot\omega = -\nu\,\omega}$

$$\dot\omega = -\frac{\nu}{J_3}\,\omega \Rightarrow \omega(t) = \omega_0\, e^{-\frac{\nu}{J_3}t}$$

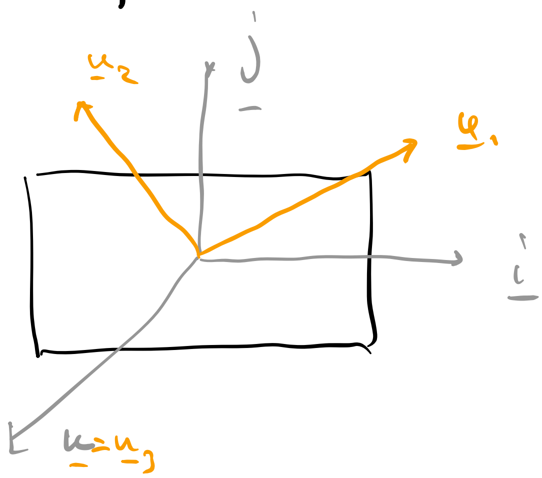
$\rightarrow$ la rotazione rallenta

Cambiamo la posizione della lamina:



l'asse di rotazione coincide con la diagonale

$$\underline{\omega} = \omega \, \underline{u}_1$$



$$S(G; \underline{u}_1, \underline{u}_2, \underline{u}_3 = \underline{k})$$

$\underline{i}, \underline{j}, \underline{k}$ Terna principale

$$I_G = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

$$\underline{L}(G) = I_G(\underline{\omega}) = \omega \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \overset{\underline{u}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$= \omega \left(I_{11} \underline{u}_1 + I_{12} \underline{u}_2 \right) = \omega \begin{pmatrix} I_{11} \\ I_{12} \\ 0 \end{pmatrix}$$

$$\frac{d}{dt} \underline{L}(G) = I_G(\dot{\underline{\omega}}) + \underline{\omega} \wedge I_G(\underline{\omega})$$

$$\omega \underline{u}_1 \wedge \omega \left(I_{11} \underline{u}_1 + I_{12} \underline{u}_2 \right)$$

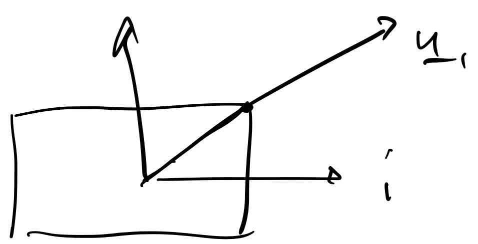
$$= \dot{\omega}\left(I_{11}\,\underline{u}_1 + I_{12}\,\underline{u}_2 \right) + \omega^2 I_{12}\,\underline{u}_3$$

$$\text{ECD} \longrightarrow \begin{cases} I_{11}\,\dot{\omega} = 0 & \underline{u}_1 \\ \underline{M}_G^e \cdot \underline{u}_2 = I_{12}\,\dot{\omega} & \underline{u}_2 \\ \underline{M}_G^e \cdot \underline{u}_3 = I_{12}\,\omega^2 & \underline{u}_3 \end{cases}$$

↑ momenti deviatori

passiamo da $\underline{u}_1, \underline{u}_2$ a $\underline{i}, \underline{j}$

$$\underline{u}_1 = \frac{a\,\underline{i} + b\,\underline{j}}{\sqrt{a^2+b^2}} \qquad \underline{u}_2 = \frac{-b\,\underline{i} + a\,\underline{j}}{\sqrt{a^2+b^2}}$$



$\underline{i}, \underline{j}, \underline{k}$

$$I_G = \begin{pmatrix} \frac{M}{12} b^2 & & \\ & \frac{M}{12} a^2 & \\ & & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$

Invece in $\underline{u}_1, \underline{u}_2$

$$I_G^{\underline{u}_1, \underline{u}_2, \underline{k}} \longrightarrow \begin{pmatrix} I_{11} & I_{12} & \cdot \\ I_{12} & I_{22} & \cdot \\ \cdot & \cdot & I_{33} \end{pmatrix}$$

$$I_{11} = \underline{u}_1 \cdot I_G(\underline{u}_1)$$

$$= \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0\right) \begin{pmatrix} \frac{Mb^2}{12} & & \\ & \frac{Ma^2}{12} & \\ & & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix}$$

$$= \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0\right) \begin{pmatrix} \frac{Mb^2}{12} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{Ma^2}{12} \frac{b}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix}$$

$$= \frac{M}{12} \frac{a^2b^2}{a^2+b^2} + \frac{M}{12} \frac{a^2b^2}{a^2+b^2} = \frac{M}{6} \frac{a^2b^2}{a^2+b^2}$$

$$I_{22} = \cdots$$

$$I_{12} = \underline{u}_1 \cdot I_G(\underline{u}_2) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{b^2+a^2}}, 0\right) \cdot$$

$$\begin{pmatrix} \frac{Mb^2}{12} & & \\ & \frac{Ma^2}{12} & \\ & & \frac{M}{12}(a^2+b^2) \end{pmatrix} \begin{pmatrix} \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix}$$

$$= \frac{1}{a^2+b^2} \begin{pmatrix} a, & b, & 0 \end{pmatrix} \begin{pmatrix} M \frac{b^2}{12} (-b) \\ M \frac{a^2}{12} a \\ 0 \end{pmatrix} =$$

$$= \frac{M}{12} ab \frac{a^2 - b^2}{a^2+b^2} \rightarrow I_{12}$$

$= 0$ se $a = b$

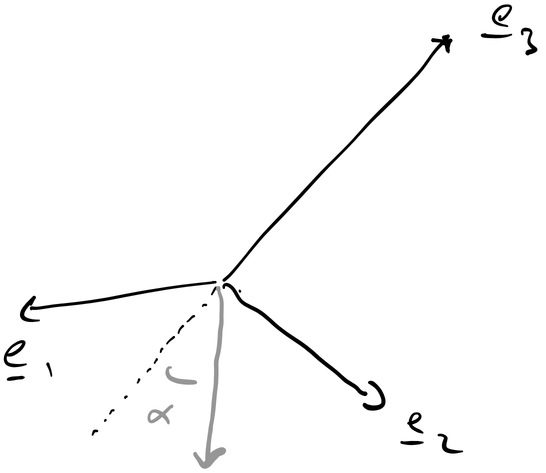
Consideriamo un rigido soggetto al proprio peso, vincolato a ruotare intorno ad un asse obliquo rispetto alla verticale

Terna fissa

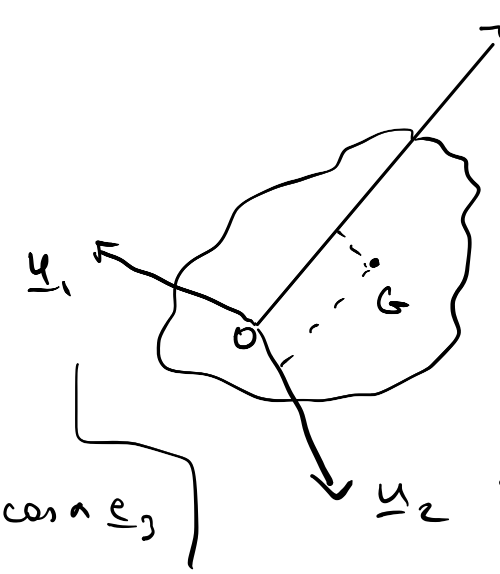
$\Sigma(O, \underline{e}_1, \underline{e}_2, \underline{e}_3)$

Terna solidale

$S(O; \underline{u}_1, \underline{u}_2, \underline{u}_3)$



$\underline{g} = g \sin\alpha\, \underline{e}_2 - g \cos\alpha\, \underline{e}_3$



$G \notin$ asse di rotazione

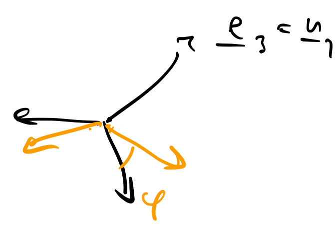
$\underline{x}_G - \underline{x}_O =$

$= y_G \underline{u}_2 + z_G \underline{u}_3$

Equazioni cardinali della dinamica

$$\begin{cases} M\underline{g} + \underline{F}_o^{2} = \dot{\underline{P}} \\ \\ (\underline{x}_G - \underline{x}_o) \wedge M\underline{g} + \underline{\mu}_o^{2} = \dfrac{d}{dt}\underline{L}(O) \end{cases}$$

Momento del peso (rispetto ad O)



$$(\underline{g})_b = R^T(\varphi)\,(\underline{g})_\Sigma =$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ g\sin\alpha \\ -g\cos\alpha \end{pmatrix}$$

$$= g \begin{pmatrix} \sin\alpha\sin\varphi \\ \cos\varphi\sin\alpha \\ -\cos\alpha \end{pmatrix} =$$

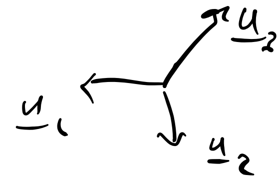
$$= g\left(\sin\alpha\sin\varphi\,\underline{u}_1 + \cos\varphi\sin\alpha\,\underline{u}_2 - \cos\alpha\,\underline{u}_3\right)$$

Momento del peso:

$$(\underline{x}_G - \underline{x}_o) \wedge M\underline{g} =$$

$$= \left(y_G \, \underline{u}_2 - z_G \, \underline{u}_3 \right) \wedge Mg \left(\sin\alpha \sin\varphi \, \underline{u}_1 + \right.$$

$$\left. + \cos\varphi \sin\alpha \, \underline{u}_2 - \cos\alpha \, \underline{u}_3 \right)$$



$$= Mg \left[y_G \sin\alpha \sin\varphi \underbrace{\underline{u}_2 \wedge \underline{u}_1}_{-\underline{u}_3} - y_G \cos\alpha \underbrace{\underline{u}_2 \wedge \underline{u}_3}_{\underline{u}_1} \right.$$

$$\left. - z_G \sin\alpha \sin\varphi \underbrace{\underline{u}_3 \wedge \underline{u}_1}_{\underline{u}_2} - z_G \cos\varphi \sin\alpha \underbrace{\underline{u}_3 \wedge \underline{u}_2}_{-\underline{u}_1} \right]$$

$$= Mg \left[\left(- y_G \cos\alpha + z_G \cos\varphi \sin\alpha \right) \underline{u}_1 + \right.$$

$$\left. \left(- z_G \sin\alpha \sin\varphi \right) \underline{u}_2 - y_G \sin\alpha \sin\varphi \, \underline{u}_3 \right]$$

Calcoliamo $\frac{d}{dt} \underline{L}(O)$

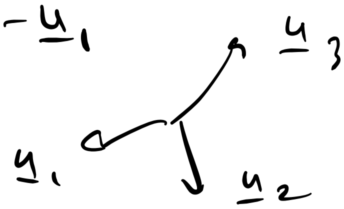
$$\underline{L}(O) = I_O(\omega \, \underline{u}_3)$$

$$\frac{d}{dt} \underline{L}(O) = \dot{\omega} \, I_O(\underline{u}_3) + \omega^2 \underbrace{\underline{u}_3 \wedge I_O(\underline{u}_3)}_{\underline{\omega} \wedge I_O(\underline{\omega})}$$

$$I_O(\underline{u}_3) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} I_{13} \\ I_{23} \\ I_{33} \end{pmatrix}$$

$$\underline{u}_3 \wedge I_O(\underline{u}_3) = \underline{u}_3 \wedge \left(I_{13}\underline{u}_1 + I_{23}\underline{u}_2 + I_{33}\underline{u}_3 \right)$$

$$= I_{13} \underbrace{\underline{u}_3 \wedge \underline{u}_1}_{\underline{u}_2} + I_{23} \underbrace{\underline{u}_3 \wedge \underline{u}_2}_{-\underline{u}_1}$$



$$= I_{13}\underline{u}_2 - I_{23}\underline{u}_1$$

$$\frac{d}{dt}\underline{L}(O) = \dot{\omega}\left(I_{13}\underline{u}_1 + I_{23}\underline{u}_2 + I_{33}\underline{u}_3 \right) + \omega^2\left(-I_{23}\underline{u}_1 + I_{13}\underline{u}_2 \right)$$

$$= \underline{u}_1\left(I_{13}\dot{\omega} - I_{23}\omega^2 \right) + \underline{u}_2\left(I_{23}\dot{\omega} + I_{13}\omega^2 \right) + \underline{u}_3\,\dot{\omega}\, I_{33}$$

$$\omega = \dot{\varphi}$$

Stiamo calcolando

$$(\underline{x}_G - \underline{x}_O) \wedge M\underline{g} + \hat{\mu}_O = \frac{d}{dt}\underline{L}(O)$$

<u>lungo $\underline{u}_3$</u>

$$\Rightarrow \ddot{\varphi}\, I_{33} = -\left(M g\, y_G \sin\alpha\right) \underline{\sin\varphi}$$

equazione del moto

$\left(-\nu\dot{\varphi}\right)$ per attrito

<u>lungo $\underline{u}_1$</u>

$$\rightarrow \mu_0^2 \cdot \underline{u}_1 = I_{13}\,\ddot{\varphi} - I_{23}\,\dot{\varphi}^2 + Mg\left(y_G \cos\alpha - z_G \cos\varphi \sin\alpha\right)$$

<u>lungo u_2</u>

$$\rightarrow \mu_0^2 \cdot \underline{u}_2 = I_{23}\,\ddot{\varphi} + I_{13}\,\dot{\varphi}^2 + Mg \sin\alpha \sin\varphi$$

Se usiamo la conservazione dell'energia

$$E = K + V$$

$$K = \frac{1}{2}\,\underline{\omega} \cdot I_0(\underline{\omega}) = \frac{1}{2}\,\omega^2 I_{33}$$

$$V = -M\underline{g} \cdot (\underline{x}_G - \underline{x}_0)$$

$$E = \frac{1}{2}\,\omega^2 I_{33} - M\underline{g} \cdot (\underline{x}_G - \underline{x}_0)$$

$$= \frac{1}{2} I_{33} \dot{\varphi}^2 - M\underline{g} \cdot (\underline{x}_G - \underline{x}_0) = E|_{t=0}$$

da cui ricaviamo $\dot{\varphi}^2$ in funzione

di φ.

→ stimare i momenti di reazione

Sappiamo studiare il moto di un

rigido con asse fisso.

→ rigido con punto fisso e

rigido libero

→ fenomeni di precessione

Consideriamo rigido con punto fisso O

$$\begin{cases} \underline{M}^e(O) = \frac{d}{dt} \underline{L}(O) & \to \text{moto} \\ \underline{R}^e + \underline{F}_0^2 = \dot{\underline{P}} & \to \text{incognite di reazione} \end{cases}$$

Consideriamo un rigido libero

$$\begin{cases} \underline{R}^e = \dot{\underline{P}} = M \underline{a}_G \\ \underline{M}^e(G) = \frac{d}{dt} \underline{L}(G) \end{cases}$$

Se assumiamo $\underline{R}$ (dipende da G) e $\underline{M}$ (angoli di Eulero) dipendono solo dalle posizioni (e velocità G e ω velocità angolare)

$\underline{R}^e = M \underline{a}_G$: eq. del moto del centro di massa a cui applichiamo una forza $\underline{R}^e$

$\underline{M}^e_{(G)} = \frac{d}{dt} \underline{L}(G)$ equazione di precessione attorno a G, pensato fisso

$$\longrightarrow \quad \underline{M}^a_{(G)} = \frac{d}{dt} \underline{L}(G)$$

$$\underline{M}^e(O) = \frac{d}{dt} \underline{L}(O)$$

formalmente uguali