Cosmology 1

2021/2022
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First intermediate test
Topic: general relativity.
Deadline: April 14, 13:00.

This exercise is based on the first tests of 2020 and 2021, some results have already been obtained in the previous tests so their answer is known and working the solution out is considered as a preliminary work.

A human-made probe of mass m is in a circular orbit at radius R around a Schwartzschild black hole of mass M; as an example, this could be SgrA*, whose mass is $M = 4 \times 10^6$ M$_\odot$ ($M_\odot = 1.99 \times 10^{30}$ kg is the solar mass). Assume that $R = 8GM$, but be prepared to vary it. The probe sends a monochromatic, continuous electromagnetic signal, of frequency $\nu_{\rm em}$. This signal is received by a distant observer at $d_{\rm obs} = 100$ pc, at rest in the reference frame where the metric has the Schwartschild form; it is distant enough to ensure that the metric is Minkowski at its location, but near enough to be able to see the movement of the probe on the sky. Its angular coordinates are $\theta = \pi/2$ and $\varphi = 0$, so it lies in the same plane of the orbit.

The aim is to compute how the wavelength of the signal from the probe changes with its apparent position on the sky, and possibly with time.

(1) Call $\vec{U}$ the four-velocity of the probe, and $\omega = U^\varphi$ its angular frequency. Work out ω as a function of R. Check that the geodesic equation for the probe only tells us that the components of $\vec{U}$ are constant.

(2) Call now $\tilde{E}$ and $\tilde{L}$ the invariants of a photon that is emitted by the probe and received by the observer. Demonstrate that these two invariants can be interpreted respectively as $\tilde{E} = h\nu_{\rm obs}$ and $\tilde{L} = b\tilde{E}$, where b is the impact parameter of the photon. Write down all the components of the photon momentum f^α and of its one-form f_α.

(3) Now the original part. We need to relate the impact parameter b with the position ϕ of the probe when the photon is emitted, given that photon geodesics are bent by gravitational lensing. One possibility, that applies (approximately) to the part of the trajectory where $-\pi/2 < \phi < \pi/2$, is to compute the variation of the variation $\Delta\varphi$ of the φ coordinate of the photon along its null geodesic, from the emission to infinity, and equate it to $-\phi$ of the probe at emission. This way the photon gets to $\varphi = 0$ at infinity. This can be done by noticing that

$$\Delta\varphi = \int \frac{d\varphi}{d\lambda} d\lambda$$

where λ is an affine parameter of the null geodesic, and that $f^r = dr/d\lambda$ and $f^\varphi = d\varphi/d\lambda$. Changing the integration variable to r it is possible to find an integral that must be solved numerically. Check for what values of the impact parameter the integral is defined, and try to interpret this result. Find a way to visualize the result (impact parameter b against the angle at emission), comparing it with what would happen in a Newtonian orbit. It can be convenient to show results for other choices of R.

(4) Now compute the redshift of the probe radiation for the distant observer, as a function of the observed position of the probe. Assume that the observer is at a distance of 100 pc and express the impact parameter as an observed angle, in arcseconds. The redshift will contain both gravitational and Doppler contribution. Compare this relation with what you would obtain with a Newtonian orbit, and interpret the differences.

(5) In what way would this experiment confirm the validity of GR?

The calculation suggested above covers only the part of the orbit that goes toward the observer, do you have suggestions on how to extend it? for instance one might try to compute for what angles φ the probe is invisible because its photons fall into the black hole.

It is also possible to compute the redshift of the probe, and its position on the sky, as a function of redshift. To do this it is necessary to calculate the time taken by photons traveling along different geodesics to get to the observer. If the work above has required less time than expected, try to perform this further step.

As a general suggestion, it may be useful in the formulas to express lengths in units of the gravitational radius GM.

Solution

In what follows distances are expressed in units of the black hole's gravitational radius GM, so $a = r/GM$ refers to the coordinate r, $A = R/GM$ to the coordinate of the probe's orbit and $\beta = b/GM$ to the photon's impact parameter b.

(1) The four-velocity of a probe in circular orbit around a black hole has been worked out in the First Test of 2021, so we refer to its solution for a more extended discussion. If $\tilde{E}^{(p)}$ and $\tilde{L}^{(p)}$ are the probe's conserved quantities, then:

$$\tilde{L}^{(p)} = (GM)\frac{a}{\sqrt{a-3}}$$

$$\tilde{E}^{(p)} = \frac{(a-2)}{\sqrt{a(a-3)}}$$

The probe four-velocity results:

$$\vec{U} \to \left(\sqrt{\frac{a}{a-3}}, 0, 0, \frac{1}{GM}\frac{1}{a\sqrt{a-3}}\right)$$

We can express $\tilde{L}^{(p)}$ in terms of the probe's angular frequency $\omega = U^{\varphi}$:

$$\tilde{L}^{(p)} = R^2\omega$$

The Christoffel symbols for the Schwartzschild metric have been worked out in the 2019 test, where it was requested to obtain the geodesic equation for a radial orbit. For a circular orbit we have that $dr/d\tau = 0$, $d\vartheta/d\tau = 0$, and the 0 and φ components of the geodesic equations trivially vanish. The r component results:

$$\frac{d^2r}{d\tau^2} = -\Gamma^1_{00}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^1_{33}\left(\frac{d\varphi}{d\tau}\right)^2 =$$

$$= -\frac{1}{GM}\left(1-\frac{2}{a}\right)\frac{1}{a(a-3)} + \frac{1}{GM}\left(1-\frac{2}{a}\right)\frac{1}{a(a-3)} = 0$$

This means that the coordinates of $\vec{U}$ are constant, as it should be in an uniform circular motion in spherical coordinates.

(2) The trajectory of a photon around a black hole has been studied in the First Test of 2020, where it was argued that $\tilde{E} = h\nu_{\text{obs}}$, where $h\nu_{\text{obs}}$ is the photon wavelength observed at infinity, and $\tilde{L} = b\,h\nu_{\text{obs}}$, where b is the photon impact parameter. Then, asking that $f_\alpha f^\alpha = 0$, one obtains:

$$f^\alpha \to h\nu_{\text{obs}} \times \left(\frac{a}{a-2},\ \sqrt{1-\frac{\beta^2}{a^2}\left(1-\frac{2}{a}\right)},\ 0,\ \frac{b}{r^2}\right)$$

$$f_\alpha \to h\nu_{\text{obs}} \times \left(-1,\ \frac{a}{a-2}\sqrt{1-\frac{\beta^2}{a^2}\left(1-\frac{2}{a}\right)},\ 0,\ b\right)$$

(3) The probe emits photons of frequency ν_{em} when it is at the coordinate ϕ. If φ is the coordinate of the photon, it will be received by the distant

observer at $\varphi = 0$ if its coordinate changes by an amount $\Delta\varphi = -\phi$. The integral to obtain $\Delta\varphi$ can be rewritten as:

$$\Delta\varphi = \int_R^\infty \frac{d\varphi}{d\lambda}\frac{d\lambda}{dr}dr = \int_A^\infty \frac{\beta\, da}{a^2\sqrt{1-\frac{\beta^2}{a^2}\left(1-\frac{2}{a}\right)}}$$

The limitation of this formulation is that the radius of the geodesic must always increase, so the photon must always travel outwards. The integrand is not defined when the argument of its square root is negative, so one can obtain a condition for the existence of the integral:

$$b < b_{\rm min} = R\sqrt{\frac{a}{a-2}}$$

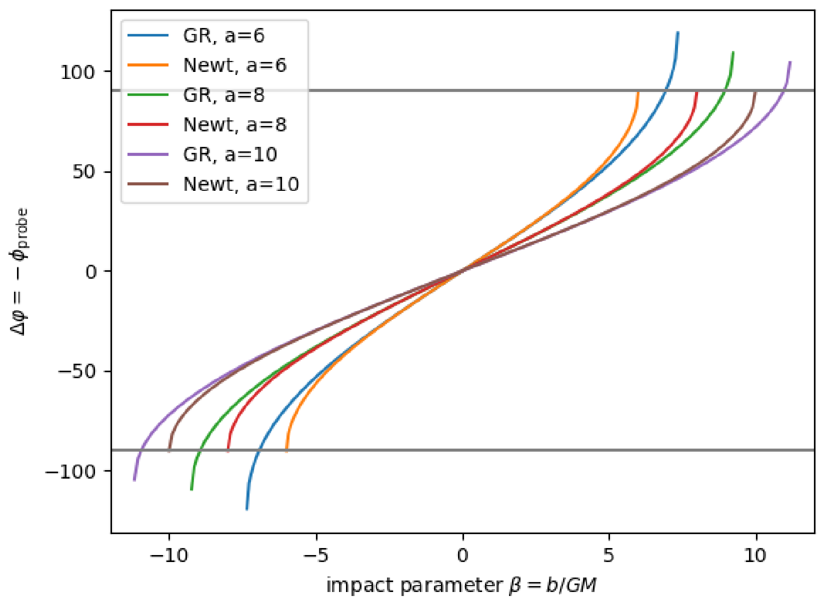
This corresponds to the impact parameter of a photon that starts tangential to the orbit's trajectory; it is impossible to have larger impact parameters (from the solution of this integral). Then $b_{\rm min}$ defines the angular extension of this part of the orbit as seen by a distant observed (see also the 2020 test).

The integral must be solved numerically, I have done it using the quadrature algorithm of the `scipy.integrate` library, using A and `np.inf` as integration intervals. Then I solved the integral for a series of b values from $-b_{\rm min}$ to $+b_{\rm min}$, because negative values of b correspond to negative values of the photon angular momentum.

The result can be compared with the corresponding Newtonian result, where the integral to solve reduces to:

$$\Delta\varphi = \int_A^\infty \frac{\beta\, da}{a^2\sqrt{1-\frac{\beta^2}{a^2}}}$$

that can should be solvable analytically (I have integrated it numerically). The figure below shows the resulting relation between angle $\Delta\varphi$ and impact parameter; angles are given in degrees and the gray lines show the ± 90 deg limits of the Newtonian case. We show the cases of $A = 6$ (last stable orbit), $A = 8$ and $A = 10$.



One can draw two conclusions from this result: (1) the apparent extension of the orbit, as seen by the distant observer, is larger in GR than in Newtonian theory, (2) thanks to gravitational lensing one can see a larger part of the orbit (using this integral). Both effects are stronger if the probe is nearer to the black hole.

(4) Because $\Delta\varphi = -\phi$, a negative angular momentum and a negative impact factor corresponds to a positive apparent position θ (the apparent angle of the probe with respect to the black hole position, positive in the direction of positive ϕ). To compute θ we must project a length $b = \beta GM$ to the distance between black hole and observer, $d_{\rm obs} = 100$ pc. I will express the angle in milliarcsec, the proportionality constant between β and θ results:

$$\theta = -\frac{GM}{c^2}\frac{1}{100\ \text{pc}}\beta = -0.394\beta\ \text{milliarcsec}$$

where the $-$ sign accounts is motivated above.

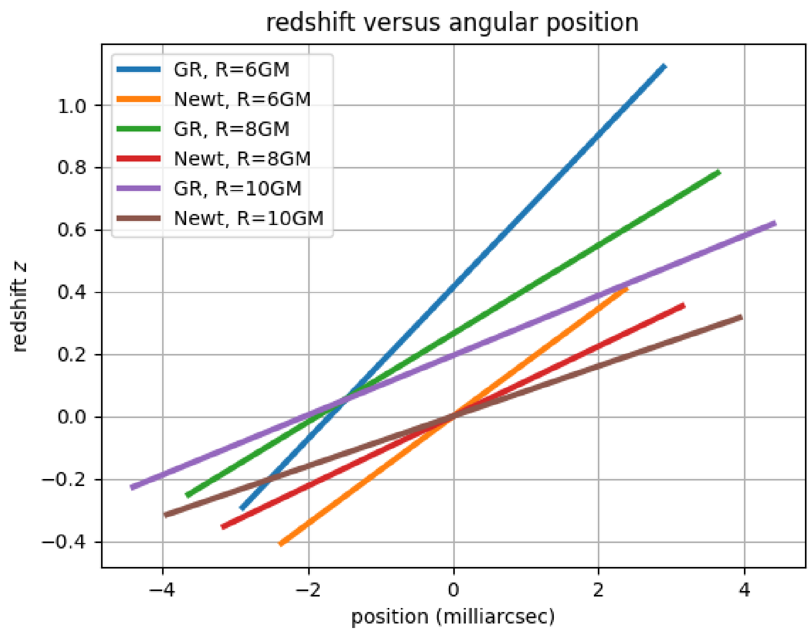
The redshift of the emitted radiation can be computed with the same principle used to compute the gravitational redshift of a probe standing at a radius r. The energy of the photon in the reference frame of the probe is:

$$h\nu_{\rm em} = -\vec{U}\cdot\vec{f} = h\nu_{\rm obs}\sqrt{\frac{a}{a-3}}\left(1 + \frac{\theta/0.394}{a^{3/2}}\right)$$

This can be used to compute the redshift of the probe:

$$z = \frac{h\nu_{\rm em}}{h\nu_{\rm obs}} - 1 = \sqrt{\frac{a}{a-3}}\left(1 + \frac{\theta/0.394}{a^{3/2}}\right) - 1$$

It is easy to generalize this result to the Newtonian and non-relativistic case, where the Doppler redshift is computed as $z = v/c = \sin\phi\sqrt{GM/Rc^2} = -(\theta/0.394)A^{-3/2}$. The figure below shows the comparison of the resulting relation between redshift and position on the sky.



The most notable difference is that, due to gravitational redshift, the redshift does not vanish when the probe is passing in front of the black hole. However, the Newtonian prediction lacks transverse Doppler effect that is significant given that the rotation speed, $v^2 = GM/R = 1/A$ is not much less than one. Another interesting property is that we need milli-arcsecond angular resolution to see the probe at the relatively moderate distance of 100 pc (compared with the distance of 8 kpc of the Sun from SgrA*).

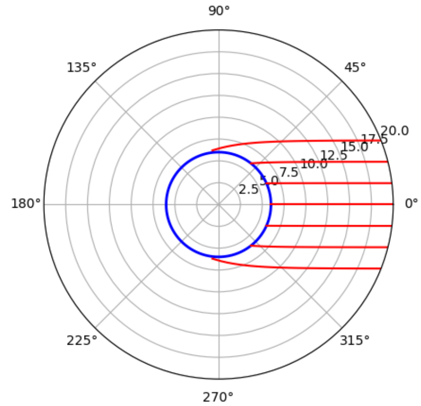
(5) There are of course many ways to answer the last question, but I think that the main "smoking gun" of GR is the presence of a redshift at $\theta = 0$ that is in excess of the transverse Doppler effect. Indeed, if $z_{\rm TD} = \gamma - 1$ (where TD stands for Transverse Doppler) then it's easy to show that for $1 + z_{\rm TD} = \sqrt{A/(A-1)}$, to be compared with $1 + z = \sqrt{A/(A-3)}$ valid for $\beta = 0$. This amounts to a redshift of $z = 0.265$ in place of $z_{\rm TD} = 0.069$ for $A = 8$.

The analysis can be extended further in several directions. As an example, one can plot the spatial part of the null geodesics to show the degree of light bending by the black hole curvature, by plotting

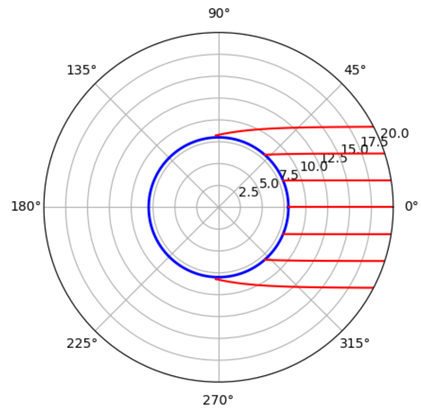
$$\varphi(r) = \int_R^r \frac{d\varphi}{d\lambda} d\lambda - \int_R^\infty \frac{d\varphi}{d\lambda} d\lambda$$

in polar coordinates. This is shown here for $R = 6$, 8 and $10GM$, where several light rays are reported.

R=6GM



R=8GM



R=10GM