Calcolo di e^{At}

utilizzando la

trasformata di Laplace

Dato il sistema LTI descritto da:

$$\begin{cases} \dot{x} = Ax + 3u \\ y = Cx \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$e^{At} = ?$$

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

$$tr(A) = 1 + 0 + 1 = 2 > 0 \Rightarrow \text{SISTEMA INSTABILE}$$

Tra i termini di e^{At} almeno uno deve $\to \infty$ quando $t \to +\infty$

$$(sI - A) = \begin{bmatrix} (s-1) & -1 & 0 \\ 0 & s & -2 \\ -1 & 0 & (s-1) \end{bmatrix} \longleftarrow$$

per calcolare il determinante per es. sviluppo lungo prima riga

percorso ⟹ scegliere altra riga o colonna per il calcolo del determinante

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \left[c_{ij} \right]^T$$

$$c_{ij} \longleftrightarrow (-1)^{i+j} \det\left[\hat{A}_{ij}\right]$$

compl. algebrici

Sviluppando il calcolo lungo la riga evidenziata

$$\det(sI - A) = (-1)^{1+1}(s-1)\begin{vmatrix} s & -2 \\ 0 & s-1 \end{vmatrix} +$$

$$+ (-1)^{1+2}(-1)\begin{vmatrix} 0 & -2 \\ -1 & s-1 \end{vmatrix} +$$

$$+ (-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} 0 & s \\ -1 & 0 \end{vmatrix}$$

sistema instabile

$$= s^3 - 2s^2 + s - 2 = (s-2)(s+j)(s-j)$$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} s & -2 \\ 0 & (s-1) \end{vmatrix} = s(s-1)$$

elemento della matrice dei complementi algebrici

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -2 \\ -1 & s-1 \end{vmatrix} = +2$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 0 & s \\ -1 & 0 \end{vmatrix} = s$$

$c_{21} = s-1 \qquad c_{22} = (s-1)^2 \qquad c_{23} = +1$

$c_{31} = +2 \qquad c_{32} = 2(s-1) \qquad c_{33} = s(s-1)$

$$(sI-A)^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s(s-1) & +2 & s \\ (s-1) & (s-1)^2 & 1 \\ 2 & 2(s-1) & s(s-1) \end{bmatrix}^T$$

$$= \frac{1}{(s-2)(s-j)(s+j)} \begin{bmatrix} s(s-1) & (s-1) & +2 \\ +2 & (s-1)^2 & 2(s-1) \\ s & +1 & s(s-1) \end{bmatrix}$$

Ora si tratta di antitrasformare tutti i termini

Ad esempio

$$[\quad]_{1,2} = \frac{s-1}{(s-2)(s^2+1)}$$

$\longrightarrow$ sviluppo in fratti semplici

$$\frac{c_1}{s-2} + \frac{c_2}{s-j} + \frac{c_2^*}{s+j}$$

$\swarrow$ completamento dei prodotti

$$\frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

per casa: provare entrambe le tecniche

Soluzione completa di

$$e^{At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
$t \cdot 1(t)$	$\frac{1}{s^2}$
$t^2 \cdot 1(t)$	$\frac{2}{s^3}$
$e^{\alpha\, t} \cdot 1(t)$	$\frac{1}{s-\alpha}$
$t \cdot e^{\alpha\, t} \cdot 1(t)$	$\frac{1}{(s-\alpha)^2}$
$\sin(\omega\, t) \cdot 1(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega\, t) \cdot 1(t)$	$\frac{s}{s^2+\omega^2}$
$t \cdot \sin(\omega\, t) \cdot 1(t)$	$\frac{2\,\omega\, s}{\left(s^2+\omega^2\right)^2}$
$t \cdot \cos(\omega\, t) \cdot 1(t)$	$\frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$
$e^{\sigma\, t} \cdot \sin(\omega\, t) \cdot 1(t)$	$\frac{\omega}{\left(s-\sigma\right)^2+\omega^2}$
$e^{\sigma\, t} \cdot \cos(\omega\, t) \cdot 1(t)$	$\frac{s-\sigma}{\left(s-\sigma\right)^2+\omega^2}$
$t \cdot e^{\sigma\, t} \cdot \sin(\omega\, t) \cdot 1(t)$	$\frac{2\,\omega\,(s-\sigma)}{\left[\left(s-\sigma\right)^2+\omega^2\right]^2}$
$t \cdot e^{\sigma\, t} \cdot \cos(\omega\, t) \cdot 1(t)$	$\frac{\left(s-\sigma\right)^2-\omega^2}{\left[\left(s-\sigma\right)^2+\omega^2\right]^2}$

Tabella 1: Segnali e corrispondenti trasformate di Laplace

$$(sI-A)^{-1} = \frac{1}{(s-2)(s-j)(s+j)} \begin{bmatrix} s(s-1) & (s-1) & +2 \\ +2 & (s-1)^2 & 2(s-1) \\ s & +1 & s(s-1) \end{bmatrix}$$

elemento $(1,1)$

$$\left[e^{At}\right]_{(1,1)} = \mathcal{L}^{-1}\left[\frac{s(s-1)}{(s-2)(s^2+1)}\right]$$

Sviluppo in fratti semplici

$$\frac{s(s-1)}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{B}{s-j} + \frac{B^*}{s+j}$$

$$A = \lim_{s \to 2} \frac{s(s-1)}{\cancel{(s-2)}(s^2+1)} \cancel{(s-2)} = 2/5$$

$$B = \lim_{s \to j} \frac{s(s-1)}{(s-2)\cancel{(s-j)}(s+j)} \cancel{(s-j)} = \frac{1}{10}(3-j)$$

$$\begin{cases} \dfrac{e^{jt}+e^{-jt}}{2} = \cos t \\ \dfrac{e^{jt}-e^{-jt}}{2j} = \operatorname{sen} t \end{cases}$$

$$B^* = \frac{1}{10}(3+j)$$

$$\mathcal{L}^{-1}\{\cdot\cdot\} = \left[\frac{2}{5}e^{2t} + \frac{1}{10}(3-j)e^{jt} + \frac{1}{10}(3+j)e^{-jt}\right]\cdot 1(t)$$

$$2 \cdot \frac{3}{10}\frac{(e^{jt}+e^{-jt})}{2} - \frac{j}{10}\frac{(e^{jt}-e^{-jt})}{2j}\cdot 2j$$

$$\left[e^{At}\right]_{(1,1)} = \mathcal{L}^{-1}\left\{\frac{s(s-1)}{(s-2)(s^2+1)}\right\} =$$

$$= \left[\frac{2}{5}e^{2t} + \frac{3}{5}\cos t + \frac{1}{5}\operatorname{sen} t\right]\cdot 1(t)$$

elemento (1,2)

$$\left[e^{At}\right]_{(1,2)} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s^2+1)}\right\}$$

utilizzo il "completamento di quadrato"

$$\frac{s-1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$\frac{s-1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$A = \lim_{s \to 2} \frac{(s-1)}{\cancel{(s-2)}(s^2+1)} \cdot \cancel{(s-2)} = \frac{1}{5}$$

sostituisco e cerco gli altri coeff. col principio di identità dei polinomi

$$(s-1) = \frac{1}{5}(s^2+1) + (s-2)(Bs+C)$$

$$s-1 = \left(\frac{1}{5}+B\right)s^2 + (C-2B)s + \left(\frac{1}{5}-2C\right)$$

$$\begin{cases} \frac{1}{5}+B = 0 \\ C-2B = 1 \\ \frac{1}{5} - 2C = -1 \end{cases} \qquad \begin{cases} B = -\frac{1}{5} \\ C = 1+2B = 3/5 \end{cases}$$

$$\frac{1}{5} - 6/5 = -1$$

$$\mathcal{L}^{-1}\left[\frac{(s-1)}{(s-2)(s^2+1)}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{5}\mathcal{L}^{-1}\left[\frac{s-3}{s^2+1}\right]$$

per la linearità di $\mathcal{L}\{\cdot\}$

$$\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}\cdot 1(t)$$

$$\mathcal{L}^{-1}\left[\frac{s-3}{s^2+1}\right] = \underbrace{\mathcal{L}^{-1}\left[\frac{s}{s^2+1^2}\right]}_{\cos t\cdot 1(t)} - 3\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s^2+1^2}\right]}_{\operatorname{sen} t\cdot 1(t)}$$

$$= \left[\cos t - 3\operatorname{sen} t\right]\cdot 1(t)$$

$$\left[e^{At}\right]_{(1,2)} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)(s^2+1)}\right\}$$

$$= \left[\frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

elemento (1,3)

$$\left[e^{At}\right]_{(1,3)} = \mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s^2+1)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s^2+1)}\right\} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1} \quad (*)$$

È l'approccio più conveniente in questo caso!

$$A = \lim_{s \to 2} \frac{2}{\cancel{(s-2)}(s^2+1)} \cancel{(s-2)} = 2/5$$

applico il principio di identità dei polinomi ai polinomi a numeratore nell'espressione ⊛

$$2 = \frac{2}{5}s^2 + \frac{2}{5} + Bs^2 + Cs - 2Bs - 2C$$

$$= \left(\frac{2}{5} + B\right)s^2 + (C - 2B)s + \left(\frac{2}{5} - 2C\right)$$

$$\begin{cases} \frac{2}{5} + B = 0 \\ C - 2B = 0 \\ \frac{2}{5} - 2C = 2 \end{cases} \qquad \begin{cases} B = -2/5 \\ C = 2B = -4/5 \\ 2/5 + 8/5 = 2 \end{cases} \qquad \begin{cases} B = -2/5 \\ C = -4/5 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s^2+1)}\right\} = \frac{1}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} +$$

$$-\frac{4}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

per linearità

$$\left[e^{At}\right]_{(1,3)} = \mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s^2+1)}\right\}$$

$$= \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t - \frac{4}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

elemento (2,1)

$$\left[e^{At}\right]_{(2,1)} = \mathcal{L}^{-1}\left\{\frac{2}{(s-2)(s^2+1)}\right\}$$

come elemento (1,3)

$$= \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t - \frac{4}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

elemento $(2,2)$

$$\left[e^{At}\right]_{2,2} = \mathcal{L}^{-1}\left\{\frac{(s-1)^2}{(s-2)(s^2+1)}\right\}$$

$$\frac{(s-1)^2}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

uso

ancora lo stesso approccio!

$$A = \lim_{s\to 2} \frac{(s-1)^2}{\cancel{(s-2)}(s^2+1)} \cdot \cancel{(s-2)} = \frac{1}{5}$$

$$(s-1)^2 = \frac{1}{5}(s^2+1) + (Bs+C)(s-2)$$

$$s^2-2s+1 = \left(\frac{1}{5}+B\right)s^2 + (C-2B)s + \left(\frac{1}{5}-2C\right)$$

$$\begin{cases} B + 1/5 = 1 \\ C - 2B = -2 \\ 1/5 - 2C = +1 \end{cases} \Rightarrow \begin{cases} B = 4/5 \\ C = -2/5 \\ 1/5 + 4/5 = 1 \end{cases} \Rightarrow \begin{matrix} B = 4/5 \\ C = -2/5 \end{matrix}$$

$$\mathcal{L}^{-1}\left\{\frac{(s-1)^2}{(s-2)(s^2+1)}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + $$

$$-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

per linearità

$$\left[e^{At}\right]_{2,2} = \mathcal{L}^{-1}\left\{\frac{(s-1)^2}{(s-2)(s^2+1)}\right\}$$

$$= \left[\frac{1}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\operatorname{sen} t\right]\cdot 1(t)$$

Elemento (2,3)

$$\left[e^{At}\right]_{2,3} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-2)(s^2+1)}\right\}$$

$$\frac{2(s-1)}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

come negli altri casi

$$A = \lim_{s\to 2} \frac{2(s-1)}{\cancel{(s-2)}(s^2+1)}\cancel{(s-2)} = 2/5$$

confronto i polinomi e numeratore ed applico il principio di identità dei polinomi per determinare i coefficienti B, C

$$2s-2 = \frac{2}{5}(s^2+1) + (Bs+C)(s-2)$$

$$2s-2 = \left(\frac{2}{5}+B\right)s^2 + (C-2B)s + \left(\frac{2}{5}-2C\right)$$

$$\begin{cases} B + \frac{2}{5} = 0 \\ C - 2B = 2 \\ \frac{2}{5} - 2C = -2 \end{cases} \Rightarrow \begin{cases} B = -\frac{2}{5} \\ C = 2 + 2B = \frac{6}{5} \\ \frac{1}{5} - \frac{6}{5} = -1 \end{cases}$$

quindi per linearità

$$\mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-2)(s^2+1)}\right\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$\left[e^{At}\right]_{2,3} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-2)(s^2+1)}\right\}$$

$$= \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t + \frac{6}{5}\operatorname{sen} t\right]\cdot 1(t)$$

elemento $(3,1)$

$$[e^{At}]_{3,1} = \mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s^2+1)}\right\}$$

stesso approccio degli altri casi

$$\frac{s}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$A = \lim_{s\to 2} \frac{s}{\cancel{(s-2)}(s^2+1)}\cdot \cancel{(s-2)} = 2/5$$

applico il principio di identità dei polinomi ai polinomi a numeratore

$$s = \frac{2}{5}(s^2+1) + (Bs+C)(s-2)$$

$$s = \left(\frac{2}{5}+B\right)s^2 + (C-2B)s + \left(\frac{2}{5}-2C\right)$$

$$\begin{cases} B + 2/5 = 0 \\ C - 2B = 1 \\ \frac{2}{5} - 2C = 0 \end{cases} \Rightarrow \begin{cases} B = -2/5 \\ C = 1 + 2B = 1/5 \\ 2/5 - 2/5 = 0 \end{cases}$$

quindi per linearità:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s^2+1)}\right\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$\left[e^{At}\right]_{3,1} = \mathcal{L}^{-1}\{\ast\} = \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t + \frac{1}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

elemento (3,2)

$$\left[e^{At}\right]_{3,2} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$$

provare ad utilizzare l'altro approccio, usando lo sviluppo in fratti semplici

$$\frac{1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$A = \lim_{s\to 2} \frac{1}{\cancel{(s-2)}(s^2+1)}\cancel{(s-2)} = 1/5$$

applico il principio di identità dei polinomi

$$1 = \tfrac{1}{5}(s^2+1) + (Bs+C)(s-2)$$

$$1 = \left(\tfrac{1}{5} + B\right)s^2 + (C-2B)s + \left(\tfrac{1}{5} - 2C\right)$$

$$\begin{cases} B + \frac{1}{5} = 0 \\ C - 2B = 0 \\ \frac{1}{5} - 2C = 1 \end{cases} \Longrightarrow \begin{cases} B = -1/5 \\ C = -2/5 \\ 1/5 + 4/5 = 1 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)\,(s^2+1)}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \\ -2/5\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$\left[e^{At}\right]_{32} = \mathcal{L}^{-1}\{\,!\,\} = \left[\frac{1}{5}e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

elemento$(3,3)$

$$\left[e^{At}\right]_{3,3} = \mathcal{L}^{-1}\left\{\frac{s(s-1)}{(s-2)(s^2+1)}\right\}$$

Come negli altri così trovo A con la formula dei residui, B e C con il principio di identità dei polinomi

$$\frac{s(s-1)}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

$$A = \lim_{s\to 2} \frac{s(s-1)}{\cancel{(s-2)}(s^2+1)} \cdot \cancel{(s-2)} = 2/5$$

$$s^2 - s = 2/5(s^2+1) + (Bs+C)(s-2)$$

$$s^2 - s = \left(\frac{2}{5} + B\right)s^2 + \left(C - 2B\right)s + \left(\frac{2}{5} - 2C\right)$$

$$\begin{cases} \frac{2}{5} + B = +1 \\ C - 2B = -1 \\ \frac{2}{5} - 2C = 0 \end{cases} \Rightarrow \begin{cases} B = 3/5 \\ C = 1/5 \\ \frac{2}{5} - 2/5 = 0 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{s(s-1)}{(s-2)(s^2+1)}\right\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$\left[e^{At}\right]_{3,3} = \mathcal{L}^{-1}\left\{\vdots\right\} = \left[\frac{2}{5}e^{2t} + \frac{3}{5}\cos t + \frac{1}{5}\operatorname{sen} t\right]\cdot 1(t)$$