Calcolo di e^{At}

utilizzando la

forma diagonale

della matrice A

Dato il sistema LTI descritto da:

$$\begin{cases} \dot{x} = Ax + 3u \\ y = Cx \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$e^{At} = ?$$

A diagonalizzabile $\Longrightarrow D = T^{-1} A T$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$T = \left[v_1 \mid v_2 \mid v_3 \right]$$

v_j autovettore associato a λ_j

Cerco autovalori ed autovettori corrispondenti :

$$p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} (\lambda-1) & -1 & 0 \\ 0 & (\lambda-0) & -2 \\ -1 & 0 & (\lambda-1) \end{vmatrix} =$$

sviluppo il determinante lungo la 1ª colonna

$$= (\lambda-1)(-1)^{1+1} \cdot \begin{vmatrix} \lambda & -2 \\ 0 & \lambda-1 \end{vmatrix} + (-1)\cdot(-1)^{3+1} \cdot \begin{vmatrix} -1 & 0 \\ \lambda & -2 \end{vmatrix}$$

$$= (\lambda-1)\left[\lambda(\lambda-1)\right] + (-1)\,|2| = \lambda(\lambda-1)^2 - 2$$

$$= \lambda\left[\lambda^2 - 2\lambda + 1\right] - 2 = \lambda^3 - 2\lambda^2 + \lambda - 2 = \Rightarrow$$

$$p_A(\lambda) = (\lambda^2+1)[\lambda - 2]$$

$\lambda_1 = -j$

$\lambda_2 = +j$

$\lambda_3 = +2$

3 autovalori distinti

$\Downarrow$

A è diagonalizzabile

Autovettori

v_3 : $\quad A v_3 = \lambda_3 v_3 \rightarrow (A - \lambda_3 I) v_3 = 0$

$$v_3 \in \ker(A - \lambda_3 I)$$

$$(A - 2I) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} =$$

$$(A-2I) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$(A-2I)v_3 = 0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} -a + b = 0 \\ 2(-b + c) = 0 \\ a - c = 0 \end{cases} \qquad \begin{cases} a = b \\ b = c \\ a = c \end{cases}$$

$$v_3 \in \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

$$\lambda_1 = -j \qquad v_1 : (A - \lambda_1 I) v_1 = 0$$

$$v_1 \in \ker(A - \lambda_1 I)$$

$$(A - \lambda_1 I) = \begin{bmatrix} (1+j) & 1 & 0 \\ 0 & +j & 2 \\ 1 & 0 & (+j+1) \end{bmatrix}$$

$$(A - \lambda_1 I) v_1 = 0 \iff \begin{cases} (1+j)a + b = 0 \\ jb + 2c = 0 \\ a + (j+1)c = 0 \end{cases}$$

$$\begin{cases} (1+j)a + b = 0 \\ jb + 2c = 0 \\ a + (j+1)c = 0 \end{cases}$$

$$b = -(1+j)a$$

$$c = -\frac{j}{2}b = \frac{j(1+j)}{2}a = \frac{-1+j}{2}a$$

Sostituisco nella 3ª equazione

$$a + (j+1)\frac{(-1+j)}{2}a = a + \left[\frac{(j)^2 - 1}{2}\right]a = a + \left(-\frac{2}{2}\right)a = 0 \quad \#$$

In definitiva

$$\begin{cases} a \text{ qualsiasi} \\ b = -(1+j)\,a \\ c = \dfrac{-1+j}{2}\,a \end{cases} \qquad v_1 \in \left\langle \begin{bmatrix} 1 \\ -(1+j) \\ \left(\dfrac{-1+j}{2}\right) \end{bmatrix} \right\rangle$$

v_2 è ovviamente il complesso coniugato di v_1

La matrice di trasformazione allora è:

$$T = \left[\, v_1 \mid v_2 \mid v_3 \,\right] = \longrightarrow$$

$$T = \begin{bmatrix} 1 & 1 & 1 \\ -(1+j) & -(1-j) & 1 \\ \left(\frac{-1+j}{2}\right) & -\left(\frac{1+j}{2}\right) & 1 \end{bmatrix} \qquad \det T = 5j$$

$$T^{-1} = -j\,\frac{1}{5} \cdot \begin{bmatrix} -\frac{1}{2}+\frac{3}{2}j & -\frac{3}{2}-\frac{1}{2}j & +2-j \\ +\frac{1}{2}+\frac{3}{2}j & +\frac{3}{2}-\frac{1}{2}j & -2-j \\ +2j & +j & +2j \end{bmatrix}$$

da cui $\quad T^{-1} A T = D$

$$T^{-1}AT = \begin{bmatrix} -j & 0 & 0 \\ 0 & +j & 0 \\ 0 & 0 & +2 \end{bmatrix}$$

$$A = TDT^{-1}$$

Ora vale che :

$$e^{At} = T e^{Dt} T^{-1}$$

$\boxed{\text{FA Parte 3 \#25}}$

$$e^{At} = T \begin{bmatrix} e^{-jt} \cdot 1(t) & 0 & 0 \\ 0 & e^{jt} \cdot 1(t) & 0 \\ 0 & 0 & e^{2t} \cdot 1(t) \end{bmatrix} T^{-1} = \Rightarrow$$

$$\left[e^{At}\right]_{(1,1)} = \left[\frac{2}{5}e^{2t} + \left(\frac{3}{10}+j\frac{1}{10}\right)e^{-jt} + \left(\frac{3}{10}-j\frac{1}{10}\right)e^{+jt}\right]\cdot 1(t)$$

formule di Eulero

$$\begin{cases} \text{sen}\,x = \dfrac{e^{jx}-e^{-jx}}{2j} \\[2ex] \cos x = \dfrac{e^{jx}+e^{-jx}}{2} \end{cases}$$

$$+\frac{\cancel{6}^{\,3}}{\cancel{10}_{\,5}}\cdot\frac{e^{jt}+e^{-jt}}{2} \qquad +\frac{\cancel{2}^{\,1}}{\cancel{10}_{\,5}}\cdot\frac{e^{jt}-e^{-jt}}{2j} \qquad -\frac{2}{10}\left(-\frac{1}{j}\right)\cdot\frac{\quad}{2}$$

$$= \left[\frac{2}{5}e^{2t} + \frac{3}{10}\left(e^{-jt}+e^{+jt}\right) - j\frac{1}{10}\left(e^{jt}-e^{-jt}\right)\right]\cdot 1(t)$$

$$= \left[\frac{2}{5}e^{2t} + \frac{3}{5}\cos t + \frac{1}{5}\,\text{sen}\,t\right]\cdot 1(t)$$

$$e^{At}\Big|_{(1,2)} = \left[\frac{1}{5}e^{2t} - e^{-jt}\left(\frac{1}{10} - \frac{3}{10}j\right) - e^{jt}\left(\frac{1}{10} + \frac{3}{10}j\right)\right] \cdot 1(t)$$

$$= \left[\frac{1}{5}e^{2t} \underbrace{- \frac{1}{10}\left(e^{jt} + e^{-jt}\right)}_{-\frac{1}{5}\cdot\frac{e^{jt}+e^{-jt}}{2}} \underbrace{- \frac{3}{10}j\left(e^{jt} - e^{-jt}\right)}_{+\frac{3}{5}\cdot\frac{e^{jt}-e^{-jt}}{2j}}\right] \cdot 1(t)$$

$$= \left[\frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\operatorname{sen} t\right] \cdot 1(t)$$

$$
\left. e^{At} \right|_{(1,3)} = \left[\frac{2}{5} e^{2t} - \frac{1}{5}(1+2j) e^{-jt} - \frac{1}{5}(1-2j) e^{jt} \right] \cdot 1(t)
$$

$$
= \left[\frac{2}{5} e^{2t} - \frac{1}{5}\left(e^{-jt} + e^{jt}\right) + \frac{2}{5} j \left(e^{jt} - e^{-jt}\right) \right] \cdot 1(t)
$$

$$
-\frac{2}{5} \cdot \frac{e^{jt} + e^{-jt}}{2} \qquad - \frac{4}{5} \cdot \frac{e^{jt} - e^{-jt}}{2j}
$$

$$
= \left[\frac{2}{5} e^{2t} - \frac{2}{5} \cos t - \frac{4}{5} \operatorname{sen} t \right] \cdot 1(t)
$$

$$e^{At}\Big|_{(2,1)} = \left[\frac{2}{5}e^{2t} - \frac{1}{5}(1+2j)e^{-jt} - \frac{1}{5}(1-2j)e^{+jt}\right]\cdot 1(t)$$

cfr. elemento (1,3)

$$= \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t - \frac{4}{5}\operatorname{sen} t\right]\cdot 1(t)$$

$$e^{At}\Big|_{(2,2)} = \left[\frac{1}{5}e^{2t} + \frac{1}{5}(2-j)e^{-jt} + \frac{1}{5}(2+j)e^{jt}\right]\cdot 1(t)$$

$$= \left[\frac{1}{5}e^{2t} + \frac{2}{5}\left(e^{jt}+e^{-jt}\right) + \frac{1}{5}j\left(e^{jt}-e^{-jt}\right)\right]\cdot 1(t)$$

$$e^{At}\Big|_{(1,2)} = \left[\frac{1}{5}e^{2t} + \frac{2}{5}\left(e^{jt}+e^{-jt}\right) + \frac{1}{5j}\left(e^{jt}-e^{-jt}\right)\right]\cdot 1(t)$$

$$+\frac{4}{5}\cdot\frac{e^{jt}+e^{-jt}}{2} \qquad -\frac{2}{5}\cdot\frac{e^{jt}-e^{-jt}}{2j}$$

$$= \left[\frac{1}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

$$e^{At}\Big|_{(1,3)} = \left[\frac{2}{5}e^{2t} - \frac{1}{5}(1-3j)e^{-jt} - \frac{1}{5}(1+3j)e^{jt}\right]\cdot 1(t)$$

$$= \left[\frac{2}{5}e^{2t} - \frac{1}{5}\left(e^{jt}+e^{-jt}\right) - \frac{3}{5j}\left(e^{jt}-e^{-jt}\right)\right]\cdot 1(t)$$

$$-\frac{2}{5}\cdot\frac{e^{jt}+e^{-jt}}{2} \qquad +\frac{6}{5}\cdot\frac{e^{jt}-e^{jt}}{2j}$$

$$e^{At}\Big|_{(2,3)} = \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t + \frac{6}{5}\,\text{sen}\,t\right]\cdot 1(t)$$

$$e^{At}\Big|_{(3,1)} = \left[\frac{2}{5}e^{2t} - \frac{1}{5}\left(1 - \frac{1}{2}j\right)e^{-jt} - \frac{1}{5}\left(1 + \frac{j}{2}\right)e^{jt}\right]\cdot 1(t)$$

$$= \left[\frac{2}{5}e^{2t} - \frac{1}{5}\left(e^{jt} + e^{-jt}\right) - \frac{1}{10}j\left(e^{jt} - e^{-jt}\right)\right]\cdot 1(t)$$

$$-\frac{2}{5}\cdot\frac{e^{jt}+e^{-jt}}{2} + \frac{\cancel{2}^{1}}{\cancel{10}_{5}}\cdot\frac{e^{jt}-e^{-jt}}{2j}$$

$$= \left[\frac{2}{5}e^{2t} - \frac{2}{5}\cos t + \frac{1}{5}\,\text{sen}\,t\right]\cdot 1(t)$$

$$e^{At}\Big|_{(3,2)} = \left[\frac{1}{5}e^{2t} - \frac{1}{5}\left(\frac{1}{2}+j\right)e^{-jt} - \frac{1}{5}\left(\frac{1}{2}-j\right)e^{jt}\right]\cdot 1(t)$$

$$= \left[\frac{1}{5}e^{2t} - \frac{1}{10}\left(e^{jt}+e^{-jt}\right) + \frac{1}{5}j\left(e^{jt}-e^{-jt}\right)\right]\cdot 1(t)$$

$$-\frac{\cancel{2}^{\,1}}{\cancel{10}_{\,5}}\cdot\frac{e^{jt}+e^{-jt}}{2} \qquad -\frac{2}{5}\cdot\frac{e^{jt}-e^{-jt}}{2j}$$

$$= \left[\frac{1}{5}e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\,\mathrm{sen}\,t\right]\cdot 1(t)$$

$$\left. e^{At} \right|_{(3,3)} = \left[\frac{2}{5} e^{2t} + \frac{1}{10}(3+j)e^{-jt} + \frac{1}{10}(3-j)e^{jt} \right] \cdot 1(t)$$

$$= \left[\frac{2}{5} e^{2t} + \frac{3}{10}\left(e^{jt} + e^{-jt}\right) - \frac{1}{10} j \left(e^{jt} - e^{-jt}\right) \right] \cdot 1(t)$$

$$\frac{\cancel{6}^{\,3}}{_5\cancel{10}} \cdot \frac{e^{jt} + e^{-jt}}{2} + \frac{\cancel{2}^{\,1}}{\cancel{10}_{\,5}} \frac{e^{jt} - e^{-jt}}{2j}$$

$$= \left[\frac{2}{5} e^{2t} + \frac{3}{5} \cos t + \frac{1}{5} \operatorname{sen} t \right] \cdot 1(t)$$