MECCANICA RAZIONALE

Equazioni di Lagrange

$$\frac{d}{dt}\frac{\partial K}{\partial \dot q_i} - \frac{\partial K}{\partial q_i} = Q_i \qquad i=1,\dots,\ell$$

$$\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad , \text{ con } \mathcal{L} = K - V \right)$$

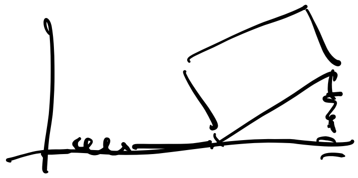
→ struttura dell'energia cinetica

$$K = \frac{1}{2}\sum_{B\in S} m_B \|\underline{v}_B\|^2$$

$$= K_0(\underline{q}(t), t) + \underline{b}(\underline{q}(t), t)\cdot \dot{\underline{q}} + \frac{1}{2}\dot{\underline{q}}\cdot A(\underline{q}(t); t)\, \dot{\underline{q}}$$

$\sin(\varphi)$

T ad esempio



$$K = \frac{1}{2} m \left(\dot x^2 + \dot\varphi^2 \left(\frac{5}{3}\ell^2\right) - \sqrt{3}\,\ell\,\dot x\,\dot\varphi \sin(\gamma+\varphi) \right)$$

$$\longrightarrow \quad \frac{1}{2}\,\dot{\underline{q}}\cdot \underbrace{A(\underline{q}(t),t)}\cdot\dot{\underline{q}}$$

$\det A > 0$

$$\hookrightarrow \quad \frac{d}{dt}\frac{\partial K}{\partial \dot q_i} - \frac{\partial K}{\partial q_i} = Q_i$$

$\dot q$ in K

$\Downarrow$

$\ddot q$ nella eq. del moto

$$\Updownarrow$$

$$A\,\ddot{\underline{q}} = \underline{F}\left(\underline{q}(t),\dot{\underline{q}}(t),t\right)$$

$$\uparrow$$

$$\boxed{\ddot{\underline{q}} = A^{-1}\,F}$$

$\rightarrow$ eq. diff. 2° ordine in forma normale

$\Rightarrow$ Teorema di Cauchy: la soluzione esiste ed è unica dati: $\underline{q}(t=0)$ e $\dot{\underline{q}}(t=0)$

⇒ DETERMINISMO.

Eq Lagrange → "sistema dinamico"

→ stabilità

→ linearizzare le eq. di moto

Sistema dinamico : esempio semplice

$$\left| \; \underline{F} = m\,\underline{a} = m \frac{d^2}{dt^2}\underline{x} \; \right|$$

→ risolvere questa eq vuol dire trovare x da $\ddot{x}$

$$\begin{cases} \dfrac{d\underline{x}}{dt} = \dfrac{\underline{p}}{m} \\[2mm] \dfrac{d}{dt}\underline{p} = \underline{F} \end{cases}$$

sistema eq. differenziali 1° ordine in $(\underline{x}, \underline{p})$

$\underline{p} = m \frac{d\underline{x}}{dt}$

$$\underline{F} = \frac{d}{dt}\underline{p} = m \frac{d^2\underline{x}}{dt^2}$$

Eq 2° ordine per $\underline{x}$ $\longleftrightarrow$ 2 eq 1° ordine per $\underline{x}$ e $\underline{p}$

$$\underline{F} = m \ddot{\underline{x}} \quad \longleftrightarrow \quad \begin{cases} \dot{\underline{p}} = \underline{F} \\ \dot{\underline{x}} = \dfrac{\underline{p}}{m} \end{cases}$$

Equazioni di Lagrange:

$$\underbrace{\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_i}} - \frac{\partial K}{\partial q_i} = Q_i$$

Definiamo

$$\boxed{p_i = \frac{\partial K}{\partial \dot{q}_i}} \qquad i = 1, \dots, \ell$$

i-esimo momento coniugato

$$K = K_0 + \sum_{i=1}^{\ell} b_i \dot{q}_i + \frac{1}{2} \sum_{i,j=1}^{\ell} \dot{q}_i A_{ij} \dot{q}_j$$

$$p_i = b_i + \sum_{j=1}^{\ell} A_{ij} \dot{q}_j$$

In notazione vettoriale: $\underline{p} = (p_1, \dots, p_\ell)$

$$\underline{p} = A\,\underline{\dot q} + \underline{b}$$

$$\Updownarrow$$

A è invertibile

$$\underline{\dot q} = A^{-1}(\underline{p} - \underline{b})$$

$\underline{q}, t$ $\quad$ $\underline{p}$ $\quad$ $\underline{q}, t$

La possiamo riscrivere come

$$\boxed{\frac{dq_i}{dt} = \gamma_i(\underline{q}, \underline{p}, t)} \qquad i = 1, \dots, \ell$$

La seconda equazione

$$\frac{d}{dt} p_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_i}\right) =$$

per le equazioni di Lagrange

$$= \left(\frac{\partial K}{\partial q_i} + Q_i\right)\Bigg|_{\dot q_i = \gamma_i}$$

$$= f_i(\underline{q}, \underline{p}, t)$$

Allora possiamo riscrivere le equazioni di Lagrange come un sistema normale di 2ℓ eq. differenziali del primo ordine nelle incognite

$$(q_1, \dots, q_\ell, p_1, \dots, p_\ell)$$

$$\begin{cases} \dfrac{d}{dt} q_i = \gamma_i(\underline{q}, \underline{p}, t) \\ \dfrac{d}{dt} p_i = f_i(\underline{q}, \underline{p}, t) \end{cases}$$

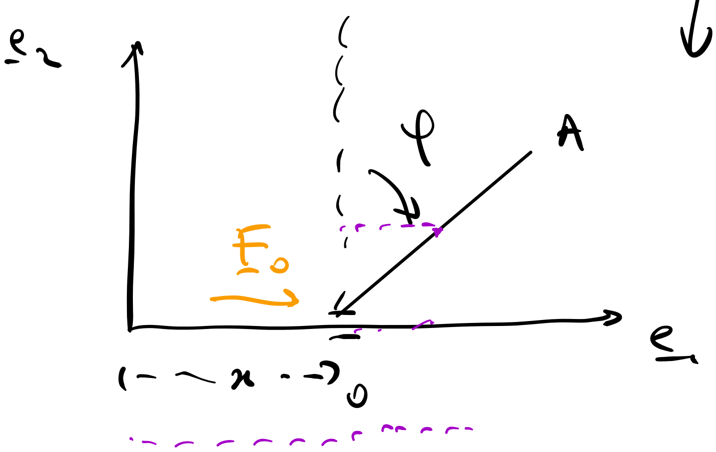
$$\underline{y} = (q_1, \dots, q_\ell, p_1, \dots, p_\ell)$$

$$\underline{G} = (\gamma_1, \dots, \gamma_\ell, f_1, \dots, f_\ell)$$

$$\boxed{\frac{d}{dt}\,\underline{y} = \underline{G}(\underline{y}, t)}$$ Sistema dinamico

Lo spazio delle $\underline{y}$ si dice anche spazio delle fasi

Esempio



$\downarrow \underline{g}$

$\overline{OA} = \ell$

piano verticale

$$\underline{F}_0 = (P\varphi + D\dot{\varphi})\underline{e}_1$$

Equazioni di Lagrange:

$$K = \frac{1}{2} m\, v_G^2 + \frac{1}{2} I_{3,G}\, \dot{\varphi}^2$$

$$\underline{x}_G = \left(x + \frac{\ell}{2}\sin\varphi\right)\underline{e}_1 + \frac{\ell}{2}\cos\varphi\, \underline{e}_2$$

$$\underline{v}_G = \left(\dot{x} + \frac{\ell}{2}\dot{\varphi}\cos\varphi\right)\underline{e}_1 - \frac{\ell}{2}\dot{\varphi}\sin\varphi\, \underline{e}_2$$

$$\underline{v}_G \cdot \underline{v}_G = \left(\dot x + \frac{\ell}{2}\dot\varphi\cos\varphi\right)^2 + \left(\frac{\ell}{2}\dot\varphi\sin\varphi\right)^2$$

$$= \dot x^2 + \frac{\ell^2}{4}\dot\varphi^2\cos^2\varphi + \ell\,\dot x\,\dot\varphi\cos\varphi + \frac{\ell^2}{4}\dot\varphi^2\sin^2\varphi$$

$$= \dot x^2 + \ell\,\dot x\,\dot\varphi\cos\varphi + \frac{\ell^2}{4}\dot\varphi^2$$

$$\frac{1}{2} I_{3,G}\,\dot\varphi^2 \longrightarrow$$

$$I_{3,G} = \frac{M\ell^2}{12}$$

$$\left(I_{O,G} = \frac{M\ell^2}{3}\right)$$

$$K = \frac{1}{2} m \left(\dot x^2 + \frac{\ell^2}{4}\dot\varphi^2 + \dot x\,\dot\varphi\,\ell\cos\varphi\right) + \frac{1}{2}\frac{M\ell^2}{12}\dot\varphi^2$$

Peso $\longrightarrow$ $V = -m\,\underline{g}\cdot\underline{x}_G = mg\frac{\ell}{2}\cos\varphi$

$$\underline{F} = (P\varphi + D\dot\varphi)\,\underline{e}_1 = F_0\,\underline{e}_1$$

Lavoro virtuale := 0 : $\underline{F}\cdot\delta\underline{x}_0 = F_0\,\underset{q}{\delta x}$

$$\begin{cases} Q_x = F_0 - \dfrac{\partial V}{\partial x} = F_0 \\[2ex] Q_\varphi = -\dfrac{\partial V}{\partial \varphi} = \underline{mg\dfrac{\ell}{2}\sin\varphi} \end{cases}$$

$$K = \frac{1}{2}m\left(\dot{x}^2 + \frac{\ell^2}{3}\dot{\varphi}^2 + \dot{x}\dot{\varphi}\,\ell\cos\varphi\right)$$

•) $$\frac{d}{dt}\frac{\partial K}{\partial \dot{x}} = \frac{d}{dt}\left[m\dot{x} + \frac{1}{2}m\dot{\varphi}\,\ell\cos\varphi\right]$$

$$= m\ddot{x} + \frac{1}{2}m\ell\,\ddot{\varphi}\cos\varphi - \frac{1}{2}m\dot{\varphi}^2\ell\sin\varphi$$

$$\underline{\frac{\partial K}{\partial x} = 0} \qquad \underline{Q_x = F_0}$$

$$\boxed{m\ddot{x} + \frac{1}{2}m\ell\,\ddot{\varphi}\cos\varphi - \frac{1}{2}m\dot{\varphi}^2\ell\sin\varphi = F_0}$$

•) $$\frac{d}{dt}\frac{\partial K}{\partial \dot{\varphi}} = \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{2}{3}\ell^2\dot{\varphi} + \dot{x}\,\ell\cos\varphi\right)\right]$$

$$= \frac{m}{3}\ell^2\ddot{\varphi} + \frac{1}{2}m\ell\,\ddot{x}\cos\varphi - \frac{1}{2}m\ell\,\dot{x}\dot{\varphi}\sin\varphi$$

$$\frac{\partial K}{\partial \varphi} = \underline{\frac{1}{2} m \ell \dot{x} \dot{\varphi} \left(-\sin \varphi\right)}$$

$$Q_\varphi = \underline{m g \frac{\ell}{2} \sin \varphi}$$

$$\frac{m}{3} \ell^2 \ddot{\varphi} + \frac{1}{2} m \ell \ddot{x} \cos \varphi - \cancel{\frac{1}{2} m \ell \dot{x} \dot{\varphi} \sin \varphi}$$
$$+ \cancel{\frac{1}{2} m \ell \dot{x} \dot{\varphi} \sin \varphi} = m g \frac{\ell}{2} \sin \varphi$$

$$\begin{cases} m \ddot{x} + \frac{1}{2} m \ell \ddot{\varphi} \cos \varphi - \frac{1}{2} m \dot{\varphi}^2 \ell \sin \varphi = F_0 \\[2ex] \frac{m}{3} \ell^2 \ddot{\varphi} + \frac{1}{2} m \ell \ddot{x} \cos \varphi = m g \frac{\ell}{2} \sin \varphi \end{cases}$$

Possiamo riscriverle in forma matriciale

$$\underbrace{m \begin{pmatrix} 1 & \frac{\ell}{2} \cos \varphi \\ \frac{\ell}{2} \cos \varphi & \ell^2/3 \end{pmatrix}} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} m \frac{\ell}{2} \dot{\varphi}^2 \sin \varphi + F_0 \\ m g \frac{\ell}{2} \sin \varphi \end{pmatrix}$$

$$A \cdot \underline{\ddot{q}} = \underline{F}$$

$$\frac{1}{2}\begin{pmatrix}\dot{x} & \dot{\varphi}\end{pmatrix}\begin{pmatrix} m & m\frac{\ell}{2}\cos\varphi \\ m\frac{\ell}{2}\cos\varphi & m\frac{\ell^2}{3}\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{\varphi}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}\dot{x} & \dot{\varphi}\end{pmatrix}\begin{pmatrix} m\dot{x} + m\frac{\ell}{2}\dot{\varphi}\cos\varphi \\ m\frac{\ell}{2}\dot{x}\cos\varphi + \dot{\varphi}\, m\frac{\ell^2}{3}\end{pmatrix}$$

$$= \frac{1}{2}\Big(m\dot{x}^2 + m\frac{\ell}{2}\dot{x}\dot{\varphi}\cos\varphi + m\frac{\ell}{2}\dot{x}\dot{\varphi}\cos\varphi$$
$$+ \dot{\varphi}^2 m\frac{\ell^2}{3}\Big) =$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{\varphi}^2\frac{\ell^2}{3} + m\ell\dot{x}\dot{\varphi}\cos\varphi\right)$$

$$K = \frac{1}{2}\begin{pmatrix}\dot{q}_1 & \dot{q}_2\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}\dot{q}_1 \\ \dot{q}_2\end{pmatrix}$$

$$= \frac{1}{2}\left(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + 2a_{12}\dot{q}_1\dot{q}_2\right)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}$$

$$A = \begin{pmatrix} m & m\frac{\ell}{2}\cos\varphi \\ m\frac{\ell}{2}\cos\varphi & m\frac{\ell^2}{3} \end{pmatrix}$$

$$\det A = m^2\frac{\ell^2}{3} - m^2\frac{\ell^2}{4}\cos^2\varphi > 0$$

Adesso riscriviamo queste equazioni come un sistema dinamico

$$\begin{cases} p_x = \dfrac{\partial K}{\partial \dot{x}} = m\dot{x} + m\dfrac{\ell}{2}\dot{\varphi}\cos\varphi \\ p_\varphi = \dfrac{\partial K}{\partial \dot{\varphi}} = m\dfrac{\ell^2}{3}\dot{\varphi} + \dfrac{1}{2}m\dot{x}\,\ell\cos\varphi \end{cases}$$

$$K = \frac{1}{2}m\left(\dot{x}^2 + \frac{\ell^2}{3}\dot{\varphi}^2 + \dot{x}\dot{\varphi}\,\ell\cos\varphi\right) \longleftarrow$$

$$K = K_0 + (\underline{b}\cdot\dot{q}) + \frac{1}{2}\dot{q}\,\underline{\underline{A}}\,\dot{q}$$

$$p_i = \frac{\partial K}{\partial \dot{q}_i} \qquad i = 1, 2 \qquad \dot{q}_1 = \dot{x}, \quad \dot{q}_2 = \dot{\varphi}$$

$$\underline{p} = A\underline{\dot{q}} + \cancel{\underline{b}} \Rightarrow \underline{\dot{q}} = A^{-1}(\underline{p} - \cancel{\underline{b}})$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = A \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = m \begin{pmatrix} 1 & \frac{\ell}{2}\cos\varphi \\ \frac{\ell}{2}\cos\varphi & \frac{\ell^2}{3} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix}$$

↑ controllore

$$\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = A^{-1} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

$$A^{-1} = \frac{1}{m\ell^2\left(\frac{1}{3} - \frac{1}{4}\cos^2\varphi\right)} \begin{pmatrix} \frac{\ell^2}{3} & -\frac{\ell}{2}\cos\varphi \\ -\frac{\ell}{2}\cos\varphi & 1 \end{pmatrix}$$

$$\begin{cases} \dot{x} = \dfrac{1}{m\ell^2\left(\frac{1}{3} - \frac{1}{4}\cos^2\varphi\right)} \left(\dfrac{\ell^2}{3} p_x - \dfrac{\ell}{2}\cos\varphi \, p_y \right) \\[2ex] \dot{\varphi} = \dfrac{1}{m\ell^2\left(\frac{1}{3} - \frac{1}{4}\cos^2\varphi\right)} \left(-\dfrac{\ell}{2}\cos\varphi \, p_x + p_y \right) \end{cases}$$

$$\frac{d}{dt} p = \left(\left(\frac{\partial k}{\partial q_i} + Q_i \right) \right)\Bigg|_{\dot{q}_i = \gamma_i}$$

$$\frac{d}{d\tau} p_x = \left(\frac{\partial K}{\partial x} + Q_x \right) \Bigg|_{\substack{\dot{x} = \gamma_1 \\ \dot{\varphi} = \gamma_2}} = F_0$$

$$\frac{d}{d\tau} p_\varphi = \left(\frac{\partial K}{\partial \varphi} + Q_\varphi \right) \Bigg|_{\substack{\dot{x} = \gamma_1 \\ \dot{\varphi} = \gamma_2}}$$

$$= \left(-\frac{1}{2} m l \dot{x} \dot{\varphi} \sin\varphi + m g \frac{l}{2} \sin\varphi \right) \Bigg|_{\substack{\dot{x} = \gamma_1 \\ \dot{\varphi} = \gamma_2}}$$

$$= \Bigg[-\frac{1}{2} m l \cdot \sin\varphi \left(\frac{1}{m l^2 \left(\frac{1}{3} - \frac{1}{4} \cos^2\varphi \right)} \left(\frac{l^2}{3} p_x - \frac{l}{2} \cos\varphi \, p_\varphi \right) \right) \left(\frac{1}{m l^2 \left(\frac{1}{3} - \frac{1}{4} \cos^2\varphi \right)} \left(-\frac{l}{2} \cos\varphi \, p_x + p_\varphi \right) \right)$$

$$+ m g \frac{l}{2} \sin\varphi \Bigg]$$

$$= \frac{-\frac{1}{2} m l \sin\varphi}{\left[m l^2 \left(\frac{1}{3} - \frac{1}{4} \cos^2\varphi \right) \right]^2} \left(\frac{l^2}{3} p_x - \frac{l}{2} \cos\varphi \, p_\varphi \right) \cdot$$

$$\cdot \left(-\frac{l}{2}\cos\varphi\, p_x + p_y\right) + mg\frac{l}{2}\sin\varphi$$

$$\begin{cases} \dot{x} = \text{---} & (p_x \quad p_y \quad \varphi \quad x) \\ \dot{\varphi} = \text{---} & (p_x \quad p_y \quad \varphi \quad x) \\ \dot{p}_x = \text{---} & (p_x \quad p_y \quad \varphi \quad x) \\ \dot{p}_y = \text{---} & (p_x \quad p_y \quad \varphi \quad x) \end{cases}$$

Commento: consideriamo il caso conservativo:

$$\mathcal{L} = \mathcal{L}(\underline{q}, \underline{\dot{q}}, t) = K - V$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad i = 1 \dots l$$

$\rightarrow$ "meccanica HAMILTONIANA"

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \longrightarrow \underline{\dot{q}} = f^i(\underline{q}, \underline{p}, t)$$

Introduciamo la funzione hamiltoniana

$$\mathcal{H} := \left(\sum_{j=1}^{l} p_j \dot{q}_j - \mathcal{L} \right) \Big|_{\dot{q}_i = \gamma_i(\underline{q}, \underline{p})}$$

" Trasformata di Legendre "

Allora il sistema dinamico è

$$\begin{cases} \dfrac{d}{dt} q_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\[2ex] \dfrac{d}{dt} p_i = - \dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases}$$

Infatti: siccome $\mathcal{H}$ è funzione di $\underline{q}$ e di $\underline{p}$ direttamente e tramite $\dot{q}_j = \gamma_j(\underline{q}, \underline{p}, t)$

- $$\frac{\partial \mathcal{H}}{\partial \dot{q}_k} = p_k - \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0$$ per la definizione di p

$$\bullet\quad \frac{\partial \mathcal{H}}{\partial p_i} = \dot q_i + \sum_{k=1}^{l} \underbrace{\frac{\partial \mathcal{H}}{\partial \dot q_k}}_{=0} \frac{\partial \dot q_k}{\partial p_i} = \dot q_i$$

$$\bullet\quad \frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{k=1}^{l} \underbrace{\frac{\partial \mathcal{H}}{\partial \dot q_k}}_{=0} \frac{\partial \dot q_k}{\partial q_i} =$$

$$= -\frac{\partial \mathcal{L}}{\partial q_i}$$

eq. di Lagrange

$$= -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i} = -\frac{d}{dt} p_i$$

$p_i = \dfrac{\partial \mathcal{L}}{\partial \dot q_i}$ per definizione

$$\Rightarrow \begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial \mathcal{H}}{\partial p_i} \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases}$$

equazioni di Hamilton

Caso di un sistema olonomo a

Vincoli fissi & forze conservative

$$\mathcal{L} = K - V = \frac{1}{2}\, \dot{\underline{q}} \cdot A \cdot \dot{\underline{q}} - V(\underline{q})$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_j A_{ij}\, \dot{q}_j = (A\, \dot{\underline{q}})_i$$

$$\mathcal{H} = \left(\underline{p} \cdot \dot{\underline{q}} - \mathcal{L} \right)_{\dot{\underline{q}} = A^{-1}\underline{p}}$$

$$= \left(\underline{p} \cdot \dot{\underline{q}} - \frac{1}{2}\, \dot{\underline{q}}\, A \cdot \dot{\underline{q}} + V(\underline{q}) \right)\Big|_{\dot{q} = A^{-1}\underline{p}}$$

$$= \left(\dot{\underline{q}} \cdot A\, \dot{\underline{q}} - \frac{1}{2}\, \dot{q}\, A\, \dot{q} + V(\underline{q}) \right)\Big|_{\dot{q} = A^{-1}\underline{p}}$$

$$= K + V$$ = l'energia meccanica.

→ $\mathcal{L} = K - V$ ⇒ eq. di Lagrange

→ $\mathcal{H} = K + V$ ⇒ sistema dinamico

Sistemi dinamici

Eq di Lagrange eq. diff 2° ordine in l variabili $(q_1 \dots q_l)$ $\longleftrightarrow$ Eq. diff. 1° ordine in $2l$ variabili $(q_1 \dots q_l, p_1 \dots p_l)$

Sistema dinamico: ha la forma

$$\begin{cases} \dfrac{d}{dt}\underline{y} = \underline{F}(\underline{y}(t), t) \\ \underline{y}(0) = \underline{y}_0 \end{cases}$$

dove lo spazio delle $\underline{y} = (\underline{q}, \underline{p})$ è anche detto spazio delle fasi.

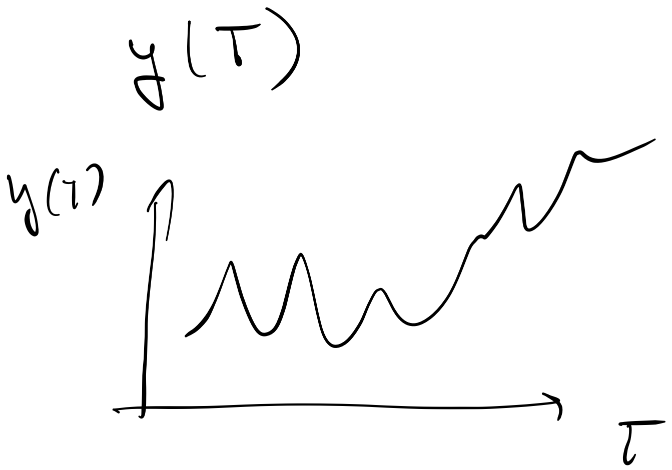
↳ esiste un'unica soluzione $\underline{y}(t; \underline{y}_0)$ per sottolineare la dipendenza dei dati iniziali.

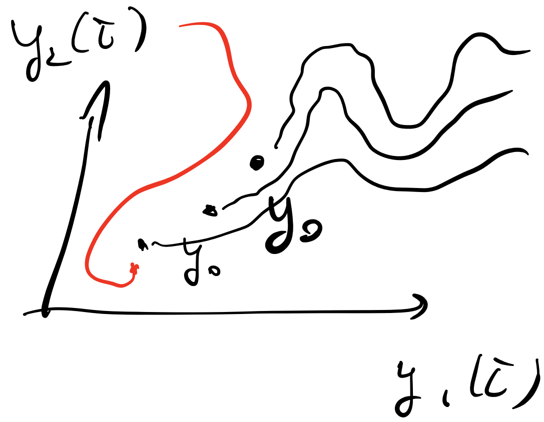
Definiamo

$$\Gamma_{\underline{y}_0} = \{ \underline{y} \in \mathbb{R}^{2\ell} : \underline{y} = \underline{y}(t; \underline{y}_0) \}$$

la traiettoria uscente da $\underline{y}_0$

→ è una curva nello spazio delle fasi (nello spazio delle $\underline{y}$)





$\underline{y} = (y_1, y_2)$

sistema olonomo → $\underline{y} = (\underline{q}, \underline{p})$

Una configurazione di equilibrio

$$\underline{q}(t) = \underline{q}_e$$

$$\underline{p}(T) = \underline{0}$$

Allora $\quad \underline{y}_E = (\underline{q}_E, \underline{0})$

$\uparrow$ costante

$$\left.\frac{d\underline{y}}{dT}\right|_{y=y_E} = 0 \qquad \left(\begin{array}{l} \dfrac{d\underline{q}_E}{dT} = \underline{0} \\ \dfrac{d\underline{p}_E}{dT} = \underline{0} \end{array} \right)$$

Allora le configurazioni di equilibrio di

$$\frac{d}{dT}\underline{y} = \underline{F}(\underline{y}(T), T)$$

si trovano quando $\underline{F}(\underline{y}(T), T) = 0$

$\longrightarrow$ soluzioni stazionarie

Se $\underline{F} = \underline{F}(\underline{y}(T))$ non dipende esplicitamente dal tempo, sistema dinamico autonomo.