MECCANICA RAZIONALE

Equazioni della dinamica :

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i$$ equazioni di Lagrange

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \mathcal{L} = K - V$$

D'Alembert : $\frac{d}{dt}\underline{p}_B$

+ PLV $\Rightarrow$ eq di Lagrange

$\rightarrow$ dinamica delle coordinate libere

Eq differenziali : conoscendo i dati iniziali $q_i(t=0)$, $\dot{q}_i(t=0)$

→ esiste ed è unica la soluzione

cioè le funzioni $q_i = q_i(\tau)$

Segue che: $K = K_0 + \underline{b} \cdot \underline{\dot{q}} + \frac{1}{2} \underline{\dot{q}} A \underline{\dot{q}}$

$\left(\underline{b} = \frac{\partial x_B}{\partial t}\right)$

al più 2° grado nelle $\dot{q}$

dove $A = \begin{pmatrix} - & - \\ - & - \\ - & - \end{pmatrix}_{q(\tau),\, \tau}$ → $\det A > 0$

Problema: come risolviamo queste equazioni?

Metodi di approssimazione

→ ci basta capire la struttura qualitativa (sistemi dinamici)

→ soluzioni analitiche ma con regione di validità "limitata"

Idea: semplifichiamo le nostre equazioni, in modo da poterle risolvere. → LINEARIZZAZIONE

Eq differenziali non-lineari → Eq lineari "buttando via i termini non lineari"

≡ Sviluppo in serie di Taylor troncato al primo ordine non banale.

Dalle equazioni di Lagrange: possiamo derivare un sistema dinamico passando da l eq. differenziali del secondo ordine in q_i $(i=1, \dots, l)$, a $2l$ eq. diff. del 1° ordine in (q_i, p_i)

Si definisce $p_i = \frac{\partial K}{\partial \dot{q}_i}$ momento coniugato

$$p_i = \sum_{j=1}^{l} A_{ij} \dot{q}_j + b_i$$ (segue dalla struttura di K)

$$\rightarrow \quad \underline{\dot{q}} = A^{-1} (\underline{p} - \underline{b})$$

ha la forma di $\dot{q}_i = \gamma_i(\underline{q}, \underline{p}, t)$

Inoltre

$$\frac{d}{dt} p_i = \frac{d}{dt} \left(\frac{\partial k}{\partial \dot{q}_i} \right) = \left(\frac{\partial k}{\partial q_i} + Q_i \right)_{\dot{q}_i = \gamma_i}$$

(↑ eq. di Lagrange)

$$= f_i(\underline{q}, \underline{p}, t)$$

$$\begin{cases} \frac{d}{dt} q_i = \gamma_i(\underline{q}, \underline{p}, t) \\ \frac{d}{dt} p_i = f_i(\underline{q}, \underline{p}, t) \end{cases}$$

Consideriamo il caso

$$K = \frac{1}{2} \, \dot{\underline{q}} \cdot A(q) \cdot \dot{\underline{q}}$$

$$\rightarrow \quad \underline{p} = A(q) \cdot \dot{\underline{q}}$$

$$\begin{cases} \dfrac{d}{d\tau} q_i = \left(A^{-1}(\underline{q}) \cdot \underline{p} \right)_i \\[2ex] \dfrac{d}{d\tau} p_i = \left(\dfrac{\partial K}{\partial q_i} + Q_i \right) \Bigg|_{\dot{\underline{q}} = A^{-1}(q)\, \underline{p}} \end{cases}$$

Linearizziamo:

$$\underline{q}(\tau) = \underline{q}_E + \eta \, \underline{x}(\tau)$$

evoluzione vicino a $\underline{q}_E$

piccolo

$$\rightarrow \quad \underline{p} = \underline{0} + \eta \, \underline{v}(\tau)$$

$$\dot{\underline{q}} = \underline{0} + \eta \, \dot{\underline{x}}(\tau)$$

1) $$\frac{d}{d\tau} q_i - \left(A^{-1}_{(q)}\, p\right)_i =$$

$$= \eta\, \dot{x}(\tau) - \left(A^{-1}(q_\varepsilon + \eta x)\, \eta\, v \right)$$

sostituito

$$= \eta \left(\dot{x}(\tau) - A^{-1}(q_\varepsilon)\, v \right) + O(\eta^2)$$

sviluppato

$$A^{-1}(q_\varepsilon + \eta x) = A^{-1}(q_\varepsilon) + \eta \left.\frac{\partial A^{-1}}{\partial \eta}\right|_{\eta=0} + \dots$$

$$\boxed{\dot{x} = A^{-1}(q_\varepsilon)\, v}$$

cont eq → $\begin{cases} \theta = \frac{\pi}{2} \\ \varphi = 0 \\ x = 0 \end{cases}$

2) $$\frac{d}{d\tau} p_i = \left(\frac{\partial K}{\partial q_i} + Q_i \right)\Bigg|_{\dot{q} = A^{-1}(q)\, p}$$

Notiamo:

$$\frac{\partial K}{\partial q_i} = \frac{\partial}{\partial q_i} \left(\frac{1}{2}\, \dot{q} \cdot A(q(t)) \cdot \dot{q} \right) =$$

$$= \frac{\partial}{\partial q_i}\left(\frac{1}{2}\, \eta\, \dot{\underline{x}}\; A(\underline{q}_\varepsilon + \eta \underline{x}) \cdot \eta \dot{\underline{x}} \right)$$

$$= O(\eta^2)$$

Rimane $Q_i(\underline{q}_\varepsilon + \eta \underline{x},\ \eta \dot{\underline{x}})$

$= 0$ perché cond. equilibrio

$$= Q_i(\underline{q}_\varepsilon, \underline{0}) + \eta \frac{d}{d\varepsilon} Q_i(\underline{q}_\varepsilon + \eta \underline{x}, \eta \dot{\underline{x}})\Big|_{\eta=0}$$

$$+ O(\eta^2)$$

$$= \eta\, Q_i^L + O(\eta^2)$$

$$Q_i^L = \frac{d}{d\eta} Q_i(\underline{q}_\varepsilon + \eta \underline{x}, \eta \dot{\underline{x}})\Big|_{\eta=0}$$

Allora:

$$\begin{cases} \dot{\underline{x}} = A^{-1}(\underline{q}_\varepsilon)\, \underline{v} \\ \dot{\underline{v}} = \underline{Q}^L \end{cases}$$

← derivo questo rispetto a t

$$\ddot{\underline{x}} = A^{-1}(\underline{q}_\varepsilon)\, \dot{\underline{v}} = A^{-1}(\underline{q}_0)\, \underline{Q}^L$$

Possiamo anche riscrivere queste equazioni come

$$A(\underline{q}_E)\,\ddot{\underline{x}} = \underline{Q}_L$$

equazioni di Lagrange linearizzate

Andiamo a vedere come è fatta $\underline{Q}_L$:

$$Q_i(\underline{q}_E + \eta\underline{x}, \eta\dot{\underline{x}}) =$$

$$\eta\left[\sum_{j=1}^{\ell} \frac{\partial Q_i}{\partial q_j}\Bigg|_{(\underline{q}_E, \underline{0})} x_j + \sum_{j=1}^{\ell} \frac{\partial Q_i}{\partial \dot{q}_j}\Bigg|_{(\underline{q}_E, \underline{0})} \dot{x}_j\right]$$

abbiamo : $\dfrac{\partial q_i}{\partial \eta} = \eta x_i$, $\dfrac{\partial \dot{q}_i}{\partial \eta} = \eta \dot{x}_i$

Quindi : $$\boxed{\underline{Q}^L = -\left(C\underline{x} + B\dot{\underline{x}}\right)}$$

C matrice costante $\qquad C_{ij} = - \left.\dfrac{\partial Q_i}{\partial q_j}\right|_{(\underline{q}_\varepsilon, \underline{0})}$

B matrice costante $\qquad B_{ij} = - \left.\dfrac{\partial Q_i}{\partial \dot{q}_j}\right|_{(\underline{q}_\varepsilon, \underline{0})}$

Equazioni di Lagrange linearizzate

$$A(\underline{q}_\varepsilon)\,\ddot{\underline{x}} + B(\underline{q}_\varepsilon)\,\dot{\underline{x}} + C(\underline{q}_\varepsilon)\,\underline{x} = \underline{0}$$

$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$

Caso conservativo : $Q_i = Q_i(\underline{q})$

in particolare $Q_i = - \dfrac{\partial V}{\partial q_i}$

Se questo è il caso:

$B_{ij} \equiv 0 \qquad C_{ij} = + \left.\dfrac{\partial}{\partial q_j}\dfrac{\partial V}{\partial q_i}\right|_{(\underline{q}_\varepsilon, \underline{0})}$

$= \text{Hess}\, V \big|_{(\underline{q}_\varepsilon, 0)}$

Ricordiamo:

$$V(\underline{q}) = \underbrace{V(\underline{q}_E)}_{\text{costante}} + \sum_{i=1}^{l} \underbrace{\left(\frac{\partial V}{\partial q_i}\right)}_{=0}\Big|_{\underline{q}_E} \eta x_i + \sum_{i,j}^{l} \frac{1}{2} \frac{\partial V}{\partial q_i \partial q_j}\Big|_{q_E} \eta x_i \, \eta x_j + \dots$$

Equazioni di Lagrange linearizzate nel caso conservativo sono:

$$A(\underline{q}_E)\,\ddot{\underline{x}} + \text{Hess}\,V\big|_{(\underline{q}_E)}\,\underline{x} = \underline{0}$$

$A(\underline{q})$ calcolata in $\underline{q}_E$ $\qquad$ $\frac{\partial V}{\partial q_i \partial q_j}$ calcolata in $\underline{q}_E$

$$\text{“}\ \ddot{x} + \alpha x = 0\ \text{”}$$

Nel caso conservativo: possiamo anche partire da $\mathcal{L} = K - V$

$$K = \frac{1}{2}\,\dot{\underline{q}} \cdot A(\underline{q}(t)) \cdot \dot{\underline{q}}$$

$$= \frac{1}{2}\left(\underline{0} + \eta\,\dot{\underline{x}}\right) A(\underline{q}_E + \eta\,\underline{x})\left(\underline{0} + \eta\,\dot{\underline{x}}\right)$$

$$= \frac{1}{2}\,\eta\,\dot{\underline{x}}\, A(\underline{q}_E)\,\eta\,\dot{\underline{x}} + \dots$$

$$V = \dots\ \frac{1}{2}\sum \left.\frac{\partial V}{\partial q_i \partial q_j}\right|_{\underline{q}_E} \eta x_i\, \eta x_j + \dots$$

Possiamo partire dalla Lagrangiana:

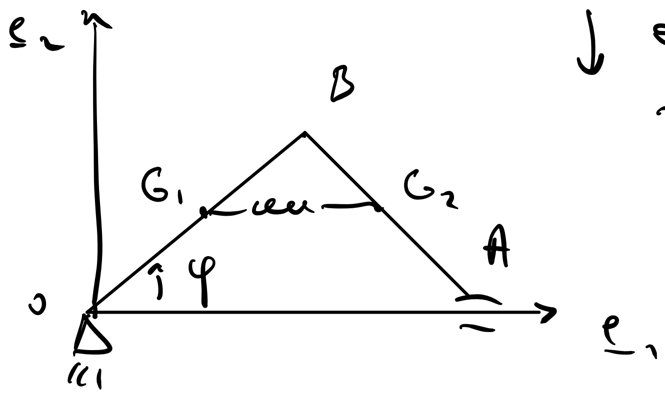
$$\tilde{\mathcal{L}} = \frac{1}{2}\,\dot{\underline{x}}\, A(\underline{q}_E)\,\dot{\underline{x}} - \frac{1}{2}\,\underline{x} \cdot \mathrm{Hess}\,V\big|_{\underline{q}_E} \cdot \underline{x}$$

Infatti le equazioni di Lagrange

$$A(\underline{q}_E)\,\ddot{\underline{x}} + \mathrm{Hess}\,V\big|_{(\underline{q}_E)}\,\underline{x} = 0$$

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}_i} - \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = 0$$

Esempio



$\downarrow \underline{g}$

OB, BA sono aste omogenee lunghezza ℓ massa m

Abbiamo già visto:

$$K_{OB} = \frac{1}{2} I_{3,O} \dot{\varphi}^2 = \frac{1}{2} \frac{m\ell^2}{3} \dot{\varphi}^2$$

$$K_{AB} = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} \frac{m\ell^2}{12} \dot{\varphi}^2$$

$$\underline{x}_{G_2} \rightarrow \frac{d}{dt} \underline{x}_{G_2} \rightarrow v_G^2$$

$$K = \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right)$$

1 grado di libertà: $\underline{\dot{q}} \rightarrow \dot{\varphi}$

$$\frac{1}{2} \underline{\dot{q}} A \underline{\dot{q}} \rightarrow \frac{1}{2} \dot{\varphi} \underbrace{\left[m\ell^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right) \right]}_{A(\varphi)} \dot{\varphi}$$

$$V = mg\ell \sin\varphi + \frac{c}{2} \ell^2 \cos^2 \varphi$$

Per linearizzare: conf. di equilibrio

$$\frac{dV}{d\varphi} = mgl\cos\varphi - cl^2\sin\varphi\cos\varphi =$$

$$= cl^2\cos\varphi\left(\frac{mg}{cl} - \sin\varphi\right)$$

$$= cl^2\cos\varphi\left(\gamma - \sin\varphi\right)$$

$$\gamma = \frac{mg}{cl}$$

Allora:

1) Se $\gamma = \frac{mg}{cl} > 1 \Rightarrow \cos\varphi = 0$

$$\varphi = \frac{\pi}{2}, -\frac{\pi}{2}$$

2) Se $\gamma = \frac{mg}{cl} < 1 \Rightarrow \varphi = \frac{\pi}{2}, -\frac{\pi}{2}$

$$\varphi = \arcsin\gamma$$

$$\varphi = \pi - \arcsin\gamma$$

Stabilità: minimi di V

$$V'' = cl\left(-\gamma\sin\varphi - \cos^2\varphi + \sin^2\varphi\right)$$

→ calcolarla nelle conf. di equilibrio.

Prendiamo $\varphi = \frac{\pi}{2}$

$$V''\left(\frac{\pi}{2}\right) = c\ell^2 (1-\gamma) = \begin{cases} <0 & \& \ \gamma > 1 \quad \text{INSTABILE} \\ >0 & \& \ \gamma < 1 \quad \text{STABILE} \end{cases}$$

Poniamo $\varphi = \frac{\pi}{2} + \underline{y}\, x$

$$\dot{\varphi} = \underline{y}\, \dot{x}$$

$$\tilde{\mathcal{L}} = \frac{1}{2} \dot{\underline{x}}\, A(\underline{q}_E) \cdot \dot{\underline{x}} - \frac{1}{2} \underline{x}\, \mathrm{Hess}\, V\Big|_{\underline{q}_E} \underline{x}$$

$$= \frac{1}{2} \dot{x} \left[m\ell^2 \left(\frac{2}{3} + 2 \sin^2\varphi \right) \right]\Bigg|_{\varphi = \frac{\pi}{2}} \dot{x}$$

$$- \frac{1}{2} x \left[c\ell \left(-\gamma \sin\varphi - \cos^2\varphi + \sin^2\varphi \right) \right]\Bigg|_{\varphi = \frac{\pi}{2}} x$$

$$= \frac{1}{2} m\ell^2 \dot{x}^2 \left(\frac{2}{3} + 2 \sin^2 \frac{\pi}{2} \right) +$$

$$- \frac{1}{2} x^2 c\ell \, (1-\gamma)$$

$$= \frac{1}{2} m\ell^2 \frac{8}{3} \dot{x}^2 - \frac{1}{2} x^2 c\ell (1-\gamma)$$

eq di Lagrange:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{8}{3} m\ell^2 \dot{x}\right) - \left(- c\ell^2(1-\gamma)x\right) = 0$$

$$\frac{8}{3} m\ell^2 \ddot{x} + c\ell^2(1-\gamma)\, x = 0$$

$$A(\underline{q}_\varepsilon)\, \ddot{\underline{x}} + \mathrm{Hess}\, V\big|_{\underline{q}_\varepsilon} \underline{x} = \underline{0}$$

[Richiamo eq diff a coefficienti costanti:

Chiamiamo $y = y(t)$ l'incognita:

$$a \frac{d^2y}{dt^2} + b\frac{dy}{dt} + c\, y = 0$$

Per risolverla prendiamo $y(t) = e^{kt}$

con k costante.

$y = e^{kt}$ $\quad \dot{y} = k e^{kt}$ $\quad \ddot{y} = k^2 e^{kt}$

$$(ak^2 + bk + c)\, e^{kt} = 0$$

è l'eq. che otteniamo sostituendo $y = e^{kt}$

equazione caratteristica

$$ak^2 + bk + c = 0$$

$$k_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad k_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Abbiamo 3 casi:

1) se $b^2 - 4ac$ è positivo

→ k_1, k_2 sono reali e distinti

e le soluzioni sono $e^{k_1 t}$, $e^{k_2 t}$

2) se $b^2 - 4ac$ è negativo

→ k_1, k_2 sono complessi e coniugati

$$k_{1,2} = \alpha \pm i\beta \qquad k = \alpha + i\beta$$

$$e^{k\tau} = e^{\alpha\tau} e^{i\beta\tau} =$$

$$= e^{\alpha\tau}(\cos\beta\tau + i\sin\beta\tau)$$

$$y(\tau) = e^{-\frac{b}{2a}\tau}(c_1\cos\beta\tau + c_2\sin\beta\tau)$$

$$\beta = \frac{\sqrt{4ac - b^2}}{2a}$$

è soluzione

3) $b^2 = 4ac \rightarrow k_1 = k_2 = -\frac{b}{2a}$

$$y(\tau) = c_1 e^{-\frac{b}{2a}\tau} + c_2 \tau e^{-\frac{b}{2a}\tau}$$

è la soluzione

Nel nostro caso

$$\frac{8}{3} m l^2 \ddot{x} + c l^2(1-\gamma) x = 0$$

$$a\ddot{x} + cx = 0$$

Quindi

$\gamma < 1 \longrightarrow$ oscillazioni armoniche

$$c_1 \cos \nu \tau + c_2 \sin \nu \tau$$

$$\nu = \sqrt{\frac{3}{8} \frac{c(1-\gamma)}{m}}$$

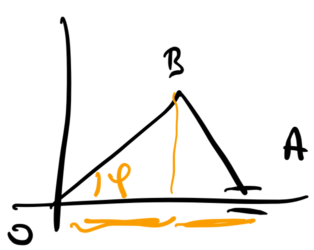
$\gamma > 1 \longrightarrow$ esponenziali

$$a_1 e^{-\nu \tau} + a_2 e^{\nu \tau}$$

$$\nu = \sqrt{\frac{3}{8} \frac{c(1-\gamma)}{m}}$$

<u>Esempio</u> stesso esempio, aggiungiamo

$$\underline{F}_A = - \nu \, \underline{v}_A \quad (\nu > 0 \text{ forza di attrito})$$



Forza generalizzata:

$$LV = \underline{F}_A \cdot \delta \underline{x}_A$$

$$= - \nu \, \dot{x}_A \, \delta x_A$$

$x_A = 2\ell \cos\varphi \qquad \delta x_A = -2\ell \sin\varphi \, \delta\varphi$

$$\dot{x}_A = -2\ell \sin\varphi\,\dot{\varphi}$$

Quindi $\underline{F}_A \cdot \delta \underline{x}_A = -\nu\, \dot{x}_A\, \delta x_A =$

$$= -\nu\,(-2\ell\sin\varphi\,\dot{\varphi})(-2\ell\sin\varphi\,\delta\varphi)$$

$$= \underbrace{-\nu\, 4\ell^2 \sin^2\varphi\,\dot{\varphi}}\,\delta\varphi = Q_\varphi\,\delta\varphi$$

$$Q_\varphi = -\nu\, 4\ell^2 \sin^2\varphi\,\dot{\varphi}$$

Linearizziamo Q_φ: attorno a $\varphi = \frac{\pi}{2}$

$$Q_\varphi = -4\nu\ell^2 \sin^2\left(\frac{\pi}{2} + \eta x\right)(\eta\dot{x})$$

$$\varphi = \frac{\pi}{2} + \eta x$$

$$\dot{\varphi} = \eta\dot{x}$$

$$= \eta\left(-4\nu\ell^2\sin^2\frac{\pi}{2}\right)\dot{x} + O(\eta^2)$$

$$Q_\varphi^{(L)} = -4\nu\ell^2\dot{x}$$

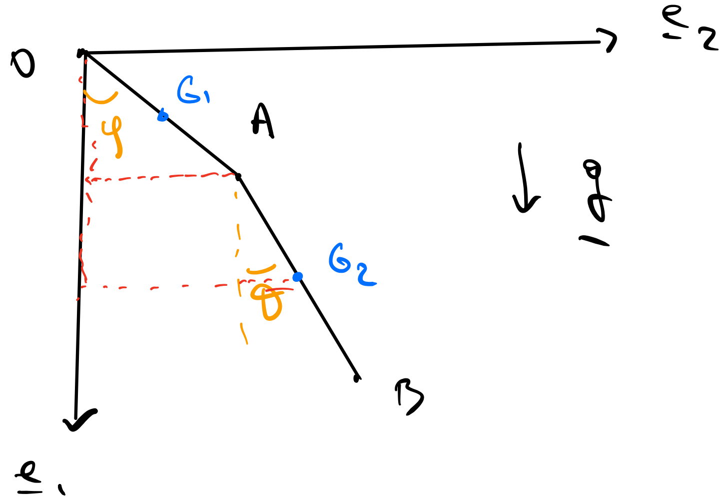
Mettiamo tutto insieme:

$$\frac{8}{3} m l^2 \ddot{x} + c l^2 (1-\gamma) x + 4 \nu l^2 \dot{x} = 0$$

Trovato prima

$$a \ddot{x} + b \dot{x} + c x = 0$$

Esercizio



$OA = l \quad m$

$\overline{AB} = 2l \quad 2m$

Energia cinetica:

$$K_{OA} = \frac{1}{2} \frac{m l^2}{3} \dot{\varphi}^2$$

$$K_{AB} = \frac{1}{2} (2m) v_{G_2}^2 + \frac{1}{2} \frac{(2m)(2l)^2}{12} \dot{\theta}^2$$

$$\underline{x}_{G_2} = (l \sin\varphi + l \sin\theta)\, \underline{e}_2 + (l \cos\varphi + l \cos\theta)\, \underline{e}_1$$

$$\underline{v}_{G_2} = (-l \sin\varphi\, \dot{\varphi} - l \sin\theta\, \dot{\theta})\, \underline{e}_1 +$$

$$+ \left(l\cos\varphi\,\dot\varphi + l\cos\theta\,\dot\theta \right) \underline{e}_2$$

$$v_{G_2}^2 = l^2 \left[\sin^2\varphi\,\dot\varphi^2 + \sin^2\theta\,\dot\theta^2 + 2\dot\varphi\dot\theta\sin\varphi\sin\theta \right.$$

$$\left. + \cos^2\varphi\,\dot\varphi^2 + \cos^2\theta\,\dot\theta^2 + 2\cos\varphi\cos\theta\,\dot\varphi\dot\theta \right]$$

$$= l^2 \left[\dot\varphi^2 + \dot\theta^2 + 4\dot\varphi\dot\theta\cos(\varphi-\theta) \right]$$

Allora

$$K = \frac{1}{2} m \frac{l^2}{3} \dot\varphi^2 + \frac{1}{2} \frac{(2m)(2l)^2}{12} \dot\theta^2 +$$

$$+ \frac{1}{2} m l^2 2 \left[\dot\varphi^2 + \dot\theta^2 + 4\dot\varphi\dot\theta\cos(\varphi-\theta) \right]$$

$$= \frac{1}{2} m l^2 \left[\frac{7}{3}\dot\varphi^2 + \frac{8}{3}\dot\theta^2 + 8\dot\varphi\dot\theta\cos(\varphi-\theta) \right]$$

$$V = - mg\frac{l}{2}\cos\varphi - 2m\left(l\cos\varphi + l\cos\theta \right) g$$

$$= - mgl\left(2\cos\theta + \frac{5}{2}\cos\varphi \right)$$

Provate per esercizio a scrivere le equazioni di Lagrange

Linearizziamo: vicino a $\varphi = 0$, $\theta = 0$

Poniamo $\varphi = \eta_1$, $\theta = \eta_2$

$$K = \frac{1}{2} m l^2 \left[\frac{7}{3} \dot{\varphi}^2 + \frac{8}{3} \dot{\theta}^2 + 8 \dot{\varphi} \dot{\theta} \cos(\varphi - \theta) \right]$$

$$= \frac{1}{2} (\dot{\varphi} \ \ \dot{\theta}) \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \end{pmatrix}$$

$$= \frac{1}{2} (\dot{\varphi} \ \ \dot{\theta}) \begin{pmatrix} m l^2 \frac{7}{3} & m l^2 4 \cos(\varphi - \theta) \\ m l^2 4 \cos(\varphi - \theta) & m l^2 \frac{8}{3} \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \end{pmatrix}$$

calcolata in $\theta = \varphi = 0$

$$\tilde{K} = \frac{1}{2} m l^2 \left(\frac{7}{3} \dot{\eta}_1^2 + \frac{8}{3} \dot{\eta}_2^2 + 8 \dot{\eta}_1 \dot{\eta}_2 \right)$$

$$V = - m g l \left(\frac{5}{2} \cos\varphi + 2 \cos\theta \right)$$

$$\frac{\partial V}{\partial \varphi} = m g l \frac{5}{2} \sin\varphi \qquad \frac{\partial V}{\partial \theta} = m g l \, 2 \sin\theta$$

$$\text{Hess} V = \begin{pmatrix} \frac{\partial^2 V}{\partial \varphi^2} & \frac{\partial V}{\partial \varphi \partial \theta} \\ \frac{\partial V}{\partial \varphi \partial \theta} & \frac{\partial^2 V}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} m g l \frac{5}{2} \cos\varphi & 0 \\ 0 & m g l \, 2 \cos\theta \end{pmatrix}$$

$$\frac{1}{2}\,\underline{x}\cdot \text{Hess}\,V\big|_{q_e}\cdot \underline{z} =$$

$$\frac{1}{2}\,(z_1 \;\; z_2)\begin{pmatrix} mgl\,\frac{5}{2} & 0 \\ 0 & mgl\,2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \frac{1}{2}\,mgl\left(\frac{5}{2}\,z_1^2 + 2\,z_2^2\right)$$

$$\tilde{\mathcal{L}} = \frac{1}{2}\,ml^2\left(\frac{7}{3}\,\dot z_1^2 + \frac{8}{3}\,\dot z_2^2 + 8\,\dot z_1\,\dot z_2\right) +$$

$$- \frac{1}{2}\,mgl\left(\frac{5}{2}\,z_1^2 + 2\,z_2^2\right)$$

da cui le eq. di Lagrange linearizzate

$$A(\underline{q}_e)\cdot\begin{pmatrix} \ddot z_1 \\ \ddot z_2 \end{pmatrix} + \text{Hess}\,V\big|_{q_e}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} ml^2\,\frac{7}{3} & 4ml^2 \\ 4ml^2 & ml^2\,\frac{8}{3} \end{pmatrix}\begin{pmatrix} \ddot z_1 \\ \ddot z_2 \end{pmatrix} + \begin{pmatrix} mgl\,\frac{5}{2} & 0 \\ 0 & 2mgl \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\begin{cases} ml^2\left(\frac{7}{3}\ddot{\theta}_1 + 4\ddot{\theta}_2\right) + mgl\frac{5}{2}\theta_1 = 0 \\ ml^2\left(\frac{8}{3}\ddot{\theta}_2 + 4\ddot{\theta}_1\right) + 2mgl\,\theta_2 = 0 \end{cases}$$

$$a\ddot{y} + b\dot{y} + cy = 0 \quad \longrightarrow \quad \begin{cases} a_1\ddot{\theta}_1 + b_1\theta_1 = 0 \\ a_2\ddot{\theta}_2 + b_2\theta_2 = 0 \end{cases}$$