MECCANICA RAZIONALE

LINEARIZZAZIONE EQUAZIONI DEL MOTO

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$$

$$\downarrow$$

$$\underbrace{A(\underline{q}_E)}\,\ddot{\underline{x}} + \underbrace{\text{Hess}\, V}_{\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\underline{q}_E}}\Big|_{\underline{q}_E} \underline{x} = 0$$

$$\to \quad A\,\ddot{\underline{x}} + C\,\underline{x} = 0$$

matrici a coefficienti costanti

Se fosse $\to$ $\alpha\, \ddot{y} + \gamma\, y = 0$

Invece : $\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}\begin{pmatrix}\ddot{x}_1 \\ \vdots \\ \ddot{x}_\ell\end{pmatrix} + \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}\begin{pmatrix}x_1 \\ \vdots \\ x_\ell\end{pmatrix} = 0$

→ Idea dei modi normali :

cambio di coordinate $\underline{x} \longrightarrow \underline{\xi}$

$$K = \frac{1}{2}\, \dot{\underline{x}}\, A\, \dot{\underline{x}} \qquad \longrightarrow \qquad K = \frac{1}{2}\, \dot{\underline{\xi}}\, \mathbb{1}\, \dot{\underline{\xi}}$$

$$V = \frac{1}{2}\, \underline{x}\, C\, \underline{x} \qquad \qquad V = \frac{1}{2}\, \underline{\xi}\, \tilde{C}\, \underline{\xi}$$

$$\tilde{C} = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_\ell \end{pmatrix}$$

$$A\ddot{\underline{x}} + C\underline{x} = 0 \iff \ddot{\xi}_i + \gamma_i \xi_i = 0 \quad \forall i = 1, \dots, \ell$$

eq fisica ⟷ coordinate normali

→ Guardiamo le eq: $\underline{\ddot{\xi}_i + \gamma_i \xi_i = 0}$

Fissiamo i → allora γ_i

- $\gamma_i > 0$: $\gamma_i = \omega_i^2$ con $\omega_i > 0$

$\underline{v}^i \sin\omega_i t \quad , \quad \underline{v}^i \cos\omega_i t$

v^i è il vettore $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ← 1 nella componente i-esima

- $\gamma_i < 0$: $\gamma_i = -\nu_i^2 \qquad \nu_i > 0$

$\underline{v}^i e^{\nu_i t} \quad , \quad \underline{v}^i e^{-\nu_i t}$

- $\gamma_i = 0$: $\underline{v}^i \ , \ \underline{v}^i t$

Soluzione di tutto il sistema

$$\underline{v}^1 \begin{cases} c_1 \sin\omega_1 t + c_2 \cos\omega_1 t \\ \text{oppure} \\ c_1 e^{\nu_1 t} + c_2 e^{-\nu_1 t} \\ \text{oppure} \\ \ldots \end{cases} + \underline{v}^2 \begin{cases} \text{---} \\ \text{---} \\ \text{---} \end{cases} + \ldots$$

Ricordiamoci : $\underline{x} = S \underline{\xi}$

$\ddot{\xi}_i + \gamma_i \xi_i = 0 \longleftrightarrow A\ddot{\underline{x}} + C\underline{x} = \underline{0}$

La soluzione $\boxed{\underline{v}^i e^{\lambda_i t}}$

$$\left(\lambda_i^2 + \gamma_i = 0 \longrightarrow \begin{cases} \gamma_i > 0 & \lambda_i = \pm i\omega_i \\ \gamma_i < 0 & \lambda_i = \pm \nu_i \\ \gamma_i = - & \text{—} \end{cases} \right)$$

$$S\left(\underline{v}^i e^{\lambda_i T}\right) = e^{\lambda_i T} S(\underline{v}^i)$$

$$= e^{\lambda_i T} \underline{u}^i$$

$$\boxed{\underline{u}^i = S(\underline{v}^i)} \qquad \underline{v}^i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \begin{array}{l} 1 \\ \text{nella} \\ \text{componente} \\ i\text{-esima} \end{array}$$

$$A\ddot{\underline{x}} + C\underline{x} = 0 \longrightarrow \lambda_i^2 A \underline{u}^i + C\underline{u}^i = 0$$

Abbiamo che:

- $\lambda_i^2 = -\gamma_i \in \mathbb{R} \qquad i = 1 \to \ell$

- $\underline{u}^i \in \mathbb{R}^\ell$, Tali che

$$\underline{u}^i A \underline{u}^j = (u^i)^T A u^j =$$

$$(v^i)^T \underbrace{S^T A S}_{\mathbb{1}_{\ell\times\ell} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}} v^j = (v^i)^T v^j$$

$$(\cdots 1 \cdots) \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$= \delta_{ij} = \begin{cases} 1 & \text{se } i=j \\ 0 & \text{altrimenti} \end{cases}$$

Quindi $\boxed{\underline{u}^i A \underline{u}^j = \delta_{ij}}$

Come visto prima: $A\ddot{\underline{x}} + C\underline{x} = 0$

segue che $(\lambda^2 A + C)\underline{u} = \underline{0}$

Quindi:

- λ^2 è radice dell'equazione caratteristica

$$\det(\lambda^2 A + C) = 0$$

Infatti:

$$\det(\lambda^2 A + C) = 0$$

$$= \prod_i (\lambda_i^2 + \gamma_i) = 0 \quad \text{se}$$

λ_i^2 è uguale ad un $-\gamma_i$

$$\left[(\lambda_1^2+\gamma_1)(\lambda_2^2+\gamma_2)\cdots(\lambda_\ell^2+\gamma_\ell)=0\right]$$

$$\boxed{\lambda_i^2 + \gamma_i = 0}$$

$$\boxed{(\lambda_1^2 + \gamma_1)(\lambda_2^2 + \gamma_2)\cdots(\lambda_\ell^2 + \gamma_\ell)} =$$

$$\det\left(\lambda^2 \mathbb{1}_{\ell\times\ell} + \tilde{C}\right)$$

$$\tilde{C} = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_\ell \end{pmatrix}$$

$$\begin{cases} \mathbb{1}_{\ell\times\ell} = S^T A S & \leftarrow \text{energia cinetica fisica} \\ \tilde{C} = S^T C S & \leftarrow \text{energia potenziale fisica} \end{cases}$$

$$= \det\left(\lambda^2 S^T A S + S^T C S\right)$$

$$= \det\left[S^T (\lambda^2 A + C) S\right]$$

$$= \det S^T \cdot \det(\lambda^2 A + C) \cdot \det(S)$$

$$= (\det S)^2 \underbrace{\det(\lambda^2 A + C)}_{=0}$$

$\det S^T = \det S \neq 0$

λ^i sono determinati da

$$\det(\lambda^2 A + C) = 0$$

Nelle coordinate normali $\ddot{\xi}_i + \gamma_i \xi = 0$

$\underline{v}^i e^{\lambda_i t} \longrightarrow \lambda_i^2 + \gamma_i = 0$ è un'eq. per λ_i^2

per ogni i.

Cerchiamo tutti i λ^2 che sono soluzioni di:

$$(\lambda^2+\gamma_1)(\lambda^2+\gamma_2)(\lambda^2+\gamma_3)\cdots(\lambda^2+\gamma_n) = 0$$

$$= \underset{\neq 0}{(\det S)^2}\ \det(\lambda^2 A + C)$$

$$\Rightarrow \boxed{\det(\lambda^2 \underline{A} + \underline{C}) = 0}$$

$$e^{\lambda t}$$

Quindi :

$A\ddot{\underline{x}} + C\underline{x} = 0$ → cerco soluzioni del tipo $\underline{x}(\tau) = \underline{u}\, e^{\lambda\tau}$

- $\det(\lambda^2 A + C) = 0$ determina λ^2 (autovalore di C relativo ad A)

- $(\lambda^2 A + C)\underline{u} = 0$ → determina $\underline{u}$ associato a λ^2

(autovettore di C relativo ad A)

Troviamo :

- $\det(\lambda^2 A + C) = 0$ ha ℓ radici reali

- $(\lambda^2 A + C)\underline{u} = 0$ ha almeno ℓ soluzioni reali & distinte e soli da $\underline{u}^i A \underline{u}^j = \delta_{ij}$

non è necessario

$$A\ddot{\underline{x}} + C\underline{x} = \underline{0} \xrightarrow{S} \ddot{\xi}_i + \gamma_i \xi_i = 0$$

$$\underline{v}^i e^{\lambda_i t}$$

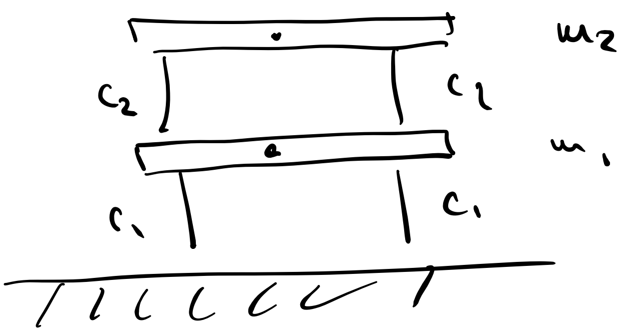
$$\lambda_i^2 + \gamma_i = 0$$

l soluzioni

$$\begin{cases} \det(\lambda^2 A + C) = 0 \\ (\lambda^2 A + C)\underline{u} = 0 \end{cases}$$

l vettori

Esempio



$c_1 = c_2 = c$

$m_1 = 2m$

$m_2 = m$

$$K = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2}(c)\, x_1^2 + \frac{1}{2} c\, (x_1 - x_2)^2$$

$$\hookrightarrow \begin{cases} \frac{m}{2c}\, 2\, \frac{d^2 x_1}{d\tau^2} + 2x_1 - x_2 = 0 \\ \frac{m}{2c}\, \frac{d^2 x_2}{d\tau^2} - x_1 + x_2 = 0 \end{cases}$$

$$A \frac{d^2}{d\tau^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\frac{m}{2c} \frac{d^2}{d\tau^2} = \frac{d^2}{d\left(\frac{2c}{m}\tau^2\right)} = \frac{d^2}{d\tau^2}$$

$$\tau = \sqrt{\frac{2c}{m}}\, \tau \qquad \tau^2 = \frac{2c}{m} \tau$$

$$A \frac{d^2}{d\tau^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$A = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}, \quad C = \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}}$$

Vogliamo soluzioni del tipo

$$\underline{x} = \underline{u}\, e^{\lambda t}$$

$$(\lambda^2 A + C)\,\underline{u} = 0$$

$$\det(\lambda^2 A + C) = 0$$

$$= \det\begin{pmatrix} 2\lambda^2 + 2 & -1 \\ -1 & \lambda^2 + 1 \end{pmatrix} = 0$$

$$\lambda^2 A + C = \lambda^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2\lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2\lambda^2 + 2 & -1 \\ -1 & \lambda^2 + 1 \end{pmatrix}$$

$$\det(\lambda^2 A + C) = 0 =$$

$$= (2\lambda^2 + 2)(\lambda^2 + 1) - 1$$

$$= 2(\lambda^2+1)^2 - 1 = 0$$

$$(\lambda^2+1) = \pm \frac{1}{\sqrt{2}} \Rightarrow \lambda^2 = -1 \pm \frac{1}{\sqrt{2}} < 0$$

Frequenze di oscillazione :

$$\lambda = \pm i\omega$$

$$\omega_1 = \left(1 - \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} \qquad \omega_2 = \left(1 + \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}$$

Ad esempio $\lambda^2 = -1 + \frac{1}{\sqrt{2}} = -\left(1 - \frac{1}{\sqrt{2}}\right)$

$$= -\omega_1^2$$

Adesso vogliamo :

$$(\lambda^2 A + C)\,\underline{u} = 0$$

$$\begin{pmatrix} 2\lambda^2+2 & -1 \\ -1 & \lambda^2+1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\begin{cases} 2(\lambda^2+1)u_1 - u_2 = 0 \\ -u_1 + (\lambda^2+1)u_2 = 0 \end{cases}$$

Prendiamo la prima:

$$2(\lambda^2+1)u_1 = u_2 \quad \to$$

quando $\lambda^2 = -1 + \frac{1}{\sqrt{2}}$ abbiamo $u_2 = \sqrt{2}\, u_1$

e quindi $\underline{u}^{(1)} = b_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ $b_1 \in \mathbb{R}$

quando $\lambda^2 = -1 - \frac{1}{\sqrt{2}}$ abbiamo

$$2\left(-1-\frac{1}{\sqrt{2}}+1\right)u_1 - u_2 = 0$$

$$\underline{u}^{(2)} = b_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad b_2 \in \mathbb{R}$$

$\ulcorner$ avremo $\to$ $u_i^T A\, u_j = \delta_{ij}$ $\lrcorner$

Abbiamo trovato:

$\underline{u}^{(1)} \sin \underline{\omega_1} t$ $\qquad$ $\underline{u}^{(1)} \cos \underline{\omega_1} t$

$\underline{u}^{(2)} \sin \underline{\omega_2} t$ $\qquad$ $\underline{u}^{(2)} \cos \underline{\omega_2} t$

La soluzione è

$$\underline{x}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \left[k_1 \sin \omega_1 \tau + k_2 \cos \omega_1 \tau \right]$$

$$+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \left[k_3 \sin \omega_2 \tau + k_4 \cos \omega_2 \tau \right]$$

$$\omega_1 = \left(1 - \frac{1}{\sqrt{2}} \right)^{\frac{1}{2}}, \qquad \omega_2 = \left(1 + \frac{1}{\sqrt{2}} \right)^{\frac{1}{2}}$$

$$\tau = \sqrt{\frac{2c}{m}}\, t$$

b_1

$$= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \left[k_1 \sin \sqrt{1 - \frac{1}{\sqrt{2}}} \sqrt{\frac{2c}{m}}\, t + k_2 \cos \sqrt{1 - \frac{1}{\sqrt{2}}} \sqrt{\frac{2c}{m}}\, t \right]$$

$$+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \left[k_3 \sin \sqrt{1 + \frac{1}{\sqrt{2}}} \sqrt{\frac{2c}{m}}\, t + k_4 \cos \sqrt{1 + \frac{1}{\sqrt{2}}} \sqrt{\frac{2c}{m}}\, t \right]$$

queste risolve $\quad A \ddot{\underline{x}} + C \underline{x} = 0$

$$u^{(1)} = b_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \qquad \underline{u}^{(2)} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} b_2$$

$$\underline{u}^{(1)} \cdot A \cdot \underline{u}^{(1)} = b_1 \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} b_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$= b_1^2 \, 4 = 1 \qquad b_1 = \frac{1}{2}$$

$$\underline{u}^{(2)} \cdot A \cdot \underline{y}^{(2)} = b_2 \begin{pmatrix} 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} b_2$$

$$= b_2^2 \, 4 = 1 \qquad b_2 = \frac{1}{2}$$

Allora la normalizzazione corretta

$$\underline{u}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \qquad \underline{u}^{(2)} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

Se $\underline{x} = S \, \underline{\xi}$

$$\begin{cases} x_1 = \frac{1}{2} (\xi_1 + \xi_2) \\ x_2 = \frac{1}{2} \sqrt{2} \, (\xi_1 - \xi_2) \end{cases}$$

relazioni fra coordinate fisiche e coordinate normali.

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \xi_1 + \xi_2 \\ \sqrt{2}\,\xi_1 - \sqrt{2}\,\xi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\xi_1 + \xi_2) \\ \frac{\sqrt{2}}{2}(\xi_1 - \xi_2) \end{pmatrix}$$

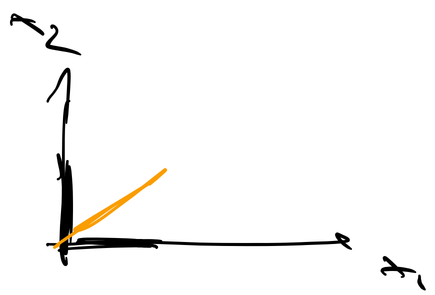
Siamo partiti da $A\ddot{\underline{x}} + C\underline{x} = 0$

x_1, x_2 erano spostamenti di G_1, G_2 dei due piani

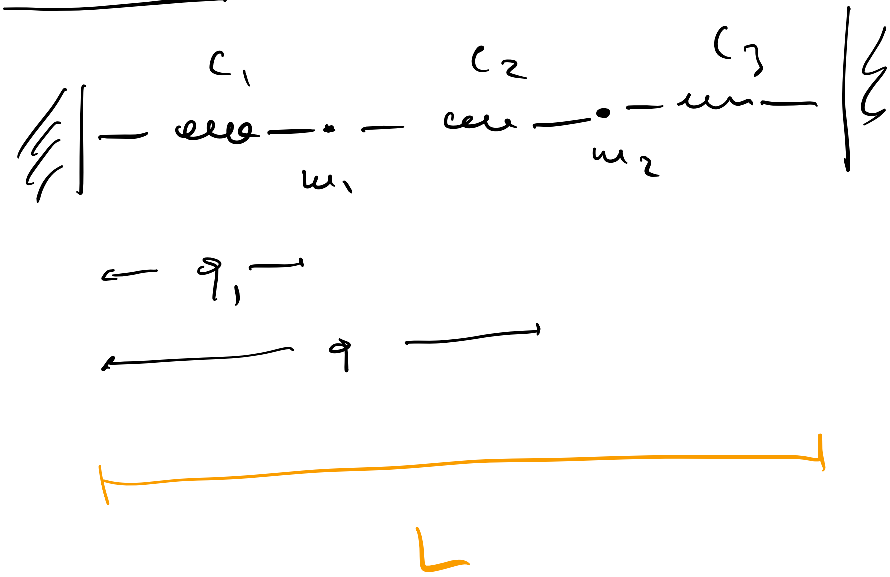
$$\underline{x} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} (k_1 \sin\omega_1 t + k_2 \cos\omega_1 t)$$
$$+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} (k_3 \sin\omega_2 t + k_4 \cos\omega_2 t)$$

$$x_1, x_2 \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\longrightarrow$ oscillazioni collettive



Esempio



$m_1 = 3m$

$m_2 = 2m$

$c_1 = 2c$

$c_2 = 2c$

$c_3 = c$

$$\underline{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$A\ddot{\underline{q}} + C\underline{q} - \underline{b} = \underline{0}$$

$$A = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad C = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 0 \\ c_3 L \end{pmatrix}$$

Configurazione di equilibrio:

$$C\underline{q}_E = \underline{b} \qquad C = \begin{pmatrix} 4c & -2c \\ -2c & 3c \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ cL \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} 4c\,q_1 - 2c\,q_2 = 0 \\ -2c\,q_1 + 3c\,q_2 = cL \end{cases} \longrightarrow \begin{cases} q_{1,E} = \dfrac{L}{4} \\ q_{2,E} = \dfrac{L}{2} \end{cases}$$

$x_1 = q_1 - \frac{L}{4}$ $\qquad$ $x_2 = q_2 - \frac{L}{2}$

l'eq. di moto per $\underline{x} = (x_1, x_2)$

$$\frac{m}{c}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \frac{d^2}{dT^2}\underline{x} + \begin{pmatrix} \frac{4c}{c} & \frac{-2c}{c} \\ \frac{-2c}{c} & \frac{3c}{c} \end{pmatrix} \underline{x} = 0$$

Stesso trucco di prima: ridefiniamo

$T \to \tau$ per eliminare m e c

$$\frac{m}{c}\frac{d^2}{dT^2} = \frac{d^2}{d\left(\frac{c}{m}T^2\right)} = \frac{d^2}{d\tau^2} \qquad \tau = \sqrt{\frac{c}{m}}\,T$$

$$\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\tilde{A}} \frac{d^2}{d\tau^2}\underline{x} + \underbrace{\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}}_{\tilde{C}} \underline{x} = \underline{0}$$

Frequenze: $\quad \det\left(\lambda^2 \tilde{A} + \tilde{C}\right) = 0$

$$\lambda^2 \tilde{A} + \tilde{C} = \begin{pmatrix} 3\lambda^2 & 0 \\ 0 & 2\lambda^2 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3\lambda^2 + 4 & -2 \\ -2 & 2\lambda^2 + 3 \end{pmatrix}$$

$$\det \begin{pmatrix} 3\lambda^2 + 4 & -2 \\ -2 & 2\lambda^2 + 3 \end{pmatrix} =$$

$$= (3\lambda^2 + 4)(2\lambda^2 + 3) - 4 =$$

$$= 6\lambda^4 + (9 + 8)\lambda^2 + 12 - 4$$

$$= 6\lambda^4 + 17\lambda^2 + 8 = 0$$

polinomio di secondo grado nella variabile λ^2

$$\lambda^2 = \frac{-17 \pm \sqrt{17^2 - 24 \cdot 8}}{12}$$

$289 - 192$

$$= \frac{-17 \pm \sqrt{97}}{12}$$

$$\omega_1 = \sqrt{\frac{17 - \sqrt{97}}{12}} \quad , \quad \omega_2 = \sqrt{\frac{17 + \sqrt{97}}{12}}$$

$\cos \sqrt{\frac{17 - \sqrt{97}}{12}} \sqrt{\frac{C}{m}}\, \tau$ ...

$\rightarrow \sin\sqrt{\frac{17-\sqrt{97}}{12}}\sqrt{\frac{c}{m}}\,t \quad , \quad \cos\sqrt{\frac{17-\sqrt{97}}{12}}\sqrt{\frac{c}{m}}\,t$

$\rightarrow \sin\sqrt{\frac{17+\sqrt{97}}{12}}\sqrt{\frac{c}{m}}\,t \quad , \quad \cos\sqrt{\frac{17+\sqrt{97}}{12}}\sqrt{\frac{c}{m}}\,t$

Adesso dobbiamo risolvere : $(\lambda^2\tilde{A}+\tilde{C})\underline{u}=0$

$$\begin{pmatrix} 3\lambda^2+4 & -2 \\ -2 & 2\lambda^2+3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=0$$

$$\underline{u}^{(1)}=\begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} \qquad \underline{u}^{(2)}=\begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix}$$

Il sistema generico

$$\begin{cases} (3\lambda^2+4)u_1-2u_2=0 \\ -2u_1+(2\lambda^2+3)u_2=0 \end{cases}$$

$$\rightarrow -2u_1+(2\lambda^2+3)u_2=0$$

$$u_1=\frac{1}{2}(2\lambda^2+3)u_2$$

1) $$\lambda^2 = \frac{-17+\sqrt{97}}{12} \qquad \underline{u}^{(1)} = \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix}$$

$$u_1^{(1)} = \frac{1}{2}\left[\cancel{2}\,\frac{-17+\sqrt{97}}{\cancel{12}_{\,6}} + 3\right] u_2^{(1)}$$

$$= \frac{1}{2}\left[\frac{1+\sqrt{97}}{6}\right] u_2^{(1)} \qquad \frac{18}{6}$$

$$\underline{u}^{(1)} = \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\frac{1+\sqrt{97}}{6}\,u_2^{(1)} \\ u_2^{(1)} \end{pmatrix} =$$

$$= b_1 \begin{pmatrix} \frac{1}{2}\,\frac{1+\sqrt{97}}{6} \\ 1 \end{pmatrix}$$

2) $$\lambda^2 = \frac{-17-\sqrt{97}}{12}$$

$$u_1 = \frac{1}{2}(2\lambda^2 + 3)u_2$$

$$= \frac{1}{2}\left[2\,\frac{-17-\sqrt{97}}{12} + 3\right] u_2$$

$$= \frac{1}{2}\left[\frac{1-\sqrt{97}}{6}\right] u_2$$

Allora $\underline{u}^{(2)} = \begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix} =$

$$= b_2 \begin{pmatrix} \frac{1-\sqrt{97}}{12} \\ 1 \end{pmatrix}$$

Allora abbiamo trovato i modi normali:

$$\begin{pmatrix} \frac{1+\sqrt{97}}{12} \\ 1 \end{pmatrix} \left[k_1 \sin \omega_1 \tau + k_2 \cos \omega_1 \tau \right]$$

$$+ \begin{pmatrix} \frac{1-\sqrt{97}}{12} \\ 1 \end{pmatrix} \left[k_3 \sin \omega_2 \tau + k_4 \cos \omega_2 \tau \right]$$

dove $\tau = \sqrt{\frac{c}{m}}\, T$ $\qquad \omega_1 = \sqrt{\frac{17-\sqrt{97}}{12}}$

$$\omega_2 = \sqrt{\frac{17+\sqrt{97}}{12}}$$

Supponiamo di avere dei dati iniziali,

$$\begin{cases} q_1(0) = 0 \\ q_2(0) = \frac{L}{2} \end{cases} \qquad \begin{cases} \dot{q}(0) = 0 \\ \dot{q}_2(0) = 0 \end{cases}$$

$$\rightarrow \quad \underline{x}(0) = \underline{q}(0) - \underline{q}_E = \begin{pmatrix} -\frac{L}{4} \\ 0 \end{pmatrix}$$

$$q_E = \begin{pmatrix} \frac{L}{4} \\ \frac{L}{2} \end{pmatrix}$$

$$\dot{x}(0) = \underline{0}$$

$$\dot{x}(0) = \begin{pmatrix} \\ \\ \end{pmatrix} \left(\omega_1 k_1 \cos \omega_1 \overset{0}{t} - \overset{\omega_1}{k_2} \sin \omega_1 \overset{0}{t} \right)$$

$$- - -$$

$$\begin{pmatrix} \\ \\ \end{pmatrix} \left(\omega_2 k_3 \cos \omega_2 \overset{0}{t} - \omega_2 k_2 \sin \omega_2 \overset{0}{t} \right)$$

$$k_1 = k_3 = 0$$

$$\underline{x}(0) = \begin{pmatrix} -\frac{L}{4} \\ 0 \end{pmatrix} \longrightarrow k_2 \text{ e } k_4$$

Se abbiamo un'eq.

$$A\ddot{\underline{x}} + C\underline{x} = 0$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

le frequenze dei modi normali

risolvendo $\det(\lambda^2 A + C) = 0$

$\rightarrow$ troviamo 2 soluzioni per

$$\lambda^2 \longrightarrow \begin{array}{ll} < 0 & \sin, \cos \\ > 0 & e^{\nu t}, e^{-\nu t} \end{array}$$

andiamo a calcolare gli autovettori relativi

$$(\lambda^2_{(i)} A + C)\begin{pmatrix} u_1^{(i)} \\ u_2^{(i)} \end{pmatrix} = 0 \qquad i = 1, 2$$

$\rightarrow$ troviamo $\underline{u}^{(i)} = \begin{pmatrix} u_1^{(i)} \\ u_2^{(i)} \end{pmatrix}$ a

meno di una costante moltiplicativa

Caso di un Termine forzante

$$A \ddot{\underline{x}} + C \underline{x} = \underline{F}(t)$$

sol particolare

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial k}{\partial \dot{x}}\right) - \dfrac{\partial k}{\partial x} = Q_x \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial k}{\partial \dot{\varphi}}\right) - \dfrac{\partial k}{\partial \varphi} = Q_\varphi \end{cases}$$

k — —

Q_x, Q_φ $\qquad\qquad$ $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial \varphi}$

$$Q_x = -\frac{\partial V}{\partial x} + \underline{\delta F_0} \cdot \delta x \, e_1$$

$$Q_y = -\frac{\partial V}{\partial y} + \cancel{\delta F \cdot \delta x} \, e_2$$

$$\delta V = \underline{F}_0 \cdot \delta \underline{x}_0 = F_0 \, \delta x_0$$

$$\underline{x}_0 = x_0 \underline{e}_1$$

$$= Q_{x_0} \, \delta x_0$$

$$K = K_0 + \underline{b} \cdot \dot{\underline{q}} + \frac{1}{2} \dot{\underline{q}} \cdot A \cdot \dot{\underline{q}}$$

$$= \frac{1}{2} \sum_b m_b v_b^2$$

$$v_b = \sum \frac{\partial \underline{x}_B}{\partial q_i} \dot{q}_i$$

$$\underline{x}_B = x_B(\underline{q}(t), t)$$

$$\underline{v}_B = \frac{d}{dt} \underline{x}_B = \sum_i \overset{①}{\frac{\partial \underline{x}_B}{\partial q_i} \frac{dq_i}{dt}} + \overset{②}{\frac{\partial \underline{x}_B}{\partial t}}$$

$i\,0$

$\dot{q}_i$

$$v_B^2 = (\quad) = ①^2 + ②^2 + 2\,①\cdot②$$

$$\underline{\frac{1}{2} \dot{q} A \dot{q}} \qquad K_0 \qquad \underline{b} \cdot \dot{\underline{q}}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$